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RAPIDLY GROWING RANDOM WALKS AND AN ASSOCIATED STOPPING TIME[1]

BY HENRY TEICHER
Rutgers University

An exponential limit distribution is obtained for stopping times associated with partial sums of independent, identically distributed random variables whose distribution function is slowly varying at infinity. It is also demonstrated that a generalized law of the iterated logarithm cannot obtain in such a case.

1. Introduction. Let $\{S_n = \Sigma_{i=1}^n X_i, n \geqslant 1\}$ constitute a random walk whose underlying i.i.d. random variables have distribution function F. Ironically, if $X_n \geqslant 0$ and $1 - F$ is slowly varying, the partial sums S_n are rapidly growing. In such a case, as pointed out by Lévy [4], $a_n S_n + c_n$ cannot have a nondegenerate limit distribution regardless of how the constants a_n, c_n are chosen. Nonetheless, it will be shown that a "pseudo limit distribution" is possible. In other words, there may well exist a nondegenerate distribution $G(x)$ and a function $b(x)$ which is nonlinear in x such that

$$\lim_{n\to\infty} P\{S_n < cb(nx)\} = G(x), \text{ all } c > 0.$$

Concomitantly, a bona fide limiting distribution exists for the collection of stopping times

$$T_x = T_x(c) = \inf\{j \geqslant 1 : S_j > cb(x)\}, \qquad x > 0, c > 0$$

which turns out to be exponential and independent of c. The limit $G(x)$ is one of the extreme value distributions.

It will also be demonstrated that such i.i.d. random variables $\{X_n, n \geqslant 1\}$ do not obey a generalized law of the iterated logarithm. In other words, there do not exist positive constants $b_n \uparrow \infty$ and a finite, nonzero α for which

$$\limsup_{n\to\infty} \frac{1}{b_n} \Sigma_{j=1}^n X_j = \alpha, \text{ a.c.}$$

2. Mainstream. In the context of random walks whose underlying distribution F sustains a (i) barely finite or (ii) barely infinite mean, it was shown in [3] how to choose normalizing constants b_n such that a one-sided iterated logarithm law and weak law of large numbers hold. The prescription for the function b_x in the case of infinite mean is precisely the one needed here to obtain a limit distribution for the normalized stopping time $(1/n)T_n$. A theorem of Darling [1] plays a basic role in the proof.

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THEOREM 1. *Let* $S_n = \Sigma_{i=1}^n X_i$, $n \geqslant 1$ *where* $\{X_n, n \geqslant 1\}$ *are nonnegative* i.i.d. *random variables with distribution function F such that* $1 - F$ *is continuous and slowly varying at* ∞. *If* $\mu(x) = \int_0^x [1 - F(y)]dy$, $x > 0$ *and*

$$b_x = \left(\frac{x}{\mu(x)}\right)^{-1}, T_x = T_x(c) = \inf\{j \geqslant 1 : S_j > cb_x\}; \tag{1}$$

then for $x > 0, y > 0, c > 0$

$$\lim_{n\to\infty} P\{T_{nx} > ny\} = \lim_{n\to\infty} P\{S_{[ny]} \leqslant cb_{nx}\} = e^{-y/x} \tag{2}$$

and, furthermore, for all $x > 0$

$$\liminf_{n\to\infty} \frac{S_n}{b_{nx}} =_{\text{a.c.}} 0, \limsup_{n\to\infty} \frac{S_n}{b_{nx}} =_{\text{a.c.}} \infty. \tag{3}$$

PROOF. Note at the outset that $x/\mu(x)$ and hence b_x is increasing to infinity. Moreover, slow variation ensures

$$\mu(x) \sim x[1 - F(x)] \tag{4}$$

whence $\mu(x)\uparrow\infty$. Via slow variation, (4) and the definition of b_x,

$$\begin{aligned} P\{S_n < cb_{nx}\} &= P\{n[1 - F(S_n)] > n[1 - F(cb_{xn})]\} \\ &= P\{n[1 - F(S_n)] > n[1 - F(b_{xn})](1 + o(1))\} \\ &= P\left\{n[1 - F(S_n)] > \frac{n\mu(b_{xn})}{b_{xn}}(1 + o(1))\right\} \\ &= P\left\{n[1 - F(S_n)] > \frac{1}{x}(1 + o(1))\right\} \\ &\to e^{-1/x} \end{aligned}$$

by a theorem of Darling [1] and so via continuity of the limit function and monotonicity of b_x

$$\lim_{n\to\infty} P\{T_{nx} > ny\} = \lim_{n\to\infty} P\{S_{[ny]} \leqslant cb_{nx}\} = e^{-y/x}.$$

Apropos of (3), for all positive x and c,

$$\begin{aligned} P\left\{\liminf_{n\to\infty} \frac{S_n}{b_{nx}} \geqslant 2c\right\} &\leqslant \lim_{n\to\infty} P\{\cap_{j=n}^{\infty}[S_j > cb_{jx}]\} \\ &\leqslant \lim_{n\to\infty} P\{S_n > cb_{nx}\} < 1 \end{aligned}$$

via (2) whence the initial portion of (3) follows from the zero-one law.

Analogously,

$$P\left\{\limsup_{n\to\infty} \frac{S_n}{b_{nx}} \geqslant c\right\} \geqslant \lim P\{\cup_{j=n}^{\infty}[S_j > cb_{jx}] \geqslant \lim P\{S_n > cb_{nx}\} > 0$$

implying the remaining portion of (3). □

The limit in (2) is a distribution function in x for all $y > 0$ and 1 minus a distribution function in y for all $x > 0$ and there is a certain resemblance between

(2) and Theorem 1 of [6]. In obtaining (2) and (3), the requirement that $X \geqslant 0$ can be weakened to $F(-x) = o(1 - F(x))$ as $x \to \infty$ [1].

Next, it will be demonstrated that random walks of this sort cannot obey a generalized law of the iterated logarithm.

THEOREM 2. *Let* $\{X_n, n \geqslant 1\}$ *be* i.i.d. *random variables with* df F. *If* $1 - F$ *is slowly varying and* $F(0) = 0$, *then for every positive sequence* $c_n \uparrow \infty$, *either* $\lim_{n\to\infty} S_n/c_n =_{\text{a.c.}} 0$ *or* $\limsup_{n\to\infty} S_n/c_n =_{\text{a.c.}} \infty$ *according as* $\Sigma_{n=1}^{\infty}[1 - F(c_n)]$ *converges or diverges.*

PROOF. The hypothesis guarantees that F is not in the domain of partial attraction of the normal distribution. In fact,

$$\frac{x^2 P\{|X| > x\}}{EX^2 I_{[|X| < x]}} \geqslant \frac{x^2[1 - F(x)]}{2\int_0^x y[1 - F(y)]\,dy} \sim 1$$

and $1 - F(c_n) \sim 1 - F(\varepsilon c_n)$, $\varepsilon > 0$. Hence, setting $S_n = \Sigma_1^n X_j$, it follows from a result of Heyde-Rogozin that for every positive sequence $c_n \uparrow \infty$, either $\Sigma_1^\infty[1 - F(c_n)] = \infty$ and

$$\limsup_{n\to\infty} \frac{S_n}{c_n} = \limsup_{n\to\infty} \frac{|S_n|}{c_n} = \infty, \text{ a.c.,} \tag{5}$$

or $\Sigma_1^\infty[1 - F(c_n)] < \infty$ and

$$\frac{1}{c_n}\left(S_n - \Sigma_{j=1}^n EXI_{[|X| < c_j]}\right) \to_{\text{a.c.}} 0. \tag{6}$$

Consequently, if

$$\nu(x) = EXI_{[|X| < x]},$$

integration by parts yields

$$\nu(x) = \int_0^x [1 - F(y)]\,dy - x[1 - F(x)] = o(x[1 - F(x)])$$

as $x \to \infty$. This implies for all sufficiently large m that

$$\Sigma_{n=m}^{\infty} \frac{\nu(c_n)}{c_n} < \Sigma_{n=m}^{\infty}[1 - F(c_n)] < \infty$$

and so Kronecker's lemma ensures

$$\frac{1}{c_n}\Sigma_{j=1}^n \nu(c_j) = o(1)$$

which in conjunction with (6), guarantees

$$\frac{S_n}{c_n} \to_{\text{a.c.}} 0$$

and proves the theorem. □

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DEPARTMENT OF STATISTICS
HILL CENTER FOR THE MATHEMATICAL SCIENCES
BUSCH CAMPUS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903