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ASYMPTOTIC COVERAGE DISTRIBUTIONS ON THE CIRCLE[1]

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Place n arcs, each of length a_n, uniformly at random on the circumference of a circle, choosing the arc length sequence a_n so that the probability of completely covering the circle remains constant. We obtain the limiting distribution of the uncovered proportion of the circle. We show that this distribution has a natural interpretation as a noncentral chi-square distribution with zero degrees of freedom by expressing it as a Poisson mixture of mass at zero with central chi-square deviates having even degrees of freedom. We also treat the case of proportionately smaller arcs and obtain a limiting normal distribution. Potential applications include immunology, genetics, and time series analysis.

1. Introduction and summary. Coverage problems arise in a wide variety of applications. The particular problem of the coverage of a circle by n random equal arcs, each of length a_n, has been associated with research in immunology (Moran and Fazekas de St. Groth (1962)), genetics (Stevens (1939)), and time series analysis (Fisher (1940)). In a previous paper (Siegel (1978a)), exact formulae for the moments and distribution of coverage were obtained for this problem. The purpose of the present paper is to explore the asymptotic behavior of the coverage distribution for large n, because the exact distributions become difficult to evaluate numerically in this case. It is hoped that these limiting distributions will prove to be useful approximations in more general coverage problems.

Definitions, notation, and a distributional representation of the coverage are given in Section 2. The asymptotic distribution of the vacancy is found in Sections 3 and 4, under different behavior of the arc length sequence. In Section 3, the arc length is chosen so that the probability of complete coverage of the circle remains fixed as n grows. The limiting distribution is found to be a mixture of a discrete mass point at zero with a positive continuous random variable, and may be interpreted in a natural way as a noncentral chi-square distribution with zero degrees of freedom. Section 4 treats proportionately smaller arc lengths, and a limiting normal distribution is obtained.

2. Definitions, notation, and a distributional representation of the coverage. Let n arcs, each of length a, be placed independently with centers uniformly distributed on the edge of a circle of circumference one. Denote these arcs by $\mathbf{A}_1, \cdots, \mathbf{A}_n$.

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The *coverage* is defined as

$$C(n, a) = \mu\left(\bigcup_{i=1}^{n} \mathbf{A}_i\right) \tag{2.1}$$

where μ denotes Lebesgue measure on the circle. $C(n, a)$ is the (random) proportion of the circle that is contained in at least one arc. The *vacancy* is defined as

$$V(n, a) = 1 - C(n, a) \tag{2.2}$$

and is the random proportion of the circle that is contained in no arc. It is introduced because it is generally easier to work with mathematically than is the coverage. For a thorough treatment of the foundations of coverage problems, the reader is referred to Ailam (1966).

The event $V(n, a) = 0$ represents complete coverage of the circle by these arcs. Its probability will be denoted by $\mathbf{P}(n, a)$, and was found by Stevens (1939) to be

$$\mathbf{P}(n, a) = \mathbf{P}(V(n, a) = 0) = \Sigma_{l=0}^{n}(-1)^l \binom{n}{l}(1 - la)_+^{n-1} \tag{2.3}$$

where $(t)_+ = \max(t, 0)$.

Fisher (1940) discovered a link between this coverage problem and the analysis of time series, from which we obtain a useful representation of the vacancy. Let $X_1, \cdots, X_n$ be independent and identically distributed χ_2^2 (or alternatively, any exponential distribution). Normalize them so that they sum to one by defining

$$Y_i = X_i / \Sigma_{j=1}^{n} X_j. \tag{2.4}$$

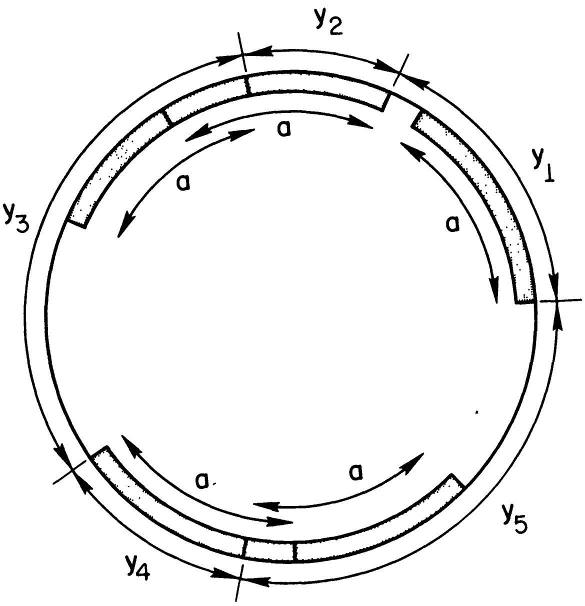

FIG. 2.1. $Y_1, \cdots, Y_n$ generate n random arcs of size a on the circle, in the case $n = 5$.

Then a distributional representation of the vacancy is given by

$$V(n, a) = \Sigma_{i=1}^{n}(Y_i - a)_+ . \tag{2.5}$$

This follows because the Y_i may be interpreted as the spacings between adjacent counter-clockwise endpoints of arcs, as is illustrated in Figure 2.1.

3. Limiting coverage distribution: constant coverage probability. In this section we find the limiting distribution of the vacancy in the case in which the coverage probability stays constant. Thus for an experiment in which n random arcs are placed, the length of each arc will be a_n and is chosen so that $\mathbf{P}(n, a_n) = \gamma$, where γ is the fixed coverage probability and lies strictly between zero and one. It is important to note that in a given experiment, all arcs placed are of the same length. The treatment of random arcs of different sizes is considerably more complicated (Siegel (1978b)).

In Theorem 3.1, the behavior of the sequence a_n is characterized, and in Theorem 3.2, the asymptotic distribution of the vacancy is found. Following this is a discussion of the interpretation of this distribution as a noncentral chi-square distribution with zero degrees of freedom.

THEOREM 3.1. *Let* $\beta = \log(1/\gamma)$. *Then*

$$\lim_{n\to\infty}\mathbf{P}\left(n, \frac{1}{n}\log\left(\frac{n}{\beta}\right)\right) = \gamma \tag{3.1}$$

and

$$\lim_{n\to\infty} n\left[a_n - \frac{1}{n}\log\left(\frac{n}{\beta}\right)\right] = 0 \tag{3.2}$$

where, we recall, a_n *satisfies* $\mathbf{P}(n, a_n) = \gamma$.

Note that (3.2) is a much stronger statement than $a_n \sim (1/n)\log(n/\beta)$, meaning their ratio tends to one in the limit. A stronger result is clearly needed because $(1/n)\log(n/\beta_1) \sim (1/n)\log(n/\beta_2)$ for *any* positive β_1 and β_2.

PROOF. Let $G_n = \max_{1\leqslant i\leqslant n} Y_i$, using the representation in Section 2. The circle is covered by n arcs of length $(1/n)\log(n/\beta)$ if and only if $G_n \leqslant (1/n)\log(n/\beta)$. Barton and David (1956) showed that

$$2n\exp(-nG_n) \to_{\mathfrak{D}} \chi_2^2. \tag{3.3}$$

Hence

$$\mathbf{P}\left(n, \frac{1}{n}\log\left(\frac{n}{\beta}\right)\right) = \mathbf{P}(2n\exp(-nG_n) \geqslant 2\beta) \to e^{-\beta} = \gamma \tag{3.4}$$

and (3.1) is established. For (3.2), we choose β_1 and β_2 that satisfy $0 < \beta_1 < \beta < \beta_2$. Then $\gamma_2 = \exp(-\beta_2) < \gamma < \exp(-\beta_1) = \gamma_1$, and therefore for sufficiently large n we have

$$\mathbf{P}\left(n, \frac{1}{n}\log\left(\frac{n}{\beta_2}\right)\right) < \mathbf{P}(n, a_n) < \mathbf{P}\left(n, \frac{1}{n}\log\left(\frac{n}{\beta_1}\right)\right) \tag{3.5}$$

using the convergence result (3.1) just established. But $\mathbf{P}(n, t)$ is an increasing function of t, so

$$\frac{1}{n}\log\left(\frac{n}{\beta_2}\right) < a_n < \frac{1}{n}\log\left(\frac{n}{\beta_1}\right) \tag{3.6}$$

and therefore (for sufficiently large n)

$$\log\left(\frac{\beta}{\beta_2}\right) < n\left[a_n - \frac{1}{n}\log\left(\frac{n}{\beta}\right)\right] < \log\left(\frac{\beta}{\beta_1}\right). \tag{3.7}$$

By choosing β_1 and β_2 close to β, we can make the outer terms in (3.7) as near to zero as we wish, completing the proof.

A proof of Theorem 3.1 can also be given using the distribution of the random number $N(a)$ of arcs of length a needed to cover the circle, when the random arcs are placed sequentially. The definition of a_n so that $\mathbf{P}(n, a_n) = e^{-\beta}$ for all n implies

$$\mathbf{P}(N(a_n) \leqslant n) = e^{-\beta} \qquad \text{for all } n. \tag{3.8}$$

Theorem 2.6 in Flatto (1973) says that

$$\lim_{n\to\infty}\mathbf{P}(N(a_n) \leqslant (\log(1/a_n) + \log\log(1/a_n) - \log(\beta))/a_n) = e^{-\beta}. \tag{3.9}$$

Setting $X_n = a_n N(a_n) + \log(a_n) - \log\log(1/a_n)$, (3.8) and (3.9) may be written as

$$\mathbf{P}(X_n \leqslant na_n + \log(a_n) - \log\log(1/a_n)) = e^{-\beta} \qquad \text{for all } n \tag{3.10}$$

and

$$\lim_{n\to\infty}\mathbf{P}(X_n \leqslant -\log(\beta)) = e^{-\beta}. \tag{3.11}$$

These last two equations force

$$na_n + \log(a_n) - \log\log(1/a_n) \to -\log\beta \tag{3.12}$$

which we will write as

$$n\left[a_n - \frac{1}{n}\log\left(\frac{n}{\beta}\right)\right] + \log[-na_n/\log(a_n)] \to 0. \tag{3.13}$$

Dividing (3.12) by $\log(a_n)$, which tends to $-\infty$, we see that

$$\log[-na_n/\log(a_n)] \to 0. \tag{3.14}$$

Equations (3.13) and (3.14) now prove (3.2). □

THEOREM 3.2. *The limiting distribution of the vacancy $V(n, a_n)$, where the arc lengths a_n are chosen so that the coverage probability remains fixed at γ, is given by*

$$nV(n, a_n) \to_{\mathfrak{D}} Y \tag{3.15}$$

where Y is the mixture

$$\begin{aligned} Y &= 0 \quad \textit{probability} \quad \gamma \\ &= Z \quad \textit{probability} \quad 1-\gamma, \end{aligned} \tag{3.16}$$

and Z is continuous with density

$$f_\beta(t) = \frac{1}{e^\beta - 1}\Sigma_{l=1}^\infty \frac{\beta^l}{l!}\frac{t^{l-1}}{(l-1)!}e^{-t} \tag{3.17}$$
$$= \frac{\beta^{\frac{1}{2}}}{e^\beta - 1}\cdot\frac{e^{-t}}{t^{\frac{1}{2}}}I_1\big(2(\beta t)^{\frac{1}{2}}\big)$$

where we recall $\beta = \log(1/\gamma)$, *and* I_1 *denotes a modified Bessel function* (*see, for example, Chapter* 9 *of Oliver* (1964)). *The cumulative distribution function of Y is*

$$\mathbf{F}_\beta(t) = \mathbf{P}(Y \leqslant t) = e^{-t}\Sigma_{k=0}^\infty \frac{t^k}{k!}\left[e^{-\beta}\Sigma_{l=0}^k \frac{\beta^l}{l!}\right] \tag{3.18}$$

where $0 \leqslant t < \infty$.

It is interesting to interpret the limiting distribution $2Y$ of $2nV(n, a_n)$ as $\chi_0^2(\beta)$, the noncentral chi-square distribution with zero (!) degrees of freedom and noncentrality parameter β. Recall the representation of $\chi_\nu^2(\beta)$, ν a positive integer, as a Poisson mixture of the central chi-squares $\chi^2_{(\nu+2K)}$ where $K \sim \mathbf{P}_0(\beta)$; see, for example, Chapter 2.4 of Searle (1971). We see from (3.16) and (3.17) that $2Y$ is a Poisson mixture of the central chi-squares χ^2_{2K} where $K \sim \mathbf{P}_0(\beta)$ so that $\mathbf{P}(K = l) = e^{-\beta}\beta^l/l!$, and we use the convention that χ_0^2 is the distribution concentrated at zero. Thus the distribution of $2Y$ naturally extends the noncentral chi-square distribution $\chi_\nu^2(\beta)$ to the case $\nu = 0$. Having zero degrees of freedom allows a mixture of discrete mass at zero (corresponding to complete coverage of the circle) with continuous variation (corresponding to partial coverage of the circle).

PROOF OF THEOREM 3.2. Set

$$b_n = \frac{1}{n}\log\left(\frac{n}{\beta}\right). \tag{3.19}$$

We will use $V(n, b_n)$ as an approximation to $V(n, a_n)$. Moments of vacancy were found by Siegel (1978a), and are

$$\mathbf{E}[nV(n, b_n)]^m = n^m\binom{m+n-1}{n}^{-1}\Sigma_{l=1}^m\binom{m}{l}\binom{n-1}{l-1}\left(1 - \frac{l}{n}\log\left(\frac{n}{\beta}\right)\right)_+^{m+n-1}. \tag{3.20}$$

It may be verified that

$$\lim_{n\to\infty} n^l\left(1 - \frac{l}{n}\log\frac{n}{\beta}\right)^{m+n-1} = \beta^l \tag{3.21}$$

by taking logs and doing a Taylor series expansion. Using this, the limit of (3.20) may be calculated, and we define

$$\mu_m = \lim_{n\to\infty}\mathbf{E}[nV(n, b_n)]^m = m!\Sigma_{l=1}^m\binom{m-1}{l-1}\frac{\beta^l}{l!}. \tag{3.22}$$

The moments of Y may also be calculated:

$$\text{(3.23)}\quad \mathbf{E}Y^m = (1-\gamma)\int_0^\infty t^m f_\beta(t)\,dt = e^{-\beta}\sum_{l=1}^\infty \frac{\beta^l}{l!\,(l-1)!}\int_0^\infty t^{m+l-1}e^{-t}\,dt,$$

where we use the monotone convergence theorem in order to exchange sum and integral. The integral is easily done, and we obtain

$$\text{(3.24)}\qquad \mathbf{E}Y^m = e^{-\beta}\sum_{l=1}^\infty \frac{\beta^l}{l!\,(l-1)!}(m+l-1)!.$$

Now if we expand $e^{-\beta}$, multiply the two series, and gather powers of β, we get

$$\text{(3.25)}\qquad \mathbf{E}Y^m = m!\sum_{l=1}^\infty\left[\sum_{k=1}^l(-1)^{k+l}\binom{m+k-1}{k-1}\binom{l}{k}\right]\frac{\beta^l}{l!}.$$

The term in brackets is $\binom{m-1}{l-1}$; to see this, simply expand the identity $(1-t)^l(1-t)^{-(m+1)} = (1-t)^{-(m-l+1)}$ and equate coefficients of t^{l-1}. Thus, comparing with (3.22) we have

$$\text{(3.26)}\qquad \mathbf{E}Y^m = \mu_m = \lim_{n\to\infty}\mathbf{E}[nV(n, b_n)]^m.$$

Convergence of moments implies convergence in distribution provided

$$\text{(3.27)}\qquad \limsup_{m\to\infty}\frac{|\mu_m|^{1/m}}{m} < \infty$$

(see Section 8.12 of Breiman (1968)). To establish (3.27), we use Stirling's formula:

$$\text{(3.28)}\qquad \mu_m = m!\sum_{l=1}^m\binom{m-1}{l-1}\frac{\beta^l}{l!} \leqslant m!\binom{m}{m/2}e^\beta \sim 2^{m+1}e^\beta m^m e^{-m}$$

from which (3.27) now follows easily. This proves that

$$\text{(3.29)}\qquad nV(n, b_n) \to_{\mathfrak{D}} Y.$$

To now prove (3.15) it will suffice to show that

$$\text{(3.30)}\qquad W_n = n[V(n, a_n) - V(n, b_n)] \to_{\mathbf{P}} 0.$$

From Theorem 3.1, $a_n = b_n + (c_n/n)$ where $c_n \to 0$. Using this and the representation of Section 2, we have

$$\text{(3.31)}\qquad \begin{aligned}\mathbf{P}(|W_n| > \varepsilon) &= \mathbf{P}\left(n\left|\sum_{i=1}^n\left\{\left(Y_i - b_n - \frac{c_n}{n}\right)_+ - (Y_i - b_n)_+\right\}\right| > \varepsilon\right)\\ &\leqslant \mathbf{P}\left(|c_n|\sum_{i=1}^n\mathbf{I}\left\{Y_i > b_n - \frac{|c_n|}{n}\right\} > \varepsilon\right)\end{aligned}$$

where $\mathbf{I}\{\mathbf{A}\}$ is 1 if $\mathbf{A}$ holds and 0 otherwise. Applying the Markov inequality to (3.31) we have

$$\text{(3.32)}\qquad \mathbf{P}(|W_n| > \varepsilon) \leqslant \frac{|c_n|}{\varepsilon}\left[n\mathbf{P}\left(Y_1 > b_n - \frac{|c_n|}{n}\right)\right].$$

It can be verified directly that if $0 \leqslant t \leqslant 1$, then

$$\text{(3.33)}\qquad \mathbf{P}(Y_1 > t) = (1-t)^{n-1}.$$

Using this, we now show that the bracketed term of (3.32) is bounded. This term is

$$nP\left(Y_1 > b_n - \frac{|c_n|}{n}\right) = n\left(1 - b_n + \frac{|c_n|}{n}\right)^{n-1}. \tag{3.34}$$

Taking logs and expanding, we have

$$\begin{aligned}\log(n) + (n-1)\log\left(1 - b_n + \frac{|c_n|}{n}\right)&\\ = \log(n) - (n-1)\left\{\frac{1}{n}\log\left(\frac{n}{\beta}\right) - \frac{|c_n|}{n} + O\left(\frac{\log n}{n}\right)^2\right\}&\\ = \log(\beta) + o(1).&\end{aligned} \tag{3.35}$$

Using this in (3.32) and recalling that $c_n \to 0$, we have

$$\mathbf{P}(|W_n| > \varepsilon) \leqslant \frac{|c_n|}{\varepsilon}O(1) \to 0 \tag{3.36}$$

completing the proof of (3.15). (3.18) follows from term-by-term integration of (3.17). □

4. Proportionately smaller arcs. The previous section treated the case of constant coverage probability, in which the arc size behaved like $(1/n)\log(n/\beta)$. Now we consider proportionately smaller arcs, of length

$$d_n = \frac{\lambda}{n}\log\left(\frac{n}{\beta}\right) \tag{4.1}$$

where $0 < \lambda < 1$. Because in this case the coverage probability $\mathbf{P}(n, d_n)$ tends to zero, as may be verified from (3.1), there are no mass points in the limiting distribution of the vacancy. The main result of this section is

THEOREM 4.1. *The vacancy $V(n, d_n)$ is asymptotically normal with mean $(\beta/n)^\lambda$ and variance $2\beta^\lambda n^{-(1+\lambda)}$. That is,*

$$\frac{V(n, d_n) - (\beta/n)^\lambda}{(2\beta^\lambda n^{-(1+\lambda)})^{\frac{1}{2}}} \to_{\mathcal{D}} \mathfrak{N}(0, 1). \tag{4.2}$$

PROOF. We will use

$$V_n^* = \Sigma_{i=1}^n\left(\frac{X_i}{2n} - d_n\right)_+ \tag{4.3}$$

as an approximation to

$$V(n, d_n) = \Sigma_{i=1}^n(X_i/(\Sigma_{j=1}^n X_j) - d_n)_+ \tag{4.4}$$

where $X_1, \cdots, X_n$ are independent and identically distributed χ_2^2, from the representation of Section 2. Since V_n^* is the sample mean of shifted and censored exponentials, it is easily seen that

$$\mathbf{E}(V_n^*) = (\beta/n)^\lambda \tag{4.5}$$

and

$$\operatorname{Var}(V_n^*) = 2\beta^\lambda n^{-(1+\lambda)}\left[1 - \tfrac{1}{2}(\beta/n)^\lambda\right]. \tag{4.6}$$

Thus the mean and variance of V_n^* and $V(n, d_n)$ from (4.2) are asymptotically identical. The proof of asymptotic normality will follow immediately from the following two lemmas. The first will show that V_n^* is asymptotically normal with the right mean and variance, and the second lemma will show that V_n^* is close enough to $V(n, d_n)$ to imply (4.2).

LEMMA 4.1. *For V_n^* defined in* (4.3) *and* (4.1),

$$\frac{V_n^* - (\beta/n)^\lambda}{(2\beta^\lambda n^{-(1+\lambda)})^{\frac{1}{2}}} \to_{\mathfrak{D}} \mathfrak{N}(0, 1). \tag{4.7}$$

PROOF. It will suffice to verify the Lindeberg condition for triangular arrays, which may be found in Loève (1960). This requires that

$$g_n(\varepsilon) = n\mathbf{E}Z_n^2\mathbf{I}\{|Z_n| \geqslant \varepsilon\} \to 0 \tag{4.8}$$

hold for each $\varepsilon > 0$, where

$$Z_n = \left[\left(\frac{X_1}{2n} - d_n\right)_+ - \frac{1}{n}\left(\frac{\beta}{n}\right)^\lambda\right]\cdot\left[\frac{2}{n}\left(\frac{\beta}{n}\right)^\lambda\left(1 - \frac{1}{2}\left(\frac{\beta}{n}\right)^\lambda\right)\right]^{-\frac{1}{2}} \tag{4.9}$$

is the first term, $((X_1/2n) - d_n)_+$, in the sum for V_n^*, normalized to have mean zero and variance $1/n$. To establish (4.8), we bound $g_n(\varepsilon)$ using the fourth moment:

$$\begin{aligned} g_n(\varepsilon) &\leqslant \frac{n}{\varepsilon^2}\mathbf{E}Z_n^4 \\ &= \frac{n^{3+2\lambda}}{4\varepsilon^2\beta^{2\lambda}\left[1 - \frac{1}{2n}(\beta/n)^\lambda\right]^2}\mathbf{E}\left[\left(\frac{X_1}{2n} - d_n\right)_+ - \frac{1}{n}\left(\frac{\beta}{n}\right)^\lambda\right]^4. \end{aligned} \tag{4.10}$$

We do a binomial expansion of the expectation term and calculate the moments from

$$\mathbf{E}\left(\frac{X_1}{2n} - d_n\right)_+^m = \frac{m!}{n^m}(\beta/n)^\lambda, \qquad m \geqslant 1. \tag{4.11}$$

Thus (4.10) becomes

$$\begin{aligned} g_n(\varepsilon) &\leqslant O(1)\cdot\left\{\frac{24\beta^\lambda}{n^{4+\lambda}} - \frac{24\beta^{2\lambda}}{n^{4+2\lambda}} + \frac{12\beta^{3\lambda}}{n^{4+3\lambda}} - \frac{3\beta^{4\lambda}}{n^{4+4\lambda}}\right\} \\ &= O\left(\frac{1}{n^{1-\lambda}}\right) \end{aligned} \tag{4.12}$$

which completes the proof. □

LEMMA 4.2. *Using the notation of this section,*

$$\frac{V(n, d_n) - V_n^*}{(n^{-(1+\lambda)})^{\frac{1}{2}}} \to_{\mathbf{P}} 0. \tag{4.13}$$

PROOF. Using the representations (4.3) and (4.4), we must show that

$$\mathbf{P}\left[\left| \Sigma_{i=1}^n \left(\frac{X_i}{n\bar{X}_n} - d_n \right)_+ - \left(\frac{X_i}{2n} - d_n \right)_+ \right| > \varepsilon n^{-(1+\lambda)/2} \right] \to 0 \tag{4.14}$$

as $n \to \infty$, for each $\varepsilon > 0$, where $\bar{X}_n = \frac{1}{n}\Sigma_{i=1}^n X_i$. For convenience, let $\mathbf{B}_n$ denote the event in (4.14). Because

$$n^{\frac{1}{2}}\left(\frac{\bar{X}_n}{2} - 1 \right) \to_{\mathfrak{D}} \mathfrak{N}(0, 1), \tag{4.15}$$

given $\delta > 0$ we can find K such that

$$\mathbf{P}\left(0 < \frac{2}{\bar{X}_n} - 1 < \frac{K}{n^{\frac{1}{2}}} \right) > \frac{1}{2}(1 - \delta) \tag{4.16}$$

and

$$\mathbf{P}\left(-\frac{K}{n^{\frac{1}{2}}} < \frac{2}{\bar{X}_n} - 1 < 0 \right) > \frac{1}{2}(1 - \delta) \tag{4.17}$$

wherever n is sufficiently large. Then $\mathbf{P}(\mathbf{B}_n)$, from (4.14), satisfies

(4.18)

$$\mathbf{P}(\mathbf{B}_n) \leqslant \mathbf{P}\left(\mathbf{B}_n, \left(1 + \frac{K}{n^{\frac{1}{2}}}\right)^{-1} < \frac{\bar{X}_n}{2} < 1 \right) + \mathbf{P}\left(\mathbf{B}_n, 1 < \frac{\bar{X}_n}{2} < \left(1 - \frac{K}{n^{\frac{1}{2}}}\right)^{-1} \right) + \delta.$$

We consider the two main terms in the right-hand side of (4.18). The first one is

(4.19)

$$T_1 = \mathbf{P}\left[\Sigma_{i=1}^n \left(\frac{X_i}{n\bar{X}_n} - d_n \right)_+ - \left(\frac{X_i}{2n} - d_n \right)_+ > \varepsilon n^{-(1+\lambda)/2}, \left(1 + \frac{K}{n^{\frac{1}{2}}}\right)^{-1} < \frac{\bar{X}_n}{2} < 1 \right].$$

The largest any single difference can be in the above sum is $KX_i/2n^{3/2}$, and this difference will be zero whenever $X_i \leqslant 2nd_n/(1 + Kn^{-\frac{1}{2}})$. Thus

$$T_1 \leqslant \mathbf{P}\left(\Sigma_{i=1}^n X_i \mathbf{I}\left\{ X_i > \frac{2nd_n}{1 + Kn^{\frac{1}{2}}} \right\} > \frac{2\varepsilon}{K} n^{1-\lambda/2} \right). \tag{4.20}$$

The Markov inequality yields

$$T_1 \leqslant \frac{Kn^{\lambda/2}}{2\varepsilon} \mathbf{E} X_1 \mathbf{I}\left\{ X_1 > \frac{2nd_n}{1 + Kn^{-\frac{1}{2}}} \right\}. \tag{4.21}$$

This expectation is straightforward to calculate, and is

$$2\left\{1+\frac{nd_n}{1+Kn^{-\frac{1}{2}}}\right\}\left(\frac{n}{\beta}\right)^{-\lambda/\left(1+Kn^{-\frac{1}{2}}\right)} \leqslant 2\{1+nd_n\}\left(\frac{n}{\beta}\right)^{-\left(\frac{1}{2}+\zeta\right)\lambda} \tag{4.22}$$

for sufficiently large n, where ζ is any number strictly between 0 and $\frac{1}{2}$. Thus the first term of (4.18) satisfies

$$T_1 \leqslant \frac{Kn^{\lambda/2}}{\varepsilon}\left\{1+\lambda\log\left(\frac{n}{\beta}\right)\right\}\left(\frac{n}{\beta}\right)^{-\left(\frac{1}{2}+\zeta\right)\lambda} = O(n^{-\zeta\lambda}\log n). \tag{4.23}$$

The second main term, T_2, of the right-hand side of (4.18) may be treated similarly to obtain

$$T_2 = O(n^{-\zeta\lambda}\log n) \tag{4.24}$$

as well. Thus

$$\mathbf{P}(\mathbf{B}_n) = \delta + O(n^{-\zeta\lambda}\log n) \tag{4.25}$$

holds for all $\delta > 0$ and $\zeta \in (0, \frac{1}{2})$, completing the proof. □

Because convergence in distribution in Theorem 4.1 does not, by itself, imply proper behavior of the moments of $V(n, d_n)$, this is treated in the following theorem.

THEOREM 4.2. *The asymptotic mean and variance of* $V(n, d_n)$ *are*

$$\mu_n = \mathbf{E}V(n, d_n) \sim (\beta/n)^{\lambda} \tag{4.26}$$

and

$$\sigma_n^2 = \mathrm{Var}(V(n, d_n)) \sim 2\beta^{\lambda}/n^{1+\lambda}. \tag{4.27}$$

PROOF. Exact formulae for the moments are available in Siegel (1978a). The first moment is

$$\mu_n = (1-d_n)^n = \left(1-\frac{\lambda}{n}\log\left(\frac{n}{\beta}\right)\right)^n. \tag{4.28}$$

Expanding $\log(\mu_n)$ in a Taylor series, we have

$$\log(\mu_n) = \log(\beta/n)^{\lambda} + O\left(\frac{1}{n}\log\left(\frac{n}{\beta}\right)^2\right) \tag{4.29}$$

which proves (4.26). The exact formula for the variance is

$$\sigma_n^2 = \frac{2}{n+1}(1-d_n)^{n+1} + \frac{n-1}{n+1}(1-2d_n)^{n+1} - (1-d_n)^{2n}. \tag{4.30}$$

The first of the three terms on the right side of (4.30) is $\sim 2\beta^{\lambda}/n^{1+\lambda}$, using the same expansion technique we just used for μ_n. To complete the proof, we will show that the remaining two terms combined are $o(1/n^{1+\lambda})$. Write the second term from (4.30) as

$$\frac{n-1}{n+1}(1-2d_n)^{n+1} = -\frac{2}{n+1}(1-2d_n)^{n+1} - 2d_n(1-2d_n)^n + (1-2d_n)^n. \tag{4.31}$$

The first term on the right is $O(1/n^{1+2\lambda})$, hence $o(1/n^{1+\lambda})$, and may be ignored. The second term is $O\left(\log\left(\frac{n}{\beta}\right)/n^{1+2\lambda}\right)$, hence also $o(1/n^{1+\lambda})$. It remains only to consider the sum of the last term in (4.30) with the last term of (4.31), namely

$$(1-2d_n)^n-(1-d_n)^{2n}. \tag{4.32}$$

Factoring (4.32) as $a^n - b^n$, it becomes

$$-d_n^2\Sigma_{i=0}^{n-1}(1-2d_n)^i(1-2d_n+d_n^2)^{n-i-1}. \tag{4.33}$$

The sum itself is bounded above by $(n-1)(1-d_n)^{2(n-1)} = O(n^{1-2\lambda})$. Thus (4.33) is $O((\log(n/\beta))^2/n^{1+2\lambda}) = o(1/n^{1+\lambda})$ as was to be shown. □

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