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PERSISTENTLY OPTIMAL PLANS FOR NONSTATIONARY DYNAMIC PROGRAMMING: THE TOPOLOGY OF WEAK CONVERGENCE CASE[1]

BY ROBERT P. KERTZ AND DAVID C. NACHMAN

Georgia Institute of Technology

In this paper, we study a nonstationary dynamic programming model $\{(S_n, \mathbb{S}_n), (A_n, \mathcal{A}_n), D_n, q_n, u : n \geq 1\}$ with standard Borel state spaces $(S_n, \mathbb{S}_n)$ and action spaces $(A_n, \mathcal{A}_n)$, upper semicontinuous admissible-action correspondences D_n, weakly continuous transition laws q_n, and Borel measurable total reward function $u : S_1 \times A_1 \times \cdots \to R_-$. We establish existence of a persistently optimal (degenerate) plan for this model under regularity and boundedness assumptions on conditional expectations of u, but require no special separable form of u such as intertemporal additivity. The methods of proof utilize results on weak convergence of probability measures and selection theorems in the context of optimization of functions over correspondences. We give two characterizations of the persistent optimality of a feasible plan: the first that the plan be both thrifty and equalizing, and the second that the plan satisfy an optimality criterion that entails period-by-period optimality.

1. Introduction. Modifications and extensions of the fundamental work of Blackwell [4] on discounted dynamic programming and of Strauch [41] on negative dynamic programming have been made by Hinderer [20], Schäl [38], Furukawa [11], Furukawa and Iwamoto [12], [13], [14], and Kreps [27], [28], [29]. In most of these models the primitive data are all stationary or the primitive data are partially nonstationary; but suitable reformulations can be made to place the model within a stationary setting, for example, as in the results of Schäl ([38], page 197) and Bertsekas ([2], page 271). In particular, the total reward function either is a sum of discounted or negative reward functions depending only on state-action-state triples involved in single stages of control, or, in the extensions, is a function with a staged structure exhibiting some degree of mathematical separability between stages. Under an expected total reward function criterion (expected utility criterion) with the "correct" combination of conditions on the primitive data, either stationary or Markov optimal plans are shown to exist. These proofs of existence of optimal plans rely on iterative, finite-horizon to infinite-horizon methods.

The extension of the classical negative dynamic programming model which is considered here allows for all of the primitive data to be nonstationary. In particular, no direct assumption is made requiring that the total reward (utility)

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function possess some staged structure. This degree of generality is natural in many dynamic choice problems in economic theory; for example, in problems of consumption and production choices over time and in the related problems of optimal economic growth ([17], Chapter XXI; [43], Chapter 9; [26]) and in studies of existence of sequences of temporary equilibria in market economies that evolve in time with agents facing uncertainty about future market conditions ([40], [15], [16], [21], [25]).

In this general setting we establish existence of persistently optimal plans (Theorem 3.4). As discussed by Dubins and Sudderth [9], Hinderer ([20], page 132), and Schäl ([38], page 197) under the appellation "strongly optimal plans," persistently optimal plans are a natural counterpart in the nonstationary setting of stationary optimal plans in the stationary setting.

Our approach to the existence question generalizes the approach developed by Jordan [22] and extended in [25]. Central to this approach is the view of the decision maker as choosing at each stage or period of the problem a probability measure for the remaining infinite future. Such choice is constrained in a natural way by the present and future admissible action correspondences. Existence of solutions and regularity properties of optimal value functions are proved using maximization theorems of Schäl [37], [38] for the choice problem of each stage. The generalization of the optimality criteria of dynamic programming is then used to establish consistency among the choices at each stage and to establish the existence of a persistently optimal plan.

The nonstationary dynamic programming model (NDPM) with general expected utility criterion is described in Section 2 with a minimum of assumptions. In this section we identify the principal objects of the analysis including the relation F^n of attainable measures on the infinite future (2.6), discuss the optimality concepts germane to our analysis, and pose the basic existence question for our model.

In Section 3, we list the Assumptions (W) for our class of problems and state the main result, Theorem 3.4, that under Assumptions (W) persistently optimal plans exist. Section 3 closes with the derivation of some preliminary consequences of these assumptions.

In Section 4, we derive the needed regularity properties of the relation F^n from the corresponding properties of the admissible action correspondences and from the assumed regularity of the laws of motion of the system. Many of the proofs of results in this section are straightforward extensions of proofs of theorems of Schäl [39]. In Section 5, we use these properties of F^n together with selection-maximization theorems of Schäl [37], [38] to prove the main result (Theorem 3.4).

In Section 6, we characterize persistently optimal plans as those plans which satisfy optimality criterion (5.1) and as those plans which are both thrifty and equalizing, where these terms, defined in (6.1) and (6.3), are analogues of those found in the gambling literature ([8], [42], [38]). We also exhibit a connection of the persistently optimal plans to the appropriate optimality equations and compare our results to those in related extensions of the negative dynamic programming model.

The interesting mathematical issues in the study of existence of solutions to (NDPM) turn on the regularity properties of the laws of motion of the system and the concomitant regularity of the admissible action correspondences. A primary concern of investigations of this kind then is the compatibility of topologies used to express regularity of measure valued functions and those used to express regularity of admissible action correspondences (set valued functions). In the present investigation we employ the topology of weak convergence on spaces of probability measures ([3], [33]) and the usual (upper) semicontinuity concepts of set valued functions taking values in (compact) subsets of a metric space ([30], [1]). In a subsequent paper [24], we investigate the existence question in the context of the ws^{∞}-topology of Schäl [39].

2. Notation, definitions, and statement of the problem. Let N denote the set of positive integers, R the set of real numbers, $R_{-} = R \cup \{-\infty\}$, and $\bar{R}$ the set of extended real numbers. For any nonempty set X we denote by $\mathfrak{P}'(X)$ the collection of nonempty subsets of X, and for $(X, \mathcal{B})$ a measurable space we denote by $\mathcal{P}(X)$ the set of all probability measures on $\mathcal{B}$. For a topological space X, we denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X. The following sets of functions will be of interest in the sequel. For X a topological space, let

(2.1) $B(X) = \{f : X \to R : f \text{ is bounded and } \mathcal{B}(X)/\mathcal{B}(R)\text{-measurable}\}$

(2.2) $\hat{B}(X) = \{f : X \to R_{-} : f \text{ is bounded above and } \mathcal{B}(X)/\mathcal{B}(R_{-})\text{-measurable}\}$

(2.3) $C(X) = \{f \in B(X) : f \text{ is continuous}\}$

(2.4) $\hat{C}(X) = \{f \in \hat{B}(X) : f \text{ is upper semicontinuous}\}$.

We assume throughout the sequel that for X a topological space, $\mathcal{P}(X)$ is a topological space endowed with the weak topology induced by $C(X)$, i.e., the topology of weak convergence [33]. We note that if X is a separable metric space, then $\mathcal{P}(X)$ is separable and metrizable ([33], Theorem 6.3, page 43), and $\mathcal{B}(\mathcal{P}(X))$ is countably generated. For measurable spaces $(\Omega, \mathcal{F})$ and $(X, \mathcal{B})$, by a transition probability from Ω to X we mean a function $\mu : \Omega \times \mathcal{B} \to [0, 1]$ such that for each $\omega \in \Omega$, $\mu(\omega, \cdot)$ is a probability measure on $\mathcal{B}$, and for each $B \in \mathcal{B}$, $\mu(\cdot, B)$ is a $\mathcal{F}/\mathcal{B}([0, 1])$-measurable function; or, equivalently, we mean a function $\mu : \Omega \to \mathcal{P}(X)$ such that the function $\omega \to \mu(\omega)(B)$ is $\mathcal{F}/\mathcal{B}([0, 1])$-measurable for each $B \in \mathcal{B}$. If X is a metric space, then every $\mathcal{F}/\mathcal{B}(\mathcal{P}(X))$-measurable function $\varphi : \Omega \to \mathcal{P}(X)$ is a transition probability from Ω to X, and if X is also separable, then $\varphi : \Omega \to \mathcal{P}(X)$ is a transition probability if and only if φ is $\mathcal{F}/\mathcal{B}(\mathcal{P}(X))$-measurable ([35], Lemma 6.1; [7], Theorems 2.1 and 3.1).

A nonstationary dynamic programming model (NDPM) is a sequence of objects $\{(S_n, \mathcal{S}_n)\ (A_n, \mathcal{A}_n), D_n, q_n, u : n \in N\}$ defined as follows:

(i) $(S_n, \mathcal{S}_n)$ is a measurable space with σ-algebra $\mathcal{S}_n$, the state space at time n.

(ii) $(A_n, \mathcal{A}_n)$ is a measurable space with σ-algebra $\mathcal{A}_n$, the action space at time n. The sets $H_n = S_1 \times A_1 \times \cdots \times S_n$ and $H^n = A_n \times S_{n+1} \times A_{n+1} \times \cdots$ are denoted as the spaces of past histories and of future histories at time n, $H^n_m = A_n \times S_{n+1} \times \cdots \times A_{m-1} \times S_m$, $m > n$, and $H_\infty = S_1 \times A_1 \times S_2 \times \cdots$. These spaces are given the product

(2.5) σ-algebras, denoted by $\mathcal{H}_n$, $\mathcal{H}^n$, $\mathcal{H}^n_m$, and $\mathcal{H}_\infty$, respectively.

(iii) D_n is a mapping $D_n : H_n \to \mathcal{A}_n \cap \mathfrak{P}'(A_n)$ in which $D_n(h_n)$ represents the set of admissible actions under the history $h_n \in H_n$.

(iv) q_n is a transition probability from $H_n \times A_n$ to S_{n+1}. The sequence $(q_n)_{n \in N}$ constitutes the laws of motion.

(v) u is a measurable mapping $u : H_\infty \to R_-$. The function u is called the total reward (or utility) function.

A plan for such a model is simply a sequence $\pi = (\pi_n)_{n \in N}$, where for each $n \in N$, π_n is a transition probability from H_n to A_n. A period-n plan is a sequence $(\pi_m)_{m \geqslant n}$ that is the tail (from n onward) of a plan. A *feasible* plan for (NDPM) is a plan $\pi = (\pi_n)_{n \in N}$ consisting of feasible controls, where π_n is a *feasible period-n control* if for each $h_n \in H_n$, $\pi_n(D_n(h_n)|h_n) = 1$, i.e., if π_n assigns probability one to the set of actions admissible at history h_n. A feasible period-n plan is then the tail from n onwards of a feasible plan. The collection of all feasible period-n controls is denoted by Π_n and the collection of all feasible period-n plans by Π^n.

To each period-n plan $(\pi_m)_{m \geqslant n}$ is associated a stochastic process $\{(\alpha_m, \zeta_m)\}_{m \geqslant n}$ of action-state pairs on H^n having probability measure $(\pi q)^n(h_n) \equiv \pi_n q_n \pi_{n+1} q_{n+1} \cdots (h_n)$ given by the product measure theorem of C. Ionescu Tulcea [32], for history $h_n \in H_n$. We point out that $(\pi q)^n$ is a transition probability from H_n to H^n, and hence we may view $(\pi q)^n$ as a function from H_n to $\mathcal{P}(H^n)$. A focal point of our analysis of (NDPM) is the set of images of these functions as π varies over the feasible plans. Formally, we are concerned with the relation (set valued function) $F^n : H_n \to \mathfrak{P}'(\mathcal{P}(H^n))$ given by

$$F^n(h_n) = \{\nu \in \mathcal{P}(H^n) : \nu = (\pi q)^n(h_n), \quad \text{some } (\pi_m)_{m \geqslant n} \in \Pi^n\} \tag{2.6}$$

that associates to each history h_n the set of attainable probability measures on the future. Of course at this point we have assumed nothing about (NDPM) that ensures that F^n is in fact nonempty valued. This ambiguity is clarified in Lemma 3.1 of Section 3. For now, we proceed as though all objects are well defined.

Let $G(F^n)$ denote the graph of the relation F^n defined in (2.6), i.e., $G(F^n) = \{(h_n, \nu) \in H_n \times \mathcal{P}(H^n) : \nu \in F^n(h_n)\}$. Let $u_n : G(F^n) \to \bar{R}$ be defined by

$$u_n(h_n, \nu) = \int_{H^n} u(h_n, h^n)\nu(dh^n). \tag{2.7}$$

We assume that u satisfies some properties which guarantee existence of these integrals, for example, $u \in \hat{B}(H_\infty)$. We refer to u_n as the expected period-n payoff function. The optimal expected period-n payoff $v_n : H_n \to \bar{R}$ is then defined by

$$v_n(h_n) = \sup_{\nu \in F^n(h_n)} u_n(h_n, \nu). \tag{2.8}$$

An optimal period-n plan is a feasible period-n plan $(\pi_m)_{m\geq n} \in \Pi^n$ such that for each $h_n \in H_n$,

$$u_n(h_n, (\pi q)^n(h_n)) = v_n(h_n). \tag{2.9}$$

A persistently optimal plan then is a feasible plan $\pi = (\pi_n)_{n\in N}$ such that for every $n \in N$, $(\pi_m)_{m\geq n}$ is an optimal period-n plan, i.e., for every $n \in N$ and every $h_n \in H_n$, (2.9) holds.

Our objective in this paper and a subsequent study [24] is to present conditions on the initial data (2.5) (i)–(v) of the (NDPM) that yield the existence of a persistently optimal plan.

3. Assumptions and preliminary results. We make the following assumptions regarding the (NDPM). For each $n \in N$, we assume:

W(i). The spaces S_n and A_n are standard Borel spaces (nonempty Borel subsets of Polish spaces) endowed with the relative (metric) topology and with $\mathcal{S}_n = \mathcal{B}(S_n)$ and $\mathcal{A}_n = \mathcal{B}(A_n)$.

W(ii). $D_n : H_n \to \mathcal{C}(A_n)$ is upper semicontinuous ([30], page 173), where $\mathcal{C}(A_n)$ denotes the nonempty compact subsets of A_n.

W(iii). $q_n : H_n \times A_n \to \mathcal{P}(S_{n+1})$ is continuous (in the product topology on $H_n \times A_n$).

W(iv). $u_n \in \hat{C}(G(F^n))$, where $G(F^n)$ has the relative topology from the product topology on $H_n \times \mathcal{P}(H^n)$.

We note first that the existence problem is well defined under these assumptions. In particular, we have the following result.

LEMMA 3.1. *Under* W(i) *and* W(ii), *for every* $n \in N$, $\Pi_n \neq \phi$, *and* $F_n(h_n) \neq \phi$ *for every* $h_n \in H_n$.

PROOF. Fix $n \in N$, and let $\Delta_n : H_n \to \mathfrak{P}'(\mathcal{P}(A_n))$ be given by

$$\Delta_n(h_n) = \{\mu \in \mathcal{P}(A_n) : \operatorname{supp}(\mu) \subseteq D_n(h_n)\}, \tag{3.2}$$

where $\operatorname{supp}(\mu)$ denotes the support of the measure μ ([33], Chapter II.2). By [19], Theorem 3, Δ_n is upper semicontinuous and compact valued on H_n. By [33], Theorem 6.2, page 43, $\mathcal{P}(A_n)$ is metrizable as a separable metric space, and hence by [10], Theorem 1, there exists a $\mathcal{H}_n/\mathcal{B}(\mathcal{P}(A_n))$-measurable selection π_n of Δ_n. It then follows from the remarks following (2.4) that $\pi_n \in \Pi_n$. Thus for every $n \in N$, $\Pi^n \neq \phi$, and hence $F^n(h_n) \neq \phi$, for every $h_n \in H_n$. □

REMARK 3.3. Since the relations D_n and Δ_n are alike under W(i) and W(ii) with respect to the hypotheses of the Engelking selection theorem [10], there exists an $\mathcal{H}_n/\mathcal{A}_n$-measurable selection of D_n, i.e., a Borel measurable function $\beta_n : H_n \to A_n$ such that $\beta_n(h_n) \in D_n(h_n)$ for each $h_n \in H_n$. In the sequel, we identify such measurable selectors of D_n as (*degenerate*) elements of Π_n.

We now state the main result of this paper.

THEOREM 3.4. *Under Assumptions* W(i), W(ii), W(iii), *and* W(iv) *there exists a (degenerate) persistently optimal plan.*

The proof of this theorem is given in Section 5 and is based on the properties of F^n which are proved in Section 4.

To simplify the derivations in the sequel, we employ the following operator terminology. For each $n \in N$ and $w \in \hat{B}(H_{n+1})$, define the functions $L_n w$ and $U_n w$ by

$$L_n w(h_n, a_n) = \int w(h_n, a_n, s_{n+1}) q_n(ds_{n+1} | h_n, a_n), \tag{3.5}$$

$$U_n w(h_n) = \sup_{a_n \in D_n(h_n)} L_n w(h_n, a_n). \tag{3.6}$$

Then L_n may be viewed as operating between $\hat{B}(H_{n+1})$ and $\hat{B}(H_n \times A_n)$. Also clearly $U_n w$ is bounded above, but in general, for any h_n, $D_n(h_n)$ may be uncountable, and the measurability of $U_n w$ cannot be ensured.

We gather some standard results for these operators below in Lemma (3.7) that are used in Sections 4 and 5. The proofs of part (i) and (ii) of Lemma (3.7) follow from [39], Lemmas 3.4 and 5.1(a), and [38], Propositions 10.2 and 10.1.3. The proof of part (iii) is essentially that used in [41], Theorem 8.2 and [20], Theorem 14.4. We note here that the result ([39], Lemma 3.4) that underlies the proof of part (i) of this lemma also implies that if the total reward function u is in $\hat{C}(H_\infty)$, then Assumption W(iv) is satisfied.

LEMMA 3.7. *Assume* W(i), W(ii), *and* W(iii). *Then*:

(i) (a) *For every* $w \in C(H_{n+1})[\hat{C}(H_{n+1})]$, $L_n w \in C(H_n \times A_n)[\hat{C}(H_n \times A_n)]$.

(b) *For every* $w \in \hat{C}(H_{n+1})$, $U_n w \in \hat{C}(H_n)$.

(ii) *For* $\{w_j\}_{j \in N} \subseteq \hat{C}(H_{n+1})$, *if* $w_j \downarrow 0$ *pointwise, then* $U_n w_j \downarrow 0$ *pointwise, as* $j \to \infty$.

(iii) *For* $n, m \in N$, $n < m$, *and for each* $w \in \hat{C}(H_m)$,

$$U_n \cdots U_{m-1} w(h_n) = \sup_{(\pi_n, \ldots, \pi_m) \in \Pi_n \times \cdots \times \Pi_m} \int w(h_n, h_m^n) \times \pi_n q_n \cdots \pi_{m-1} q_{m-1}(dh_m^n | h_n)$$

for every $h_n \in H_n$.

REMARK 3.8. With regard to part (i) (b) of Lemma (3.7), we note that in general it is not the case that $U_n w \in C(H_n)$ even when $w \in C(H_{n+1})$ and consequently $L_n w \in C(H_n \times A_n)$. In general, further regularity (such as lower semicontinuity) of the relation D_n is required for continuity of $U_n w$ (see [1], pages 115–116 and [18], pages 29–30).

4. Regularity of F^n. The objective of this section is to establish that the relation F^n of (2.6) is upper semicontinuous and compact valued (Theorem 4.12). Except in Lemmas (4.2) and (4.4), we assume W(i), W(ii), and W(iii) throughout. We begin with some useful characterizations of $G(F^n)$ and of elements of $F^n(h_n)$.

For $m, n \in N$, $m > n$, define the relation $E_m^n : H_n \to \mathfrak{P}'(H_m^n)$ by

(4.1)

$$E_m^n(h_n) = \{h_m^n = (a_n, s_{n+1}, \cdots, s_m) \in H_m^n : a_j \in D_j(h_n, \cdots, s_j), n \leqslant j \leqslant m-1\}$$

for all $h_n \in H_n$. Under Assumptions W(i) and W(ii), E_m^n has a closed graph.

Given $n \in N$ and $\nu \in \mathcal{P}(H^n)$, for every $m \geqslant n$, let ν_m^1 and ν_m denote the marginal probability measures under ν of the (coordinate) random vectors $(\alpha_n, \zeta_{n+1}, \cdots, \alpha_m, \zeta_{m+1})$ and $(\alpha_n, \zeta_{n+1}, \cdots, \alpha_m)$, respectively, where these mappings are the projections of H^n onto the appropriate factor spaces. We then have the following representations of $G(F^n)$.

LEMMA 4.2. *Under* W(i) *and* W(ii),

(a) $G(F^n) = \{(h_n, \nu) \in H_n \times \mathcal{P}(H^n) : \nu_m^1(E_{m+1}^n(h_n)) = 1$, *and* $\nu_m^1 = \nu_m q_m(h_n)$, *for every* $m \geqslant n\}$.

(b) $G(F^n) = \{(h_n, \nu) \in H_n \times \mathcal{P}(H^n) : \nu = (\pi q)^n(h_n)$ *for some period-n plan* $(\pi_m)_{m \geqslant n}$ *for which*

$$\pi_n q_n \cdots \pi_m q_m(E_{m+1}^n(h_n)|h_n) = 1, \text{ for every } m \geqslant n\}. \tag{4.3}$$

PROOF. The proof that $G(F^n)$ is contained in each of these sets is straightforward. The proof that $G(F^n)$ contains each of these sets is closely analogous to the proof of Lemma 13.1, part (b), of [20], page 94; see also Lemma 7.2 of [41], page 884. In particular, the proof rests on a theorem for decomposition of probability measures on products of *SB*-spaces ([20], Corollary 12.5, page 88) and on a corollary to the Blackwell and Ryll-Nardzewski selection theorem ([5], page 223), Corollary 12.7 of [20], page 89. □

We can interpret part (b) of the lemma in the following sense: for period-n plans $(\pi_m)_{m \geqslant n}$ which represent the probability measure $\nu \in \mathcal{P}(H^n)$ in that $\nu = (\pi q)^n(h_n)$, the additional requirement that there be such a period-n plan which is feasible is equivalent to the requirement that there be such a period-n plan which satisfies (4.3) for every $m \geqslant n$.

By a measurable selection for the relation F^n, we mean a function, say $m_n : H_n \to \mathcal{P}(H^n)$, that is $\mathcal{H}_n / \mathcal{B}(\mathcal{P}(H^n))$-measurable, and hence a transition probability, such that for each $h_n \in H_n$, $m_n(h_n) \in F^n(h_n)$. The following lemma gives a consistency result for measure-valued functions m_i that are connected by feasible period-k controls, $i \leqslant k < n$, to measurable selections for F^n.

LEMMA 4.4. *Under* W(i) *and* W(ii), *if m_n is a measurable selection for F^n and $\pi_k \in \Pi_k$, $k = 1, \cdots, n-1$, then the function $m_i = \pi_i q_i \cdots \pi_{n-1} q_{n-1} m_n$ is a measurable selection for F^i, $i = 1, \cdots, n-1$.*

PROOF. Given m_n, a measurable selection for F^n, and $\pi_k \in \Pi_k$, $k = 1, \cdots, n-1$, then $m_i = \pi_i q_i \cdots \pi_{n-1} q_{n-1} m_n$ is a transition probability from H_i to H^i by the product measure theorem of C. Ionescu Tulcea ([32], page 162). We show that for each $h_i \in H_i$, $m_i(h_i) \in F^i(h_i)$.

The proof is a modification of parts (b) and (c) in the proof of Theorem 14.1 in [20], page 99. For probability measure p on $\mathcal{H}_n$, pm_n is a probability measure on $\mathcal{H}_\infty$ with associated coordinate-representation process $(\zeta_1, \alpha_1, \zeta_2, \alpha_2, \cdots)$. From Corollary 12.7 in [20], we have the representation $pm_n = p\sigma_n\gamma_n\sigma_{n+1}\gamma_{n+1} \cdots$ where v_k is a regular conditional probability of ζ_{k+1} given $(\zeta_1, \alpha_1, \cdots, \zeta_k, \alpha_k)$ under pm_n and σ_k is a regular conditional probability of α_k given $(\zeta_1, \alpha_1, \cdots, \zeta_k)$ under pm_n satisfying $\sigma_k(D_k(h_k)|h_k) = 1$ for all $h_k \in H_k$, $k \geqslant n$. Then there exists $T \in \mathcal{H}_n$ for which $p(T) = 0$ and $m_n(h_n) = \sigma_n\gamma_n\sigma_{n+1}\gamma_{n+1} \cdots (h_n)$ for all $h_n \in T^c$. Now, since $m_n(h_n) \in F^n(h_n)$ for each $h_n \in H_n$, we have the representation $m_n(h_n) = (\pi^{h_n}q)^n(h_n)$, with $(\pi_m^{h_n})_{m\geqslant n} \in \Pi^n$, for each $h_n \in H_n$. Combining these two representations and Lemma A8 of [20], we obtain $m_n(h_n) = (\sigma q)^n(h_n)$, with $(\sigma_m)_{m\geqslant n} \in \Pi^n$, for each $h_n \in T^c$. Finally, take $p = \delta_{\{h_i\}}\pi_i q_i \cdots \pi_{n-1}q_{n-1}$, where $\delta_{\{h_i\}}$ is the probability measure degenerate at h_i. Observe that for every $\Omega \in \mathcal{H}^i$,

$$\pi_i q_i \cdots \pi_{n-1}q_{n-1}m_n(\Omega|h_i)$$

$$= \int_{T^c(h_i)} m_n(\Omega(h_n^i)|h_i, h_n^i)\pi_i q_i \cdots \pi_{n-1}q_{n-1}(dh_n^i|h_i)$$

$$= \int_{T^c(h_i)} \sigma_n q_n \sigma_{n+1} \cdots (\Omega(h_n^i)|h_i, h_n^i)\pi_i q_i \cdots \pi_{n-1}q_{n-1}(dh_n^i|h_i)$$

$$= \pi_i q_i \cdots \pi_{n-1}q_{n-1}\sigma_n q_n \sigma_{n+1} \cdots (\Omega|h_i),$$

where $t^c(h_i)$ and $\Omega(h_n^i)$ are, respectively, the h_i-section of t^c and the h_n^i-section of Ω. Thus $m_i(h_i) = \pi_i q_i \cdots \pi_{n-1}q_{n-1}\sigma_n q_n \sigma_{n+1} \cdots (h_i) \in F^i(h_i)$. □

The proofs of the following two lemmas are analogous to those of Lemmas 5.4 and 5.5 of Schäl ([39], page 361).

LEMMA 4.5. *If Λ_n is relatively compact in H_n, then $F^n(\Lambda_n)$ is relatively compact in $\mathcal{P}(H^n)$.*

PROOF. From Lemma 4.2 of Schäl ([39], page 358), it suffices to show that for each $m > n$ the set of probability measures $\{\nu \circ (\eta_m^n)^{-1}; \nu \in F^n(\Lambda_n)\}$ is relatively compact in $\mathcal{P}(H_m^n)$; here $\eta_m^n = (\alpha_n, \zeta_{n+1}, \cdots, \zeta_m)$, the coordinate-representation vector on H_m^n. For each $\nu \in F^n(\Lambda_n)$, $\nu \circ (\eta_m^n)^{-1} = \pi_n q_n \cdots \pi_{m-1}q_{m-1}(h_n)$, for some $h_n \in \Lambda_n$ and some $\pi_i \in \Pi_i$, $n \leqslant i < m$. Hence from Lemma 3.2 of Schäl ([39], page 353) the task reduces to showing for each $m > n$, and any sequence $\{f_j\}_{j\in N} \subset C(H_m^n)$ with $f_j \downarrow 0$ pointwise, that

$$\lim_{j\to\infty}\sup_{h_n\in\Lambda_n}\sup_{\pi_i\in\Pi_i,\, n\leqslant i<m}\int_{H_m^n} f_j(h_m^n)\pi_n q_n \cdots \pi_{m-1}q_{m-1}(dh_m^n|h_n) = 0. \tag{4.6}$$

From Lemma (3.7) (iii) we have that (4.6) is equivalent to

$$\lim_{j\to\infty}\sup_{h_n\in\Lambda_n} U_n \cdots U_{m-1}f_j(h_n) = 0. \tag{4.7}$$

Now, from Lemma (3.7) (i) and (ii) we obtain that $U_n \cdots U_{m-1}f_j \in \hat{C}(H_n)$ and $U_n \cdots U_{m-1}f_j \downarrow 0$ pointwise as $j \to \infty$. From Dini's theorem, given as Proposition

9.2.11 in Royden ([36], page 162), we obtain

$$\lim_{j\to\infty}\sup_{h_n\in\overline{\Lambda}_n} U_n \cdots U_{m-1} f_j(h_n) = 0,$$

where $\overline{\Lambda}$ is the closure of Λ_n in H_n. Then (4.7) follows. ▯

LEMMA 4.8. *$G(F^n)$ is closed in $H_n \times \mathcal{P}(H^n)$.*

PROOF. For each $m > n$, let $\{w_{mk}\}_{k\in N} \subset C(H^n_{m+1})$ be a sequence of functions which separate elements of $\mathcal{P}(H^n_{m+1})$ (see the proof of Theorem 2.6.2 in [33], page 43). For $\nu \in \mathcal{P}(H^n)$, we set $\hat{w}_{mk}(\nu) = \int w_{mk}(h^n_{m+1})\nu^1_m(dh^n_{m+1})$ and $\overline{w}_{mk}(h_n, \nu) = \int w_{mk}(h^n_{m+1})\nu_m q_m(dh^n_{m+1}|h_n)$ where ν^1_m and ν_m are defined at the beginning of this section. Let

$$(4.9) \qquad \Delta_{mk} = \{(h_n, \nu) \in H_n \times \mathcal{P}(H^n); \quad \text{for each } j = n, \cdots, m \quad \nu^1_j(E^n_{j+1}(h_n)) = 1 \quad \text{and} \quad \hat{w}_{mk}(\nu) = \overline{w}_{mk}(h_n, \nu)\}$$

where $E^n_{j+1}(h_n)$ is defined in (4.1). Using (4.2) (a) and the separating property of the sequences $\{w_{mk}\}_{k\in N}$, we obtain that

$$(4.10) \qquad G(F^n) = \cap_{m\geqslant n} \cap_{k\in N} \Delta_{mk}.$$

Thus it suffices to show that for each $m \geqslant n$ and $k \in N$, Δ_{mk} is closed in $H_n \times \mathcal{P}(H^n)$. Since for $j = n, \cdots, m$, E^n_{j+1} has a closed graph, it follows from [23], Corollary (3.27) and the continuity of the mappings $\hat{w}_{mk}$ and $\overline{w}_{mk}$ that Δ_{mk} is closed. ▯

COROLLARY 4.11. *If $\Lambda_n \subseteq H_n$ is closed, then $\{(h_n, \nu); h_n \in \Lambda_n$ and $\nu \in F^n(h_n)\}$ is closed.*

THEOREM 4.12. *F^n is upper semicontinuous and compact valued on H_n.*

PROOF. Let $\{h_{nk}\}_{k=0,1,\cdots}$ be a sequence from H_n with $\lim_{k\to\infty} h_{nk} = h_{n0}$ and let $m^k(h_{nk}) \in F^n(h_{nk})$ for each $k \in N$. Lemma (4.5) ensures that $\{m^k(h_{nk})\}_{k\in N}$ has a subsequence $\{m^{k'}(h_{nk'})\}$ such that $m^{k'}(h_{nk'})$ converges to some $\nu \in \mathcal{P}(H^n)$ weakly as $k' \to \infty$. Corollary (4.11) gives that $\nu \in F^n(h_{n0})$. ▯

It follows from Theorem 4.12 that the relation F^n may be viewed as a function defined on H_n taking values in $\mathcal{C}(\mathcal{P}(H^n))$, the nonempty compact subsets of $\mathcal{P}(H^n)$. Since H^n, and hence $\mathcal{P}(H^n)$, is a separable metric space, the space $\mathcal{C}(\mathcal{P}(H^n))$ endowed with the Hausdorff metric topology is a separable metric space (see Theorems 3.3, 3.6, 4.5, and 4.9 of [31]). Let $\mathfrak{S}$ denote the Borel subsets of $\mathcal{C}(\mathcal{P}(H^n))$. Then we have from Theorem 4.12 and [6], page 359, the following.

COROLLARY 4.13. *F^n is $\mathcal{H}_n/\mathfrak{S}$-measurable.*

5. Existence of persistently optimal plans. The main objective of this section is to prove Theorem 3.4. We need the following definition. We say that a plan

$\pi = (\pi_n)_{n \in N} \in \Pi^1$ satisfies the *optimality criterion* if for each $n \in N$ and $h_n \in H_n$,

$$(5.1) \quad \int v_{n+1}(h_n, a_n, s_{n+1})\pi_n q_n(d(a_n, s_{n+1})|h_n) = \sup_{a_n \in D_n(h_n)} \int v_{n+1}(h_n, a_n, s_{n+1}) q_n(ds_{n+1}|h_n, a_n),$$

where v_{n+1} is defined as in (2.8). We first show there is a (feasible) plan satisfying the optimality criterion and a measurable selection m_n^* of F^n which satisfies $u_n(h_n, m_n^*(h_n)) = v_n(h_n)$ for all $h_n \in H_n$, for each $n \in N$ (Theorems 5.2 and 5.3). Using these results, we then prove Theorem 3.4. Throughout this section, we assume W(i), W(ii), W(iii), and W(iv).

For a measurable map $z_n : H_n \to A_n$, we denote by δ_{z_n} the corresponding transition probability degenerate at $z_n(h_n)$ for each $h_n \in H_n$. If z_n is a measurable selection for D_n, then δ_{z_n} is a feasible period-n control, i.e., $\delta_{z_n} \in \Pi_n$.

THEOREM 5.2. *For each $n \in N$, there exists a measurable selection z_n^* for D_n such that the feasible plan $\pi = (\delta_{z_n^*})_{n \in N}$ satisfies the optimality criterion* (5.1).

PROOF. From Lemma (3.1), Theorem (4.12), and W(iv), together with Proposition 10.2 of [38], page 189, we obtain that v_n, defined in (2.8), is in $\hat{C}(H_n)$, for each $n \in N$. Thus, from (3.7) (i) (a) $L_n v_{n+1} \in \hat{C}(H_n \times A_n)$, for each $n \in N$, and we can apply Theorem 12.1 of [38], page 191, to obtain the mappings $(z_n^*)_{n \in N}$ with associated plan $\pi = (\delta_{z_n^*})_{n \in N} \in \Pi^1$ which satisfies the optimality criterion. □

THEOREM 5.3. *For each $n \in N$, there exists a measurable selection m_n^* for F^n which satisfies $u_n(h_n, m_n^*(h_n)) = v_n(h_n)$ for all $h_n \in H_n$.*

PROOF. Let $\tilde{u}_n = u_n$ on $G(F^n)$ and $= -\infty$ off of $G(F^n)$; from Lemma 4.8 and W(iv), $\tilde{u}_n \in \hat{C}(H_n \times \mathcal{P}(H^n))$. Thus from results (3.1), (4.12), and (4.13) and Theorem 12.1 of [38], page 191, there exists a measurable selection m_n^* for F^n satisfying $u_n(h_n, m_n^*(h_n)) = \sup_{\nu \in F^n(h_n)} \tilde{u}_n(h_n, \nu) = v_n(h_n)$ for all $h_n \in H_n$. □

We note that we have not proved existence of a persistently optimal plan in Theorem (5.3), for as h_n varies, the feasible period-n plan used in the representation of $m_n^*(h_n)$ as an element of $F^n(h_n)$ may vary.

PROOF OF THEOREM 3.4. For each $n \in N$, let z_n^* be the measurable mapping from Theorem 5.2 for which the associated plan $\pi^* = (\delta_{z_n^*})_{n \in N}$ satisfies the optimality criterion (5.1), and let m_n^* be the measurable selection for F^n from Theorem 5.3. We show that

$$(5.4) \qquad u_n(h_n, m_n^*(h_n)) = u_n(h_n, (\delta_{z^*} q)^n(h_n)),$$

for all $h_n \in H_n$, where $(\delta_{z^*} q)^n = \delta_{z_n^*} q_n \delta_{z_{n+1}^*} q_{n+1} \cdots$. It will then follow from Theorem 5.3 that π^* is a persistently optimal plan.

For $k = 0, 1, \cdots$, define $w_{n+k} : H_n \to R_-$ by

$$(5.5) \qquad w_n(h_n) = u_n(h_n, m_n^*(h_n)) = v_n(h_n)$$

and for $k \in N$,

$$w_{n+k}(h_n) = u_n\big(h_n, \delta_{z_n^*}q_n \cdots \delta_{z_{n+k-1}^*}q_{n+k-1}m_{n+k}^*(h_n)\big). \tag{5.6}$$

From results (4.4), (5.2), and (5.3), it follows that

$$\big(h_n, \delta_{z_n^*}q_n \cdots \delta_{z_{n+k-1}^*}q_{n+k-1}m_{n+k}^*(h_n)\big) \in G(F^n)$$

for each $h_n \in H_n$. We show that

$$w_{n+k} \leqslant w_{n+k+1} \text{ on } H_n, \qquad \textit{for each } k \geqslant 0, \tag{5.7}$$

and

$$\limsup_{k\to\infty} w_{n+k}(h_n) \leqslant u_n\big(h_n, (\delta_{z^*}q)^n(h_n)\big), \qquad \text{for each } h_n \in H_n. \tag{5.8}$$

To obtain (5.7), fix $k \geqslant 0$, let $h_{n+k} \in H_{n+k}$, let $\nu \in F^{n+k}(h_{n+k})$, and let $(\pi_m)_{m \geqslant n+k} \in \Pi^{n+k}$ be such that $\nu = (\pi q)^{n+k}(h_{n+k})$. From Theorem 5.2 it follows that for every $a_{n+k} \in D_{n+k}(h_{n+k})$,

$$\begin{aligned} &\int v_{n+k+1}(h_{n+k}, z_{n+k}^*(h_{n+k}), s_{n+k+1})q_{n+k}(ds_{n+k+1}|h_{n+k}, z_{n+k}^*(h_{n+k})) \\ &\qquad \geqslant \int v_{n+k+1}(h_{n+k}, a_{n+k}, s_{n+k+1})q_{n+k}(ds_{n+k+1}|h_{n+k}, a_{n+k}), \end{aligned}$$

and hence by definition of v_{n+k+1}, we have that

$$\begin{aligned} &\int v_{n+k+1}(h_{n+k}, a_{n+k}, s_{n+k+1})\delta_{z_{n+k}^*}q_{n+k}(d(a_{n+k}, s_{n+k+1})|h_{n+k}) \\ &\qquad \geqslant \int v_{n+k+1}(h_{n+k}, a_{n+k}, s_{n+k+1})\pi_{n+k}q_{n+k}(d(a_{n+k}, s_{n+k+1})|h_{n+k}) \\ &\qquad \geqslant \int u(h_{n+k}, h^{n+k})\nu(dh^{n+k}). \end{aligned} \tag{5.9}$$

Since $\nu \in F^{n+k}(h_{n+k})$ was arbitrary, it follows from (5.9) that for each $h_{n+k} \in H_{n+k}$,

$$\int v_{n+k+1}(h_{n+k}, a_{n+k}, s_{n+k+1})\delta_{z_{n+k}^*}q_{n+k}(d(a_{n+k}, s_{n+k+1})|h_{n+k}) \geqslant v_{n+k}(h_{n+k}). \tag{5.10}$$

From Theorems 5.2 and 5.3 and from (5.10) then, we have that for each $h_n \in H_n$,

$$\begin{aligned} w_{n+k+1}(h_n) &= \int u(h_n, h^n)\delta_{z_n^*}q_n \cdots \delta_{z_{n+k}^*}q_{n+k}m_{n+k+1}^*(dh^n|h_n) \\ &= \int v_{n+k+1}(h_n, h_{n+k+1}^n)\delta_{z_n^*} \cdots \delta_{z_{n+k}^*}q_{n+k}(dh_{n+k+1}^n|h_n) \\ &\geqslant \int v_{n+k}(h_n, h_{n+k}^n)\delta_{z_n^*}q_n \cdots \delta_{z_{n+k-1}^*}q_{n+k-1}(dh_{n+k}^n|h_n) \\ &= w_{n+k}(h_n), \end{aligned}$$

and (5.7) holds. (5.8) then follows immediately from W(iv), (5.5), (5.6), and (5.7).

Together, (5.7) and (5.8) imply that for each $h_n \in H_n$, (5.4) holds, and hence that π^* is a persistently optimal plan. □

6. Characterizations of persistently optimal plans. In this section we mention two characterizations of persistently optimal plans which are immediate consequences of the previous results. We also comment on some relationships between this work and that in the literature. Throughout this section we assume W(i), W(ii), W(iii), and W(iv).

The first characterization is motivated by the analysis of optimal plans in [8] and [42] and requires the following definitions.

DEFINITION 6.1. A plan $\pi = (\pi_n)_{n \in N} \in \Pi^1$ is thrifty if for each $n \in N$

$$v_n(h_n) = \int v_{n+1}(h_n, a_n, s_{n+1}) \pi_n q_n(d(a_n, s_{n+1}) | h_n), \tag{6.2}$$

for all $h_n \in H_n$.

DEFINITION 6.3. A plan $\pi = (\pi_n)_{n \in N} \in \Pi^1$ is equalizing if for each $n \in N$

$$u_n(h_n, (\pi q)^n(h_n)) \geqslant \limsup_k \int v_{n+k}(h_n, h^n_{n+k}) \pi_n q_n \cdots \pi_{n+k-1} q_{n+k-1}(dh^n_{n+k} | h_n), \tag{6.4}$$

for all $h_n \in H_n$.

The key element in the proof of the first characterization is the following result.

LEMMA 6.5. *For each* $\pi = (\pi_m)_{m \geqslant n} \in \Pi^n$ *and* $h_n \in H_n$,

$$\left\{ \int v_{n+k}(h_n, h^n_{n+k}) \pi_n q_n \cdots \pi_{n+k-1} q_{n+k-1}(dh^n_{n+k} | h_n) \right\}_{k=0,1,\cdots} \tag{6.6}$$

is a nonincreasing sequence (*with zero*th *term given by* $v_n(h_n)$) *which is bounded below by* $u_n(h_n, (\pi q)^n(h_n))$.

PROOF. From results (4.4) and (5.3) we have for $\pi_{n+k} \in \Pi_{n+k}$

$$\begin{aligned} v_{n+k}(h_{n+k}) &\geqslant u_{n+k}(h_{n+k}, \pi_{n+k} q_{n+k} m^*_{n+k+1}(h_{n+k})) \\ &= \int v_{n+k+1}(h_{n+k}, a_{n+k}, s_{n+k+1}) \pi_{n+k} q_{n+k}(d(a_{n+k}, s_{n+k+1}) | h_{n+k}) \end{aligned}$$

for each $h_{n+k} \in H_{n+k}$; hence the sequence in (6.6) is nonincreasing. The sequence is bounded below by $u_n(h_n, (\pi q)^n(h_n))$ since for each $k \geqslant 0$ $v_{n+k}(h_{n+k}) \geqslant u_{n+k}(h_{n+k}, (\pi q)^{n+k}(h_{n+k}))$, for each $h_{n+k} \in H_{n+k}$. □

THEOREM 6.7. *A feasible plan is persistently optimal if and only if it is both thrifty and equalizing.*

PROOF. Suppose $\pi \in \Pi^1$ is persistently optimal. Then $v_n(h_n) = u_n(h_n, (\pi q)^n(h_n))$ for all $h_n \in H_n$, for each $n \in N$, and hence from Lemma 6.5, $v_n(h_n) = \int v_{n+k}(h_n, h^n_{n+k}) \pi_n q_n \cdots \pi_{n+k-1} q_{n+k-1}(dh^n_{n+k} | h_n) = u_n(h_n, (\pi q)^n(h_n))$ for each $k \in N$. This implies that π is thrifty and equalizing. On the other hand, if $\pi \in \Pi^1$ is thrifty and equalizing, then

$$\begin{aligned} v_n(h_n) &= \int v_{n+k}(h_n, h^n_{n+k}) \pi_n q_n \cdots \pi_{n+k-1} q_{n+k-1}(dh^n_{n+k} | h_n) \\ &= \limsup_j \int v_{n+j}(h_n, h^n_{n+j}) \pi_n q_n \cdots \pi_{n+j-1} q_{n+j-1}(dh^n_{n+j} | h_n) \\ &\leqslant u_n(h_n, (\pi q)^n(h_n)) \end{aligned}$$

for all $h_n \in H_n$, for each $n \in N$, and hence from Lemma 6.5 π is persistently optimal. □

Observe that in the proof of existence of a persistently optimal plan (Theorem 3.4) we use results (5.2) and (5.3) to show the candidate is thrifty (5.7) and we use (5.3) and the stability implied by W(iv) to show the plan is equalizing (5.8).

The second characterization of persistently optimal plans is in terms of plans which satisfy the optimality criterion (5.1).

THEOREM 6.8. *A feasible plan is persistently optimal if and only if it satisfies the optimality criterion* (5.1).

PROOF. That a feasible plan satisfying the optimality criterion is persistently optimal follows as in the proof of Theorem 3.4. Suppose then that $\pi^* = (\pi_n^*)_{n \in N}$ is a persistently optimal plan; and for $h_n \in H_n$ fixed, take $\hat{a}_n \in D_n(h_n)$. Define $\hat{\pi}_n$ by $\hat{\pi}_n(\hat{h}_n) = \delta_{\{\hat{a}_n\}}$ if $\hat{h}_n = h_n$ and $\hat{\pi}_n(\hat{h}_n) = \pi_n^*(\hat{h}_n)$ if $\hat{h}_n \neq h_n$. Then $\hat{\pi}_n \in \Pi_n$, and $\hat{\pi} = (\pi_1^*, \cdots, \pi_{n-1}^*, \hat{\pi}_n, \pi_{n+1}^*, \cdots)$ satisfies $(\hat{\pi}q)^n(h_n) \in F^n(h_n)$. Thus by (2.9) and (6.2)

$$\begin{aligned}\int v_{n+1}(h_n, a_n, s_{n+1})\pi_n^* q_n(d(a_n, s_{n+1})|h_n) &= u_n(h_n, (\pi^* q)^n(h_n)) \\ &\geqslant u_n(h_n, (\hat{\pi} q)^n(h_n)) \\ &= \int v_{n+1}(h_n, \hat{a}_n, s_{n+1})q_n(ds_{n+1}|h_n, \hat{a}_n).\end{aligned}$$

Since $\hat{a}_n$ was arbitrary in $D_n(h_n)$, we obtain that π^* satisfies the optimality criterion. □

The concept of optimality equations for dynamic programming has been discussed at length by Hinderer [20] (see also [34]). A sequence of functions $(w_n)_{n \in N}$ with $w_n : H_n \to \bar{R}$ is said to satisfy the optimality equations of the (NDPM) if for every $n \in N$ and $h_n \in H_n$,

(6.9) $$w_n(h_n) = \sup_{a_n \in D_n(h_n)} \int w_{n+1}(h_n, a_n, s_{n+1})q_n(ds_{n+1}|h_n, a_n).$$

The following result relates persistently optimal plans to solutions of these equations.

PROPOSITION 6.10. *Suppose $\pi^* = (\pi_n^*)_{n \in N} \in \Pi^1$ is a persistently optimal plan. Then the sequence $(w_n^*)_{n \in N}$, defined by $w_n^*(h_n) = u_n(h_n, (\pi^* q)^n(h_n))$ for each $h_n \in H_n$, satisfies the optimality equations* (6.9). *If, in addition, a sequence $(w_n)_{n \in N}$ of the form $w_n(h_n) = u_n(h_n, (\pi q)^n(h_n))$, for all $h_n \in H_n$ and some $\pi \in \Pi^1$, satisfies the optimality equations, then $w_n(h_n) \leqslant w_n^*(h_n)$, for all $h_n \in H_n$.*

PROOF. If π^* is persistently optimal plan it follows from (6.8) and (2.9) that

$$\begin{aligned} w_n^*(h_n) &= \int \left\{ \int u(h_n, a_n, s_{n+1}, h^{n+1})(\pi^* q)^{n+1}(dh^{n+1}|h_n, a_n, s_{n+1}) \right\} \\ &\quad \times \pi_n^* q_n(d(a_n, s_{n+1})|h_n) \\ &= \int v_{n+1}(h_n, a_n, s_{n+1})\pi_n^* q_n(d(a_n, s_{n+1})|h_n) \\ &= \sup_{a_n \in D_n(h_n)} \int v_{n+1}(h_n, a_n, s_{n+1})q_n(ds_{n+1}|h_n, a_n) \\ &= \sup_{a_n \in D_n(h_n)} \int w_{n+1}^*(h_n, a_n, s_{n+1})q_n(ds_{n+1}|h_n, a_n) \end{aligned}$$

for all $h_n \in H_n$. The second part follows from the definition of a persistently optimal plan. $\square$

REMARK 6.11. Finally we observe some contexts in which the Assumptions W hold and hence to which Theorem 3.4 applies, when the total reward function has representation $u = \Sigma_{n=1}^{\infty} r_n$, where $r_n \in \hat{B}(H_n)$. Assumptions W hold if W(i), W(ii), and W(iii) hold and if $u = \Sigma_{j=1}^{\infty} r_j$ where $r_j \in \hat{C}(H_j)$ for each $j \in N$ and one of the following uniformity conditions hold:

$$\Sigma_{j=1}^{\infty} \|r_j\| < \infty \tag{6.12}$$

$$\Sigma_{j=1}^{\infty} \|r_j^+\| < \infty \tag{6.13}$$

$$\lim_{M\to\infty} \sup_{N>M} \sup_{h_\infty \in H_\infty} \Sigma_{j=M+1}^{N} r_j(h_j) = 0 \tag{6.14}$$

(6.15) (i) $\int \Sigma_{j=1}^{\infty} r_j^+(h_n, h_j^n) \nu(dh^n) < \infty$ for each $(h_n, \nu) \in G(F^n)$; and
(ii) $\lim_{M\to\infty} \sup_{N>M} \sup_{(h_n, \nu) \in G(F^n)} \int \Sigma_{j=M+1}^{N} r_j(h_n, h_j^n) \nu(dh^n) = 0.$

In each case, upper semicontinuity of u_n in W(iv) can be verified by using Proposition 10.1.1 of Schäl [38]. Conditions (6.12) and (6.13) are the convergence (C) and negative (EN) cases respectively in [20]. Conditions (6.14) and (6.15) are motivated by conditions (GA) and (C) in Schäl [38]. We note that conditions (GA), (C), and (W) of Schäl [38] do imply that $u_n(h_n, \cdot) \in \hat{C}(F^n(h_n))$ for each $h_n \in H_n$ but appear not to imply (6.15) (ii) or upper semicontinuity of u_n in W(iv). That is, in assuming that the total reward function has the additive, separating form, Schäl is able to prove optimality results without the uniformity in the h_n variable which we require through our Assumption W(iv). In particular, we use this additional requirement to obtain $v_n \in \hat{C}(H_n)$ for each $n \in N$, and existence of a plan which satisfies the optimality criterion and which is thrifty. On the other hand, Schäl avoids this additional requirement by using the iterative, finite-horizon to infinite-horizon method, together with the regularity and boundedness assumptions on $\{r_j\}_{j\in N}$ and $\{q_j\}_{j\in N}$ in the finite horizon. His counterpart to the optimal expected period-n payoff apparently need not be upper semicontinuous or bounded above unless some additional requirement implying W(iv) is imposed (see Corollary (16.3) of Schäl [38]).

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SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332

COLLEGE OF INDUSTRIAL MANAGEMENT
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332