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A RELIABILITY GROWTH MODEL INVOLVING DEPENDENT COMPONENTS

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Earlier papers have shown how to convert competing risk models involving *dependent* random variables into models involving only independent random variables, while simultaneously preserving the distribution of the minimum and the probabilities of the various failure patterns. In the present paper, we consider a sequence of such conversions occurring at successive points in chronological time in which the independent random variables are becoming stochastically larger. We obtain results which essentially demonstrate that the limiting distributions in the sequence of dependent models "correctly" correspond to the limiting distributions in the sequence of independent models.

These results have applications in reliability growth models and in biomedical competing risk models in which the competing risks are increasing with age; in these models dependency is permitted among the random variables.

1. Introduction. In several earlier papers, Tsiatis (1975), Miller (1977), and Langberg, Proschan and Quinzi (1978) show how to convert various competing risk models involving *dependent* random variables into models involving only *independent* random variables, while simultaneously preserving the distribution of the minimum and the probabilities corresponding to certain "failure patterns". Explicit equations are presented by Langberg, Proschan, and Quinzi (1978) which yield the distributions of the independent variables. In a more recent paper, Langberg, Proschan, and Quinzi (1977) develop statistical estimators of parameters of interest in competing risk models in which causes or times of death (in the biomedical context) or of failure (in the reliability context) are not necessarily independent.

In the present paper we present a result which should prove to be basic in converting reliability growth models involving dependent failure times into equivalent models involving only independent random variables. Analogous applications exist in the biomedical field in which survival functions may be decreasing, rather than increasing, with chronological time. (Throughout the paper we use "decreasing" in place of "nonincreasing" and "increasing" in place of "nondecreasing"). The probabilistic theorem presented will be useful in inference in reliability growth models involving dependent failure times.

2. The reliability growth model. The notation and terminology are as in Langberg, Proschan and Quinzi (1978).

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Consider a series system of n components undergoing improvement as time passes. At fixed chronological time u, $0 \leqslant u < u_0$ (u_0 possibly infinite) component i has random life length $T_i(u)$ for $i = 1, \cdots, n$, where $T_1(u), \cdots, T_n(u)$ are not necessarily mutually independent. We say that *failure pattern* I occurs if the simultaneous failures of the components in subset I of $\{1, \cdots, n\}$ and of no other components causes (i.e., coincides with) the failure of the system. Define

$$\xi(\mathbf{T}(u)) \begin{cases} = I \text{ if failure pattern } I \text{ occurs at time } u \\ = \varnothing \text{ otherwise.} \end{cases}$$

Let $\mathbf{S}(u)$ and $\mathbf{T}(u)$ represent the vectors of component life lengths of two systems whose system life lengths are $S(u)$ and $T(u)$, respectively. We say that the two systems are *equivalent in life lengths and patterns* at time u ($\mathbf{S}(u) =_{LP} \mathbf{T}(u)$) if $P(S(u) > t, \xi(\mathbf{S}(u)) = I) = P(T(u) > t, \xi(\mathbf{T}(u)) = I)$ for each $t \geqslant 0$ and each $I \subset \{1, \cdots, n\}$. (Throughout the paper, I is always taken to be *nonempty*.) Thus, two systems which are equivalent in life length and patterns are such that (i) their life lengths have the same distribution and (ii) the corresponding failure patterns in the two systems have the same probability of occurrence.

The main result of Langberg, Proschan and Quinzi (1978) may be restated in the context of our model as follows:

2.1. THEOREM. *For fixed u, $0 \leqslant u < u_0$, let $T(u) = \min(T_i(u), 1 \leqslant i \leqslant n)$ denote the life length of an n-component series system, where $T_i(u)$ represents the life length of component i, $i = 1, \cdots, n$. Define $\bar{F}_{u,I}(t) = P(T(u) > t, \xi(\mathbf{T}(u)) = I)$, $F_{u,I}(t) = P(T(u) \leqslant t, \xi(\mathbf{T}(u)) = I)$, $\bar{F}_u(t) = P(T(u) > t)$, and $\alpha(F_u) = \sup\{x : \bar{F}_u(x) > 0\}$. Then the following statements hold:*

(i) *A necessary and sufficient condition for the existence of a set of independent random variables $(H_I, I \subset \{1, \cdots, n\})$ which satisfy $\mathbf{H}(u) =_{LP} \mathbf{T}(u)$, where $H(u) = \min(H_I(u), I \subset \{1, \cdots, n\})$, is that the sets of discontinuities of the $F_{u,I}$ be pairwise disjoint.*

(ii) *The distributions of $H_I(u)$, $I \subset \{1, \cdots, n\}$, are uniquely determined on the interval $[0, \alpha(F_u))$ as follows:*

$$\bar{G}_{u,I}(t) \equiv P(H_I(u) > t) = \exp\left[-\textstyle\int_0^t \left(dF^C_{u,I}/\bar{F}_u\right)\right] \quad (2.1)$$

$$\Pi_{a_u(I,j)<t} \bar{F}_u(a_u(I,j))/\left[\bar{F}_u(a_u(I,j)) + f_u(a_u(I,j))\right],$$

$0 \leqslant t < \alpha(F_u)$, where $F^C_{u,I}$ is the continuous part of $F_{u,I}$, $\{a_u(I,j)\}$ is the set of discontinuities of $F_{u,I}$ and $f_{u,I}(a_u(I,j))$ is the size of the jump of $F_{u,I}$ at $a_u(I,j)$.

(Note that $G_{u,I}$, $F_{u,I}$, and F_I may place mass at infinity.)

The survival probabilities $\bar{G}_{u,I} \equiv 1 - G_{u,I}$ for $H_I(u)$, the time until a shock occurs which simultaneously destroys subset I, for $I \subset \{1, \cdots, n\}$, are obtained by solving the identities:

$$\Pi_{I \subset \{1, \cdots, n\}} \bar{G}_{u,I}(t) = \bar{F}_u(t), \qquad t \geqslant 0, \quad (2.2)$$

and

(2.3) $$\int_t^\infty \Pi_{J\neq I}\bar{G}_{u,J}\,dG_{u,I} = \bar{F}_{u,I}(t), \qquad t \geq 0.$$

Thus at each fixed instant u of chronological time, we may "replace" the original series system of *dependent* components by a set of mutually *independent* sources of shock; a given source of shock fails a corresponding subset of components. Moreover, the replacement is so chosen as to yield the same joint distribution of failure pattern and system life length as possessed by the original system. Finally, the distribution of the time of each type of shock occurrence is explicitly specified in (2.1).

Suppose now that the system is experiencing reliability growth; specifically, assume that:

(i) $\bar{G}_{u,I}$ is increasing in u, $0 \leq u < u_0$ for each $I \subset \{1, \cdots, n\}$, so that $H_I(u)$ is stochastically increasing in u, $0 \leq u < u_0$ for each $I \subset \{1, \cdots, n\}$. That is, each type of shock is occurring with decreasing frequency as chronological time passes. This, in turn, implies that the system life length $T(u)$ is stochastically increasing in u. Since $\bar{G}_{u,I}(t)$ is monotone increasing in $u < u_0$, it follows immediately that the $\bar{G}_{u,I}(t)$ converge to inf $\sup_{x<t;\, u<u_0}\bar{G}_{u,I}(x)$ for each $t \geq 0$, $I \subset \{1, \cdots, n\}$; call this limit $\bar{G}_I(t)$. $\bar{G}_I(t)$ is a survival function, with mass possibly at infinity.

Three basic questions now arise:

(1) Does the joint distribution $F_{u,I}$ of failure pattern and life length converge to a joint distribution as $u \to u_0$ for each $I \subset \{1, \cdots, n\}$?

(2) Assume such limits (call them F_I) exist. Does it follow that when F_I replaces $F_{u,I}$ and $\Sigma_I F_I$ replaces F_u in (2.1), the resulting survival function is $\bar{G}_I$ (as we would hope)?

(3) Does the system lifelength distribution F_u converge to a lifelength distribution F as $u \to u_0$. If so, does F satisfy (2.2), where the subscript u is omitted?

In the next section we prove that under mild, reasonable conditions, affirmative answers exist for all three questions.

3. Limit results in the reliability growth model. In this section we give convenient sufficient conditions to yield affirmative answers to questions (1), (2) and (3) at the end of Section 2, and present proofs of our results.

We are considering the model of Section 2. We make the following two additional assumptions:

(ii) The sets of discontinuities of the $G_{u,I}$ are pairwise disjoint for each fixed time point u, $0 \leq u < u_0$.

(iii) The sets of discontinuities of the G_I are pairwise disjoint. The main result may now be stated:

3.1. THEOREM. *Assume the reliability growth model specified in Section* 2 *and assume* (i), (ii) *and* (iii). *Then*

(1) $\lim_{u\to u_0}F_{u,I}(t)$ *exists for each* $t \geq 0$ *and each* $I \subset \{1, \cdots, n\}$,

(2) $\lim_{u\to u_0}F_{u,I}(t) = F_I(t)$ *for each* $t \geq 0$ *and each* $I \subset \{1, \cdots, n\}$, *where the* F_I *(replacing* $F_{u,I}$*) satisfy* (2.3) *with subscript u omitted,*

(3) $\lim_{u \to u_0} F_u(t)$ *exists and* $= F(t)$ *for each* $t \geqslant 0$; *moreover,* $\bar{F}_u(t)$ *is increasing upward to* $\bar{F}(t)$ *as u increases upward to* u_0, *where F (replacing* F_u*) satisfies* (2.2) *with subscripts u omitted.*

To prove Theorem 3.1, we will make use of

3.2. LEMMA. *Let* u_k *be an increasing sequence converging to* u_0. *Then for each* $t \geqslant 0$ *and* $I \subset \{1, \cdots, n\}$, $\limsup_{k \to \infty} \bar{F}_{u_k, I}(t) \leqslant \bar{F}_I(t)$.

PROOF OF LEMMA. By (2.3), we may write:

$$\bar{F}_{u_k, I}(t) = \int_t^\infty \Pi_{J \neq I} \bar{G}_{u_k, J}\, dG_{u_k, I} \leqslant \int_t^\infty \Pi_{J \neq I} \bar{G}_J\, dG_{u_k, I}$$

since each $\bar{G}_{u_n, J} \leqslant \bar{G}_J$. Next note that

$$\int_t^\infty \Pi_{J \neq I} \bar{G}_J\, dG_{u_k, I} = \int_t^\infty \left[\bar{G}_{u_k, I}(t) - \bar{G}_{u_k, I}(x^-) \right] d_x \left[1 - \Pi_{J \neq I} \bar{G}_J(x) \right]$$

$$\leqslant \int_t^\infty \left[\bar{G}_{u_k, I}(t) - \bar{G}_{u_k, I}(x) \right] d_x \left[1 - \Pi_{J \neq I} \bar{G}_J(x) \right].$$

But by the Lebesgue dominated convergence theorem,

$$\lim_{k \to \infty} \int_t^\infty \left[\bar{G}_{u_k, I}(t) - \bar{G}_{u_k, I}(x) \right] d_x \left[1 - \Pi_{J \neq I} \bar{G}_J(x) \right]$$

$$= \int_t^\infty \left[\bar{G}_I(t) - \bar{G}_I(x) \right] d_x \left[1 - \Pi_{J \neq I} \bar{G}_J(x) \right]$$

$$= \int_t^\infty \Pi_{J \neq I} \bar{G}_J(x)\, dG_I(x) = \bar{F}_I(t),$$

by assumption (iii). The desired conclusion now follows. ▯

We may now prove Theorem 3.1.

PROOF OF THEOREM 3.1 (1), (2). Let $\{u_k\}$, $0 \leqslant u_k < u_0$, be an arbitrary sequence converging to u_0 and let $t \geqslant 0$. By (2.2),

$$\Sigma_{I \subset \{1, \cdots, n\}} \bar{F}_{u_k, I}(t) = \Pi_{I \subset \{1, \cdots, n\}} \bar{G}_{u_k, I}(t)$$

$$\limsup_{k \to \infty} \Pi_{I \subset \{1, \cdots, n\}} \bar{G}_{u_k, I}(t) = \limsup_{k \to \infty} \Sigma_{I \subset \{1, \cdots, n\}} \bar{F}_{u_k, I}(t).$$

But

$$\limsup_{k \to \infty} \Sigma_{I \subset \{1, \cdots, n\}} \bar{F}_{u_k, I}(t) \leqslant \Sigma_{I \subset \{1, \cdots, n\}} \limsup_{k \to \infty} \bar{F}_{u_k, I}(t)$$

$$\leqslant \Sigma_{I \subset \{1, \cdots, n\}} \bar{F}_I(t),$$

by Lemma 3.2. Now by assumption (iii) and the appropriate version of (2.2) we have:

$$\Sigma_{I \subset \{1, \cdots, n\}} \bar{F}_I(t) = \Pi_{I \subset \{1, \cdots, n\}} \bar{G}_I(t).$$

Recalling that $\bar{G}_I(t) = \lim_{k \to \infty} \bar{G}_{u_k, I}(t)$ from Section 2 and summarizing the inequalities above, we may state:

$$\Pi_{I \subset \{1, \cdots, n\}} \bar{G}_I(t) \leqslant \Sigma_{I \subset \{1, \cdots, n\}} \limsup_{k \to \infty} \bar{F}_{u_k, I}(t) \leqslant \Sigma_{I \subset \{1, \cdots, n\}} \bar{F}_I(t)$$

$$= \Pi_{I \subset \{1, \cdots, n\}} \bar{G}_I(t).$$

The desired conclusions (1) and (2) now follow immediately.

PROOF OF THEOREM 3.1 (3). By (2.2), $\lim_{u\to u_0}\bar{F}_u(t)$ exists since each $\lim_{u\to u_0}\bar{G}_{u,I}$ exists. Next note that

$$\bar{F}(t) = \Sigma_{I\subset\{1,\cdots,n\}}\bar{F}_I(t) = \Pi_{I\subset\{1,\cdots,n\}}\bar{G}_I(t) = \lim_{u\to u_0}\Pi_{I\subset\{1,\cdots,n\}}\bar{G}_{u,I}(t)$$
$$= \lim_{u\to u_0}\Sigma_{I\subset\{1,\cdots,n\}}\bar{F}_{u,I}(t) = \lim_{u\to u_0}\bar{F}_u(t).$$

The monotonicity of $\bar{F}_u(t)$ in u is a consequence of the monotonicity of each $\bar{G}_{u,I}$. □

4. Extensions, modifications and generalizations.

4.1. REMARK. In the reliability growth model discussed above, we assumed that reliability growth was occurring during the interval of chronological time $[0, u_0)$. Obviously, the conclusions of Theorem 3.1 may be obtained by assuming that reliability growth occurs at the time points of any subset U of $[0, \infty)$ (or, for that matter, over the entire half-line). As examples, consider:

(a) $U = \{u, 2u, 3u, \cdots\}$, where $u > 0$,
(b) $U = \{0 \leqslant u_1 < u_2 < u_3 < \cdots\}$,
(c) $U = \{[0, u_0] < u_1 < u_2 < \cdots\}$,

etc. Moreover, the set U need not be deterministic, but may be random.

4.2. REMARK. In the model of Section 2 and the corresponding result, Theorem 3.1, we assumed that shock intervals from each source were stochastically increasing with chronological time. It is clear that by reversing inequalities and the direction of monotonicity, we can obtain a dual to Theorem 3.1 in which the intervals between shocks from each source are stochastically decreasing with chronological time. System reliability (or in the biomedical context, organism survival probability) would decrease with chronological time.

4.3. REMARK. The model of Section 2 and the limit results of Section 3 were formulated in terms of reliability growth. An equally important and useful application exists in the context of competing risks of death of a biological organism from a variety of diseases, accidents or other causes. In this context, Remark 4.2 is especially relevant. We consider a situation in which an organism is subject to death as a result of any of a number of diseases or combinations of diseases. As time passes, the organism becomes more vulnerable to each of the diseases. Note that the model now appropriate differs from the model of Section 2 only in the direction of the monotonicity; i.e., the different types of shocks are becoming stochastically more frequent, rather than, as in the first model, stochastically less frequent.

4.4. REMARK. In the reliability growth model, we are able to deduce monotonicity of $\bar{F}_u$, system reliability as a function of chronological time u. However it is *not* necessarily true that each $\bar{F}_{u,I}$ is monotone in u. This is a consequence of the fact that as some modes of failure become less likely, others may become more likely.

4.5. REMARK. Assumption (iii) of Theorem 3.1 can *not* be obtained as a consequence of assumptions (i) and (ii). A simple counterexample is available to verify this assertion.

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