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ON AN ANALOGUE OF KOMLÓS' THEOREM FOR STRATEGIES

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Using a technique developed by Chen, we obtain an analogue of Komlós' theorem and a simple proof of a strong law of large numbers for strategies.

1. Introduction. Let X be a nonempty set with the discrete topology, $H = X^\infty$ the countable product space equipped with the product topology and X^* the collection of all finite sequences of elements of X including the empty sequence. A strategy σ is a map from X^* into the space of all finitely additive probability measures on the power set $P(X)$ of X. Following Dubins and Savage [5] and Purves and Sudderth [7], there is a field $\mathcal{Q}(\sigma)$ on H containing the Borel σ-field of H and a finitely additive probability measure, again denoted by σ, satisfying some natural conditions.

Let Y_n, $n \geqslant 1$ be a sequence of real valued coordinate maps defined on H, i.e., $Y_n(h)$ depends only on the nth coordinate of $h \in H$. A number of limit theorems were established in the literature for the sequence Y_n, $n \geqslant 1$ under the strategy σ. See [2], [3], [4] and [7]. These results for strategies are generalizations of the conventional limit theorems for sequences of real random variables. It seems that many of the almost sure convergence theorems for countably additive probability measures also hold for finitely additive probability measures determined by strategies. As we shall see later in this paper, there are instances when generalizations fail to hold. The basic problem remaining is to determine precisely the class of results which carry over to strategies.

Komlós [6], Theorem 1, page 218 proved the following result.

KOMLÓS' THEOREM. *Let f_n, $n \geqslant 1$ be a sequence of real random variables defined on a probability space $(\Omega, \mathfrak{A}, P)$ and satisfying*

$$\sup_{n \geqslant 1} E|f_n| < \infty.$$

Then there exists a subsequence f_{n_k}, $k \geqslant 1$ of f_n, $n \geqslant 1$ and an integrable random variable f such that for any subsequence f_n^*, $n \geqslant 1$ of this subsequence, it is true that

$$\frac{1}{n}\Sigma_{k=1}^n f_k^*, \ n \geqslant 1 \textit{ converges to } f \textit{ almost surely.}$$

In this paper we obtain an analogue of this result in the framework of strategies.

Chen [3], Theorem 4.1, page 250, proved the following strong law of large numbers.

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CHEN'S THEOREM. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H, Y_n, $n \geqslant 1$ a sequence of coordinate mappings on H. If $\sigma(Y_n) = 0$ for every $n \geqslant 1$, and, for some constant $r \geqslant 1$, $\Sigma_{n\geqslant 1}\sigma\{|Y_n|^{2r})/n^{1+r}\} < \infty$, then*

$$\sigma\left\{h \in H; \lim_{n\to\infty}\frac{1}{n}\Sigma_{k=1}^{n}Y_k(h) = 0\right\} = 1.$$

In this paper, we give an alternative proof of this result. Some results on extensions of Komlós' theorem based on the work of Chatterji [1] are also given.

The basic tool we use here is measurable strategy introduced by Purves and Sudderth [7], Section 6, page 270. See also Theorem 2.1 of [2], page 211.

The following proposition plays a central role in the results to be proved below.

PROPOSITION 1.1. *Let $\sigma = \nu_1 \times \nu_2 \times \cdots$ be an independent strategy and Y_n, $n \geqslant 1$ a sequence of coordinate mappings from H to $\mathbb{R}$. Assume that for each $n \geqslant 1$, there exists $p_n > 0$ such that $\sigma(|Y_n|^{p_n}) < \infty$. Then there exists a σ-field $\mathfrak{B}^*$ on H, a countably additive probability measure $\hat{\sigma}$ on $\mathfrak{B}^*$, a sequence $m(1) < m(2) < m(3) < \cdots$ of positive integers, and a sequence Z_n, $n \geqslant 1$ of $\mathfrak{B}^*$-measurable real random variables defined on H satisfying the following properties.*

(i) $\mathfrak{B}^* \subset \mathfrak{A}(\sigma)$.

(ii) $\hat{\sigma} = \sigma$ *on* $\mathfrak{B}^*$.

(iii) $\sigma\{\liminf_{n\geqslant 1} A_n^c\} = 1$, *where*

$$A_n = \left\{h \in H; |Y_n(h) - Z_n(h)| > \left(\tfrac{1}{2}\right)^{m(n)}\right\}, \qquad n \geqslant 1$$

(iv) $|Z_n| \leqslant m(n)$ *for every* $n \geqslant 1$.

PROOF. Since $\sigma(|Y_n|^{p_n}) < \infty$, we can find a natural number $m(n)$ satisfying $\sigma\{h \in H; |Y_n(h)| \geqslant m(n)\} \leqslant 1/n^2$ for every $n \geqslant 1$. Assume, without loss of generality, $m(1) < m(2) < \cdots$. Define a simple function Z_n on H by

$$
\begin{aligned}
Z_n(h) &= -m(n), && \text{if } Y_n(h) \leqslant -m(n)\\
&= \frac{-m(n)2^{m(n)} + j}{2^{m(n)}}, && \text{if } \frac{-m(n)2^{m(n)} + j - 1}{2^{m(n)}} < Y_n(h) \leqslant \frac{-m(n)2^{m(n)} + j}{2^{m(n)}},\\
& && j = 1, 2, \cdots, m(n)2^{m(n)}\\
&= \frac{m(n)2^{m(n)} - j}{2^{m(n)}}, && \text{if } \frac{m(n)2^{m(n)} - j}{2^{m(n)}} \leqslant Y_n(h) < \frac{m(n)2^{m(n)} - j + 1}{2^{m(n)}}\\
& && j = 1, \cdots, m(n)2^{m(n)} - 1, m(n)2^{m(n)}\\
&= m(n), && \text{if } Y_n(h) \geqslant m(n).
\end{aligned}
$$

Since Y_n depends only on the nth coordinate, Z_n depends only on the nth coordinate. Since Z_n is simple, it induces a finite partition of X in a natural way. Let $\mathfrak{B}_n$ be the smallest σ-field on X generated by this partition. $\mathfrak{B}_n$ is obviously a finite σ-field on X. The given strategy σ (denoted by $\hat{\sigma}$) then becomes a transition probability with respect to the sequence $\mathfrak{B}_n$, $n \geqslant 1$. Let $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2 \times \cdots$ be the product σ-field on H and $\mathfrak{B}^*$ the completion of $\mathfrak{B}$ with respect to $\hat{\sigma}$. We have now a probability space $(H, \mathfrak{B}^*, \hat{\sigma})$ and a sequence Z_n, $n \geqslant 1$ of $\mathfrak{B}^*$-measurable real random variables satisfying (i) and (ii) (see Theorem 2.1 of Chen [2], page 211). Note that

$$\sigma\left\{h \in H;\, |Y_n(h) - Z_n(h)| > \frac{1}{2^{m(n)}}\right\} = \sigma(A_n)$$
$$\leqslant \sigma\{h \in H;\, |Y_n(h)| \geqslant m(n)\} \leqslant \frac{1}{n^2}.$$

From this, we conclude that $\Sigma_{n\geqslant 1}\sigma(A_n) < \infty$. Since A_n depends only on the nth coordinate, we can apply Borel-Cantelli lemma to the sequence A_n, $n \geqslant 1$ to conclude that $\sigma(\limsup_{n\to\infty} A_n) = 0$ (see Purves and Sudderth [7], Theorem 1, page 274).

2. An analogue of Komlós' theorem and its extensions.

THEOREM 2.1. *Let $\sigma = \nu_1 \times \nu_2 \times \cdots$ be an independent strategy, and Y_n, $n \geqslant 1$ a sequence of coordinate mappings from H to $\mathbb{R}$ satisfying*

$$\sup_{n\geqslant 1}\sigma(|Y_n|^p) < \infty \text{ for some } 0 < p < 2.$$

Then there exists a subsequence Y_{n_k}, $k \geqslant 1$ of Y_n, $n \geqslant 1$ and a map $Y : H \to \mathbb{R}$ satisfying the following properties:

(i) $\sigma(|Y|^p) < \infty$

(ii) $\frac{1}{n^{1/p}}\Sigma_{i=1}^n(Y_i^* - Y)$, $n \geqslant 1$ *converges to* 0 *almost surely* $[\sigma]$ *for any subsequence* Y_n^*, $n \geqslant 1$ *of* Y_{n_k}, $k \geqslant 1$.

(iii) *If* $0 < p < 1$, *one can choose* $Y \equiv 0$.

PROOF. Let Z_n, $n \geqslant 1$ be the sequence given by Proposition 1.1. We will show that $\sup_{n\geqslant 1}\sigma(|Z_n|^p) < \infty$. Note that for any two real numbers a, b,

$$|a + b|^p \leqslant |a|^p + |b|^p \qquad \text{if } 0 < p < 1$$
$$\leqslant 2^{p-1}(|a|^p + |b|^p) \qquad \text{if } p \geqslant 1.$$

$$\begin{aligned}\sigma(|Z_n|^p) &= \int |Z_n|^p\, d\sigma \\ &= \int_{\{|Y_n| < m(n)\}} |Z_n - Y_n + Y_n|^p\, d\sigma + \int_{\{|Y_n| \geqslant m(n)\}} |Z_n|^p\, d\sigma \\ &\leqslant c_p \int_{\{|Y_n| < m(n)\}} |Z_n - Y_n|^p\, d\sigma + c_p \int_{\{|Y_n| < m(n)\}} |Y_n|^p\, d\sigma \\ &\quad + [m(n)]^p \sigma\{|Y_n| \geqslant m(n)\},\end{aligned}$$

where $c_p = 1$ if $0 < p \leqslant 1$; and $c_p = 2^{p-1}$ if $p \geqslant 1$. Thus

$$\sigma(|Z_n|^p) \leqslant c_p[1/2^{m(n)}]^p + c_p \sup_{n\geqslant 1}\sigma(|Y_n|^p)$$
$$+ [m(n)]^p \sigma\{|Y_n| \geqslant m(n)\}.$$

It suffices to show that $\sup_{n \geqslant 1}[m(n)]^p \sigma\{|Y_n| \geqslant m(n)\} < \infty$. This follows from the inequality

$$\int |Y_n|^p \, d\sigma \geqslant \int_{\{|Y_n| \geqslant m(n)\}} |Y_n|^p \, d\sigma \geqslant [m(n)]^p \sigma\{|Y_n| \geqslant m(n)\}.$$

Now, Komlós' theorem applies when $p = 1$ and Chatterji's theorem applies when $p \neq 1$ but $0 < p < 2$ for the sequence Z_n, $n \geqslant 1$ (see Chatterji [1], Theorem 1, page 235). There exists a $\mathfrak{B}^*$-measurable random variable Y and a subsequence Z_{n_k}, $k \geqslant 1$ of Z_n, $n \geqslant 1$ satisfying the following properties.

(i) $\sigma(|Y|^p) < \infty$.

(ii) If Z_n^*, $n \geqslant 1$ is any subsequence of Z_{n_k}, $k \geqslant 1$, then

$$\lim_{n \to \infty}[1/n^{1/p}]\Sigma_{i=1}^n (Z_i^* - Y) = 0 \text{ a.e. } [\sigma].$$

(iii) If $0 < p < 1$, one can choose $Y \equiv 0$.

Now, we claim that Y_{n_k}, $k \geqslant 1$ is the desired subsequence. Let Y_n^*, $n \geqslant 1$ be any subsequence of Y_{n_k}, $k \geqslant 1$. Let $A = \{h \in H;\ \lim_{n \to \infty}[1/n^{1/p}]\Sigma_{i=1}^n (Z_i^*(h) - Y(h)) = 0\}$. Then $\sigma(A) = 1$. Let $B = \liminf A_n^c$. By Proposition 1.1, $\sigma(B) = 1$. Hence $\sigma(A \cap B) = 1$. We will show that, if $h \in A \cap B$,

$$\lim_{n \to \infty}[1/n^{1/p}]\Sigma_{i=1}^n (Y_i^*(h) - Y(h)) = 0.$$

For,

$$\left| \frac{[Y_1^*(h) - Y(h)] + [Y_2^*(h) - Y(h)] + \cdots [Y_n^*(h) - Y(h)]}{n^{1/p}} \right|$$
$$\leqslant \left| \frac{[Y_1^*(h) - Z_1^*(h)] + [Y_2^*(h) - Z_2^*(h)] + \cdots + [Y_n^*(h) - Z_n^*(h)]}{n^{1/p}} \right|$$
$$+ \left| \frac{[Z_1^*(h) - Y(h)] + [Z_2^*(h) - Y(h)] + \cdots + [Z_n^*(h) - Y(h)]}{n^{1/p}} \right|$$
$$\leqslant 1/n^{1/p} b(h) + |[1/n^{1/p}]\Sigma_{i=1}^n (Z_i^*(h) - Y(h)| \to 0 \quad \text{as} \quad n \to \infty,$$

where $b(h)$ is a constant depending only on h. The first part of the above inequality follows from Proposition 1.1 (iii).

This completes the proof.

REMARK. (i) The following example shows that the Borel-Cantelli lemma is not valid for strategies (see Purves and Sudderth [7], last complete paragraph on page 274; see also Chen [4], Example 1, page 344). Let $X = \{1, 2, 3, \cdots\}$, and the strategy σ be defined as follows. $\sigma_0(A) = 0$ if A is a finite subset of X and the conditional strategy $\sigma[n]$ assigns mass 1 to the history $(n, n, n, \cdots)$ in H for all $n \geqslant 1$. Let $K_n = \{h = (h_1, h_2, \cdots) \in H;\ h_n \leqslant n\}$, $n \geqslant 1$. It is easy to check that $\sigma(K_n) = 0$ for every $n \geqslant 1$ and $\sigma(\limsup_{n \to \infty} K_n) = 1$. Thus $\Sigma_{n \geqslant 1}\sigma(K_n) < \infty$ and yet $\sigma(\limsup_{n \to \infty} K_n) = 1$.

(ii) Theorem 2.1, as it stands, when $p = 1$, is not a generalization of Komlós' theorem. In Komlós' theorem, there is no assumption of independence imposed on the sequence f_n, $n \geqslant 1$ of random variables. A generalization of Komlós' theorem would be the validity of Theorem 2.1 for any strategy σ not necessarily independent. However, the following example based on Remark (i) shows that Theorem 2.1 is not valid for any strategy σ of (i).

Let $Y_n = n^2 I_{K_n}$, $n \geqslant 1$, where I stands for indicator function. Note that $\sigma(|Y_n|) = \sigma(Y_n) = 0$ for every $n \geqslant 1$. For every $h \in \limsup_{n\to\infty} K_n$ and for any subsequence Y_n^*, $n \geqslant 1$ of Y_n, $n \geqslant 1$.

$$\frac{Y_1^*(h) + Y_2^*(h) + \cdots + Y_n^*(h)}{n}, \quad n \geqslant 1 \text{ is not convergent.}$$

(iii) In view of the above example, it remains open whether Komlós' theorem is true without the assumption of independence if the Y_n's are uniformly bounded.

3. A simple proof of a strong law of large numbers due to Chen. We will just cover the case $r = 1$. The case when $r > 1$ can be disposed of in a similar way.

Let Z_n, $n \geqslant 1$ be the sequence of simple functions given by Proposition 1.1. It suffices to show that

$$\Sigma_{n\geqslant 1}\frac{\sigma[Z_n - \sigma(Z_n)]^2}{n^2} < \infty.$$

For then we can reduce the convergence problem to the sequence Z_n, $n \geqslant 1$ in the realm of countably additive set-up and then successfully argue for the sequence Y_n, $n \geqslant 1$. The idea is essentially contained in the proof of Theorem 2.1. As $\sigma(Z_n^2) \geqslant \sigma[Z_n - \sigma(Z_n)]^2$, it is enough to show that

$$\Sigma_{n\geqslant 1}\frac{\sigma(Z_n^2)}{n^2} < \infty.$$

But this can be checked quite easily.

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