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MARKOV-DEPENDENT σ-FIELDS AND CONDITIONAL EXPECTATIONS

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Basterfield showed that if $X \in L \log L$ and $\{\mathcal{F}_n\}$ form a sequence of independent σ-fields, then $E(X|\mathcal{F}_n) \to EX$ a.s. His proof uses the theory of Orlicz spaces. We generalize Basterfield's theorem to the case of Markov-dependent σ-fields and also weaken the restrictions on X. Our approach is different from Basterfield's in that it is martingale-theoretic.

1. Let (Ω, Σ, P) be a probability space and $\{\mathcal{F}_n\}$ a sequence of sub-σ-fields of Σ. The $\mathcal{F}_n$ are said to be Markov-dependent (see, e.g., [4]) if, for all $n \geq 1$,

$$E(f|\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_n) = E(f|\mathcal{F}_n)$$

and

$$E(g|\mathcal{F}_n, \mathcal{F}_{n+1}, \cdots) = E(g|\mathcal{F}_n)$$

for all L_1 functions f and g such that f is $\mathcal{B}(\mathcal{F}_n, \mathcal{F}_{n+1}, \cdots)$ measurable and g is $\mathcal{B}(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_n)$ measurable ($\mathcal{B}(\cdot)$ is the sub-σ-field of Σ generated by the σ-fields in parenthesis).

The principal result of this paper (Theorem 1) shows that if Y_n is a sequence of random variables converging to a random variable Y a.s., then, under certain conditions we may assert $E(Y_n|\mathcal{F}_n) \to E(Y|\mathcal{T})$ a.s. where $\mathcal{T}$ is the tail σ-field of the Markov-dependent sequence $\{\mathcal{F}_n\}$, i.e., $\mathcal{T} = \cap_{n \geq 1} \mathcal{B}(\mathcal{F}_n, \mathcal{F}_{n+1}, \cdots)$.

As a corollary we deduce a theorem of Basterfield [1]: if the $\{\mathcal{F}_n\}$ are independent, then $E(X|\mathcal{F}_n) \to EX$ a.s. provided that $E|X| \log^+|X| < \infty$. Independence of $\{\mathcal{F}_n\}$, a special case of Markov-dependence, can be written as: if $F_i \in \mathcal{F}_i$ for $i \leq k$, then $P(\cap_{i \leq k} F_i) = \Pi_{i \leq k} P(F_i)$. Basterfield's proof uses the theory of Orlicz spaces whereas we use an entirely probabilistic approach based on the martingale convergence theorem. The nub of the argument is an elementary lemma about conditional expectations when both the functions and the σ-fields are varying. This martingale approach seems to yield much more with much less work than the Orlicz space attack.

Gundy, in reviewing Basterfield's paper [3], indicated that the hypothesis $X \in L \log L$ (that is, $E|X|\log^+|X| < \infty$) could be weakened to $\sup_n |E(X|\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_n)| \in L_1$ with Basterfield's proof still going through to obtain his result. Gundy's suggested extension is also one of our corollaries.

Our random variables will always be in L_1 and all our σ-fields will be sub-σ-fields of Σ. We are only careful to write "a.s." in statements of results; otherwise we take the usual liberties. We write submartingales (U_n, Δ_n) to indicate that the random variables U_n are adapted to the nested σ-fields Δ_n.

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The following lemma seems to be known. We include a proof for completeness.

2. LEMMA. *Let X_n be a sequence of random variables, and $\mathfrak{B}_n$ an increasing or decreasing sequence of σ-fields with $\mathfrak{B}_n \to \mathfrak{B}$. Then*

(a) $\liminf_n E(X_n|\mathfrak{B}_n) \geq E(\liminf_n X_n|\mathfrak{B})$ a.s. *if* $X_n \geq Z \in L_1$.

(b) *If* $X_n \leq Z \in L_1$, *then*

$$\limsup_n E(X_n|\mathfrak{B}_n) \leq E(\limsup_n X_n|\mathfrak{B}) \text{ a.s.}$$

(c) *If* $\sup_n|X_n| \in L_1$, *and* $X_n \to X$ a.s., *then*

$$\lim_n E(X_n|\mathfrak{B}_n) = E(X|\mathfrak{B}) \text{ a.s.}$$

PROOF. Let $n \geq n_0$. Then

$$E(X_n|\mathfrak{B}_n) \leq E(\sup_{k \geq n} X_k|\mathfrak{B}_n) \leq E(\sup_{k \geq n_0} X_k|\mathfrak{B}_n). \tag{1}$$

If $X_n \leq Z \in L_1$, the right side of (1) converges to $E(\sup_{k \geq n_0} X_k|\mathfrak{B})$ by the martingale theorem as $n \to \infty$, so that

$$\limsup_n E(X_n|\mathfrak{B}_n) \leq E(\sup_{k \geq n_0} X_k|\mathfrak{B}).$$

Letting $n_0 \to \infty$ proves (b). We may also write

$$E(X_n|\mathfrak{B}_n) \geq E(\inf_{k \geq n} X_k|\mathfrak{B}_n) \geq E(\inf_{k \geq n_0} X_k|\mathfrak{B}_n)$$

and an argument similar to the above shows (a). Under the hypotheses of (c) both (a) and (b) are true with

$$\liminf_n E(X_n|\mathfrak{B}_n) \geq E(X|\mathfrak{B}) \geq \limsup_n E(X_n|\mathfrak{B}_n),$$

proving (c). Notice that there is no problem in (a), (b) or (c) with the finiteness of the conditional expectation of the limits, since under the stated conditions the limits belong to L_1.

The main result follows. Recall the definition of $\mathfrak{T}$ from Section 1.

THEOREM 1. *Let the sequence of σ-fields $\{\mathfrak{F}_n\}$ be Markov-dependent, and let $\mathfrak{T}$ be the tail σ-field of $\{\mathfrak{F}_n\}$. Let $\{Y_n\}$ be a sequence of random variables with $\sup_n|Y_n| \in L_1$ and $Y_n \to Y$ a.s.. Suppose that either* (a) *Y_n is $\mathfrak{B}(\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_n)$ measurable for each n, or* (b) *Y_n is $\mathfrak{B}(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \cdots)$ measurable for each n. Then,*

$$E(Y_n|\mathfrak{F}_n) \to E(Y|\mathfrak{T}) \text{ a.s.}$$

In case (b), *the limit* $E(Y|\mathfrak{T}) = Y$ a.s.

PROOF. To simplify the notation, put $\mathfrak{B}(\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_n) = \mathfrak{H}_n$, $\mathfrak{B}(\mathfrak{F}_1, \mathfrak{F}_2, \cdots) = \mathfrak{H}$, $\mathfrak{B}(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \cdots) = \mathfrak{T}_n$. Assume (a), and use the lemma and Markov-dependence to obtain

$$E(Y_n|\mathfrak{F}_n) = E(Y_n|\mathfrak{T}_n) \to E(Y|\mathfrak{T}).$$

Assume (b). Then

$$E(Y_n|\mathfrak{F}_n) = E(Y_n|\mathfrak{H}_n) \to E(Y|\mathfrak{H}),$$

but the limit Y is $\mathfrak{T}$ measurable under (b), so $E(Y|\mathfrak{H}) = Y = E(Y|\mathfrak{T})$. This concludes the proof of Theorem 1.

COROLLARY 1. *Let $X \in L_1$. If the sequence $\{\mathfrak{F}_n\}$ is Markov-dependent each of the following conditions is sufficient for $E(X|\mathfrak{F}_n) \to E(X|\mathfrak{T})$ a.s.:*
(a) $X \in L \log L$.
(b) $\sup_n |E(X|\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_n)| \in L_1$.
(c) $\sup_n |E(X|\mathfrak{F}_n, \mathfrak{F}_{n+1}, \cdots)| \in L_1$.

PROOF. It is known that (a) implies (b) and (c) ([2], page 317 and Jensen's inequality) so it is sufficient to prove the implications for (b) and (c). Using our shorthand notation, put $Y_n = E(X|\mathcal{H}_n)$. Under (b), we have $Y_n \to Y = E(X|\mathcal{H})$ by the martingale theorem and $\sup_n |Y_n| \in L_1$. Theorem 1 may be applied under case (a), so

$$E(X|\mathfrak{F}_n) = E(E(X|\mathcal{H}_n)|\mathfrak{F}_n) = E(Y_n|\mathfrak{F}_n) \to E(E(X|\mathcal{H})|\mathfrak{T}) = E(X|\mathfrak{T}),$$

proving the sufficiency of both (a) and (b). If (c) holds, set $Y_n = E(X|\mathfrak{T}_n) \to E(X|\mathfrak{T})$. Then (b) of Theorem 1 is satisfied and the conclusion is

$$E(X|\mathfrak{F}_n) = E(E(X|\mathfrak{T}_n)|\mathfrak{F}_n) = E(Y_n|\mathfrak{F}_n) \to E(E(X|\mathfrak{T})|\mathfrak{T}) = E(X|\mathfrak{T}),$$

completing the proof of Corollary 1.

COROLLARY 2. *Let $X \in L_1$ and let the sequence $\{\mathfrak{F}_n\}$ be independent. Then each of the following conditions is sufficient for $E(X|\mathfrak{F}_n) \to EX$ a.s.:*
(a) $X \in L \log L$ *(Basterfield's theorem).*
(b) $\sup_n |E(X|\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_n)| \in L_1$ *(Gundy's extension).*
(c) $\sup_n |E(X|\mathfrak{F}_n, \mathfrak{F}_{n+1}, \cdots)| \in L_1$.

PROOF. Apply Corollary 1, and observe that in the independent case $\mathfrak{T}$ is trivial, so that $E(X|\mathfrak{T}) = EX$.

REMARK. Gundy observes in [3] that (b) is properly weaker than (a). The implication by (c) appears new even in the independent case. The referee has noted (c) is properly weaker than (a); take X $\mathfrak{F}_1$ measurable and in L_1 but not in $L \log L$.

THEOREM 2. *Let $(Y_n, \mathcal{G}_n, n \geqslant 1)$ be a submartingale with $\mathcal{G}_n$ either increasing or decreasing and $\mathcal{G}_n \supset \mathfrak{F}_n$ for each $n \geqslant 1$, where the sequence $\{\mathfrak{F}_n\}$ is Markov-dependent. If, in the increasing case, we have $\sup_n |E(Y_n|\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_n)| \in L_1$ or, in the decreasing case, we have*

$$\sup_n |E(Y_n|\mathfrak{F}_n, \mathfrak{F}_{n+1}, \cdots)| \in L_1,$$

then $E(Y_n|\mathfrak{F}_n) \to Z$ a.s., where Z is $\mathfrak{T}$ measurable and in L_1. If, in addition, we have $\sup_n |Y_n| \in L_1$, then $Y_n \to Y$ a.s. and

$$E(Y_n|\mathfrak{F}_n) \to E(Y|\mathfrak{T}) \text{ a.s.}$$

PROOF. In the increasing case $\mathcal{G}_n \supset \mathcal{H}_n$ and $(E(Y_n|\mathcal{H}_n), \mathcal{H}_n)$ is seen to be a submartingale because $(Y_n, \mathcal{G}_n)$ is. The hypothesis implies $E(Y_n|\mathcal{H}_n)$ converges to an L_1 function Z_1 and Theorem 1 gives

$$E(Y_n|\mathfrak{F}_n) = E(E(Y_n|\mathcal{H}_n)|\mathfrak{F}_n) \to E(Z_1|\mathfrak{T}) = Z.$$

In the decreasing case $\mathcal{G}_n \supset \mathfrak{T}_n$ and $(E(Y_n|\mathfrak{T}_n), \mathfrak{T}_n)$ is a submartingale which converges, and Theorem 1 implies

$$E(Y_n|\mathcal{F}_n) = E(E(Y_n|\mathcal{T}_n)|\mathcal{F}_n) \to E(Z_1|\mathcal{T}) = Z.$$

Under the restriction $\sup_n|Y_n| \in L_1$, $Y_n \to Y$ by the martingale theorem, and by the lemma $E(Y_n|\mathcal{H}_n) \to E(Y|\mathcal{H})$, so that in the increasing case

$$Z = E(E(Y|\mathcal{H})|\mathcal{T}) = E(Y|\mathcal{T})$$

and in the decreasing case

$$E(Y_n|\mathcal{T}_n) \to E(Y|\mathcal{T})$$

so

$$Z = E(E(Y|\mathcal{T})|\mathcal{T}) = E(Y|\mathcal{T}).$$

This completes the proof of Theorem 2.

As an application of the preceding, let $\{X_n, n \geqslant 0\}$ be a Markov process, and let f be defined on the sample space of the variables X_n with $f \in L \log L$. Then $E(f|X_n) \to E(f|\mathcal{T})$. In particular, if the process is ϕ-recurrent and aperiodic in the sense of [5], $\mathcal{T}$ is trivial, and $E(f|X_n) \to Ef$. Here we have taken $\mathcal{F}_n$ to be the σ-field generated by X_n and applied part (a) of Corollary 1.

REFERENCES

[1] Basterfield, J. G. (1972). Independent conditional expectations and L log L. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **21** 233–240.

[2] Doob, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[3] Gundy, R. F. (1973) *Mathematical Reviews* **46** No. 8285.

[4] Loeve, M. (1955). *Probability Theory*. Van Nostrand.

[5] Orey, S. (1971). *Limit Theorems for Markov Chain Transition Probabilities*. Van Nostrand Reinhold, New York.

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