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ON THE INTEGRABILITY OF $\sup|S_n/n|$ FOR SUBSEQUENCES

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Let $\{S_n;\ n \geq 1\}$ denote the partial sums of i.i.d. random variables and let $\{n_k;\ k \geq 1\}$ be a (strictly) increasing subsequence of the positive integers. We determine necessary and sufficient conditions for $E\sup_k|S_{n_k}/n_k| < \infty$.

1. Introduction. Let $\{X_n;\ n \geq 1\}$ be a sequence of i.i.d. random variables and let $\{S_n;\ n \geq 1\}$ denote their partial sums. In 1937 Marcinkiewicz and Zygmund [8] proved that

$$E\sup_n|S_n/n|^p < \infty, \qquad p \geq 1, \tag{1.1}$$

provided

$$E|X_1|\lg|X_1| < \infty, \quad p = 1 \quad \text{and} E|X_1|^p < \infty, \quad p > 1. \tag{1.2}$$

(Here and throughout $\lg x = \max\{1, \log x\}$, $\lg_2 x = \lg\lg x$, etc.)

In 1962, Burkholder [1] proved the necessity for $p = 1$. For $p > 1$ this is obvious.

Several other proofs of this result have been given and the problem has also been generalized to other normalizing sequences and to random variables with multidimensional indices; see [5] and further references given there.

In this paper we are considering the generalization which consists of taking the supremum over subsequences of the natural numbers. In this direction the following result was established by Gabriel [4], Lemma 3.4.

(1.3) Let $\{X_n;\ n \geq 1\}$ be i.i.d. nonnegative random variables. For every (strictly) increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers such that

$$\inf_k \frac{n_k}{n_{k+1}} = c > 0,$$

the following are equivalent:

(i) $EX_1 \lg X_1 < \infty$,

(ii) $E\sup_k S_{n_k}/n_k < \infty$.

REMARK. The nonnegativity is only used in the proof of (ii) $\Rightarrow$ (i).

Jean-Pierre Gabriel has encouraged me to investigate this problem for the case $c = 0$. The resulting theorem is presented in Section 2 together with some examples and remarks. Proofs are given in Sections 3 and 4. Section 5, finally, contains an additional result.

2. Results. In this section we present the theorem and some examples and remarks.

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(2.1). THEOREM. *Let* X *and* $\{X_n;\ n \geqslant 1\}$ *be* i.i.d. *random variables. Let* $\{n_k\}_{k=1}^{\infty}$ *be an increasing subsequence of the positive integers; let* ψ *be its inverse*, i.e.,

$$\psi(x) = \min\{k;\ n_k \geqslant [x]\} = \operatorname{card}\{k;\ n_k \leqslant [x]\}; \tag{2.2}$$

and set $\gamma_k = n_{k-1}/n_k$, $k = 2, 3, \cdots$. *Then*

$$E \sup_k |S_{n_k}/n_k| < \infty \tag{2.3}$$

if and only if

$$E|X|\lg|X| < \infty, \quad \textit{when } \liminf_{k\to\infty}\gamma_k > 0 \tag{2.4.a}$$

$$E|X|\psi(|X|) < \infty, \quad \textit{when } \limsup_{k\to\infty}\gamma_k < 1. \tag{2.4.b}$$

Examples and remarks.

(2.5) It is clear that the two cases overlap. In fact, if $0 < \liminf_{k\to\infty} \gamma_k \leqslant \limsup_{k\to\infty} \gamma_k < 1$, it is easy to check that $\psi(x) \sim \log x$ (i.e., both cases yield the same conclusion).

(2.6) If $P(X \geqslant 0) = 1$ and $\liminf_{k\to\infty} \gamma_k > 0$ one rediscovers (1.3).

(2.7) The cases $n_k = 2^k$ and $n_k = k^d$, $d = 2, 3, \cdots$ yield $\gamma_k = \frac{1}{2}$ and $2^{-d} \leqslant \gamma_k \to 1$ respectively, i.e., (2.3) holds if and only if $E|X|\lg|X| < \infty$.

(2.8) For $n_k = k!$ one has $\gamma_k = k^{-1} \to 0$ and $\psi(x) \sim \lg x/\lg_2 x$ and so (2.3) holds if and only if $E|X|\lg|X|/\lg_2|X| < \infty$.

(2.9) For $n_k = 2^{2^{\cdot^{\cdot^{2^k}}}}$ with m 2's, $m = 1, 2, \cdots$ we obtain $\gamma_k \to 0$ and $\psi(x) \sim \lg_m x$, i.e., (2.3) holds if and only if $E|X|\lg_m|X| < \infty$.

(2.10) A device, useful in many proofs, is to show that a desired conclusion holds for a given fixed subsequence of the positive integers and that the object under investigation behaves "nicely" between the points of subdivision. For a classical example, see Chung [2], Theorem 5.1.2, page 103, where the strong law of large numbers is proved via the subsequence $n_k = k^2$. In fact, the most common subsequences are those of (2.7) above and it is therefore no surprise that we obtain the same result for subsequences which do not grow too rapidly as for the original case, i.e., when the supremum is taken over all n. It may be noted that the essence of the proof of (2.3) $\Rightarrow$ (2.4.a) is to show that

$$E \sup_k |S_{n_k}/n_k| < \infty \Rightarrow E \sup_n |S_n/n| < \infty.$$

(2.11) The case $\liminf_{k\to\infty} \gamma_k = 0$ and $\limsup_{k\to\infty} \gamma_k = 1$ is not covered by the theorem. The following two examples show that there is no general solution for that case.

For the first example, let $n_{2k} = k!$ and $n_{2k+1} = k!+1$, $k = 3, 4, 5, \cdots$ (with $n_k = k$, $k = 1, 2, \cdots, 5$). Since

$$E \sup_k |S_{n_k}/n_k| \leqslant E \sup_k |S_{n_{2k}}/n_{2k}| + E \sup_k |S_{n_{2k+1}}/n_{2k+1}|$$

it follows from (2.8) that $E|X|\lg|X|/\lg_2|X| < \infty$ is sufficient for (2.3) to hold. (Since $\sup_k |S_{n_k}/n_k| \geqslant \sup_k |S_{n_{2k}}/n_{2k}|$ it is necessary too.)

For the other example, let $I_k = \{i;\ k! < i \leqslant (k+1)!\}$, $k = 1, 2, 3, \cdots$, and set $B_1 = \bigcup_{k=0}^{\infty} I_{2k+1}$ and $B_2 = \bigcup_{k=1}^{\infty} I_{2k}$. Let $\{n_k'\}_{k=1}^{\infty}$ and $\{n_k''\}_{k=1}^{\infty}$ be the elements of B_1 and B_2 respectively (in increasing order). Both subsequences satisfy the conditions $\limsup_{k\to\infty} \gamma_k = 1$ and $\liminf_{k\to\infty} \gamma_k = 0$. Furthermore,

$$EV_T \leqslant EV' + EV'' \leqslant 2\max\{EV', EV''\}, \tag{2.12}$$

where

$$V_T = \sup_n|S_n/n|, \qquad V' = \sup_{n\in B_1}|S_n/n| \quad \text{and} \quad V'' = \sup_{n\in B_2}|S_n/n|.$$

Now, choose $\{n_k\}_{k=1}^{\infty}$ to be the subsequence corresponding to the larger of EV' and EV''. It follows from (2.12) that

$$E\ \sup_k|S_{n_k}/n_k| < \infty \Rightarrow EV_T < \infty,$$

which, in view of [1], implies that $E|X|\lg|X| < \infty$ is a necessary condition. (Since $\sup_k|S_{n_k}/n_k| \leqslant \sup_n|S_n/n|$ it is also sufficient.)

3. Proof of (2.4) ⇒ (2.3). Throughout c and C denote arbitrary constants. Set $n_0 = 0$ and define

$$X_n' = X_n I\{|X_n| \leqslant n\}, \qquad X_n'' = X_n - X_n';$$

$$Y_k = \Sigma_{i=n_{k-1}+1}^{n_k} X_i = S_{n_k} - S_{n_{k-1}}, \qquad Y_k' = \Sigma_{i=n_{k-1}+1}^{n_k} X_i', \qquad Y_k'' = Y_k - Y_k';$$

$$W = \sup_k|n_k^{-1}\cdot Y_k|, \qquad W' = \sup_k|n_k^{-1}\cdot Y_k'|, \qquad W'' = \sup_k|n_k^{-1}\cdot Y_k''|;$$

$$V = \sup_k|n_k^{-1}\cdot S_{n_k}| = \sup_k|n_k^{-1}\cdot \Sigma_{i=1}^{k} Y_i|.$$

It is clearly no restriction to assume $EX = 0$.

(2.4.a) ⇒ (2.3). Since $V \leqslant \sup_n|S_n/n|$ the conclusion follows immediately from the result of Marcinkiewicz and Zygmund [8], mentioned in the introduction.

(2.4.b) ⇒ (2.3).

$$E|X| < \infty \Rightarrow \Sigma_{k=1}^{\infty} n_k^{-2}\cdot \mathrm{Var}(Y_k') < \infty. \tag{3.1}$$

PROOF.

$$\Sigma_{k=1}^{\infty} n_k^{-2}\cdot \mathrm{Var}(Y_k') = \Sigma_{k=1}^{\infty} n_k^{-2}\cdot \Sigma_{i=n_{k-1}+1}^{n_k} \mathrm{Var}(X_i')$$
$$\leqslant \Sigma_{k=1}^{\infty}\Sigma_{i=n_{k-1}+1}^{n_k} i^{-2}\cdot \mathrm{Var}(X_i') = \Sigma_{n=1}^{\infty} n^{-2}\cdot \mathrm{Var}(X_n') \leqslant C\cdot E|X|.$$

For the last inequality see, e.g., [2], 126–127, the proof of the strong law of large numbers.

$$E|X| < \infty \Rightarrow EW' < \infty. \tag{3.2}$$

PROOF. We wish to show that $\int_{x_0}^{\infty} P(W' > x)dx < \infty$ for some $x_0 > 0$. Set $F(x) = P(X \leqslant x)$. By using the fact that $EY_k = 0$ we obtain

$$|EY_k'| = |\Sigma_{i=n_{k-1}+1}^{n_k} EX_i'| = |\Sigma_{i=n_{k-1}+1}^{n_k} \int_{|x|>i} x\ dF| \leqslant (n_k - n_{k-1})\int_{|x|>n_{k-1}} |x|dF$$
$$\leqslant n_k \int_{|x|>n_{k-1}} |x|dF = o(n_k) \text{ as } k \to \infty.$$

Thus, for $x \geqslant 1$, $0 < \varepsilon < 1$, k large,

$$P(|Y_k'| > n_k x) \leqslant P(|Y_k' - EY_k'| > n_k(x - \varepsilon)) \leqslant (n_k(x - \varepsilon))^{-2} \cdot \operatorname{Var}(Y_k').$$

Consequently

$$P(W' > x) \leqslant \Sigma_{k=1}^{\infty} P(|Y_k'| > n_k x) \leqslant C \cdot (x - \varepsilon)^{-2} \cdot \Sigma_{k=k_0}^{\infty} n_k^{-2} \operatorname{Var}(Y_k')$$
$$\leqslant C \cdot (x - \varepsilon)^{-2} \cdot E|X|$$

by (3.1) and so the integral above is finite.

(3.3) $$E|X|\psi(|X|) < \infty \Rightarrow EW'' < \infty.$$

Proof.

$$EW'' = E \sup_k |n_k^{-1} \cdot Y_k''| \leqslant \Sigma_{k=1}^{\infty} n_k^{-1} \cdot E|Y_k''| \leqslant \Sigma_{k=1}^{\infty} n_k^{-1} \cdot \Sigma_{i=n_{k-1}+1}^{n_k} E|X_i''|$$
$$\leqslant \Sigma_{k=1}^{\infty} n_k^{-1} \cdot (n_k - n_{k-1}) \cdot E|X| I\{|X| > n_{k-1}\} \leqslant \Sigma_{k=1}^{\infty} E|X| I\{|X| > n_{k-1}\}$$
$$\leqslant E|X| + \Sigma_{k=1}^{\infty} E|X| I\{|X| > n_k\} = E|X| + \Sigma_{k=1}^{\infty} \Sigma_{j=k}^{\infty} \int_{n_j < |x| \leqslant n_{j+1}} |x| \, dF$$
$$= E|X| + \Sigma_{j=1}^{\infty} (\Sigma_{k=1}^{j} 1) \int_{n_j < |x| \leqslant n_{j+1}} |x| \, dF$$
$$= E|X| + \Sigma_{j=1}^{\infty} j \int_{n_j < |x| \leqslant n_{j+1}} |x| \, dF = E|X| + \Sigma_{j=1}^{\infty} \psi(n_j) \int_{n_j < |x| \leqslant n_{j+1}} |x| \, dF$$
$$\leqslant E|X| + \Sigma_{j=1}^{\infty} \int_{n_j < |x| \leqslant n_{j+1}} |x| \psi(|x|) \, dF$$
$$\leqslant E|X| + E|X|\psi(|X|).$$

(3.4) $$E|X|\psi(|X|) < \infty \Rightarrow EW < \infty.$$

This is immediate from (3.2) and (3.3) since $W \leqslant W' + W''$.

(3.5) $$E|X|\psi(|X|) < \infty \Rightarrow EV < \infty, \quad \text{i.e., (2.4.b)} \Rightarrow \text{(2.3)}.$$

Proof. From the strong law of large numbers we have $P(V < \infty) = 1$. Since, by (3.4), $EW < \infty$, the conclusion follows from Hoffmann-Jørgensen [6], Corollary 3.4, page 167.

The proof of this part is complete.

4. Proof of (2.3) $\Rightarrow$ (2.4). We first note that $E|X| < \infty$ is an obvious necessary condition. It is therefore no restriction to assume that $EX = 0$. Further, since uniformly bounded random variables have moments of all orders it is no restriction to assume that X is unbounded and also that $0 < P(|X| < 1) < 1$ (cf. [4], [5]).

(4.1) $$E|X| < \infty \Rightarrow A = \Pi_{k=1}^{\infty} P(|Y_k| \leqslant n_k) > 0.$$

Proof. We wish to show that $\Sigma_{k=1}^{\infty} P(|Y_k| > n_k) < \infty$, which is well known to be equivalent to the assertion that $A > 0$.

From (3.2) above we know that, for large k,

$$P(|Y_k'| > n_k) \leqslant (n_k(1 - \varepsilon))^{-2} \cdot \operatorname{Var}(Y_k').$$

Furthermore,

$$P(|Y_k|>n_k)\leqslant P(|Y_k'|>n_k)+\Sigma_{i=n_{k-1}+1}^{n_k}P(|X_i|>i).$$

Thus

$$\Sigma_{k=k_0}^{\infty}P(|Y_k|>n_k)\leqslant\Sigma_{k=k_0}^{\infty}(n_k(1-\varepsilon))^{-2}\cdot\operatorname{Var}(Y_k')+\Sigma_{k=k_0}^{\infty}\Sigma_{i=n_{k-1}+1}^{n_k}P(|X_i|>i)$$
$$\leqslant c\Sigma_{k=1}^{\infty}n_k^{-2}\cdot\operatorname{Var}(Y_k')+\Sigma_{n=1}^{\infty}P(|X|>n)\leqslant c\cdot E|X|.$$

The last inequality follows from (3.1) and from the fact that $E|X|<\infty\Leftrightarrow\Sigma_{n=1}^{\infty}P(|X|>n)<\infty$.

(2.3) $\Rightarrow$ (2.4.a). We first show that (2.3) implies that $E\sup_n|S_n/n|<\infty$ for symmetric distributions, from which (2.4.a) follows according to Burkholder's result [1], mentioned in the introduction. After this the symmetry assumption is removed.

Thus, suppose that the distribution is symmetric. Since $W\leqslant 2V$ we know that $EW<\infty$.

Let $n_{k-1}<n\leqslant n_k$. Then

$$|S_n/n|\leqslant|(S_n-S_{n_{k-1}})/n_{k-1}|+|S_{n_{k-1}}/n_{k-1}|,$$

i.e.,

$$\sup_{n_{k-1}<n\leqslant n_k}|S_n/n|\leqslant\sup_{n_{k-1}<n\leqslant n_k}|(S_n-S_{n_{k-1}})/n_{k-1}|+|S_{n_{k-1}}/n_{k-1}|.$$

Further, since $\sup_n|S_n/n|=\sup_k\sup_{n_{k-1}<n\leqslant n_k}|S_n/n|$ we obtain

$$E\sup_n|S_n/n|\leqslant E\sup_k\sup_{n_{k-1}<n\leqslant n_k}|(S_n-S_{n_{k-1}})/n_{k-1}|+E\sup_k|S_{n_{k-1}}/n_{k-1}|$$
$$\leqslant\Sigma_{k=1}^{\infty}E\sup_{n_{k-1}<n\leqslant n_k}|(S_n-S_{n_{k-1}})/n_{k-1}|+EV.$$

Since $EV<\infty$ by assumption we must show that the last sum is finite or, equivalently, that

$$\Sigma_{k=1}^{\infty}\Sigma_{m=1}^{\infty}P\big(\sup_{n_{k-1}<n\leqslant n_k}|S_n-S_{n_{k-1}}|>m\cdot n_{k-1}\big)<\infty. \tag{4.2}$$

Since $\liminf_{k\to\infty}\gamma_k>0$ we have $\gamma_k>\varepsilon$ for k large, which together with Lévy's inequality, yields

$$P\big(\sup_{n_{k-1}<n\leqslant n_k}|S_n-S_{n_{k-1}}|>m\cdot n_{k-1}\big)\leqslant P\big(\sup_{n_{k-1}<n\leqslant n_k}|S_n-S_{n_{k-1}}|>m\varepsilon n_k\big)$$
$$\leqslant 2P\big(|S_{n_k}-S_{n_{k-1}}|>m\varepsilon n_k\big)=2P(|Y_k|>m\varepsilon n_k).$$

Thus, (4.2) holds true if

$$\Sigma_{k=1}^{\infty}\Sigma_{m=1}^{\infty}P(|Y_k|>m\cdot n_k)<\infty. \tag{4.3}$$

However, (4.3) follows easily from the facts that $A>0$ (see (4.1)) and $EW<\infty$. This is seen as follows:

$$P(W>m)=\Sigma_{k=1}^{\infty}\big(\Pi_{i=1}^{k-1}P(|Y_i|\leqslant n_i\cdot m)\big)\cdot P(|Y_k|>n_km)$$
$$\geqslant A\cdot\Sigma_{k=1}^{\infty}P(|Y_k|>n_km).$$

Finally,

$$\infty>EW\geqslant\Sigma_{m=1}^{\infty}P(W>m)\geqslant A\Sigma_{m=1}^{\infty}\Sigma_{k=1}^{\infty}P(|Y_k|>n_km)$$

and (4.3) is proved.

We now know that (2.3) $\Rightarrow$ (2.4.a) for symmetric distributions. To desymmetrize we use standard arguments. If X^s, V^s, etc. denote the symmetrized random variables we have $EV < \infty \Rightarrow EV^s < \infty$ (since $EV^s \leqslant 2EV$) $\Rightarrow E|X^s|\lg|X^s| < \infty$. One way to conclude that $E|X|\lg|X| < \infty$ is to note that $\{X, X^s\}$ constitutes a martingale and therefore, by convexity, $\{|X|\lg|X|, |X^s|\lg|X^s|\}$ constitutes a submartingale. The conclusion now follows from the fact that a submartingale has increasing expectations.

(2.3) $\Rightarrow$ (2.4.b). As before we assume that $0 < P(|X| < 1) < 1$, $EX = 0$ and unboundedness, but no symmetry. Recall that $A = \Pi_{k=1}^{\infty} P(|Y_k| \leqslant n_k) > 0$ by (4.1).

Since $\limsup_{k\to\infty} \gamma_k < 1$ we have $\gamma_k < 1 - \varepsilon$ for large k and so

$$n_k \varepsilon \leqslant n_k - n_{k-1} (\leqslant n_k), \qquad k \geqslant k_0. \tag{4.4}$$

By using the previous methods, together with an argument by Erdös and Katz, see [3] and [7], page 317, we obtain

$$\begin{aligned} P(W > m) &\geqslant A \cdot \Sigma_{k=1}^{\infty} P(|Y_k| > n_k \cdot m) \geqslant A \Sigma_{k=k_0}^{\infty} P(|Y_k| > (n_k - n_{k-1})m/\varepsilon) \\ &= A \Sigma_{k=k_0}^{\infty} P(|S_{n_k - n_{k-1}}| > (n_k - n_{k-1})m/\varepsilon) \\ &\geqslant c \Sigma_{k=k_0}^{\infty} (n_k - n_{k-1}) P(|X| > c(n_k - n_{k-1})m) \\ &\geqslant c\varepsilon \Sigma_{k=k_0}^{\infty} n_k P(|X| > c \cdot n_k m) \geqslant c \cdot \Sigma_{k=1}^{\infty} n_k P(|X| > c \cdot n_k m), \end{aligned}$$

whence, using partial summation,

$$\begin{aligned} \infty > EW &\geqslant c \Sigma_{m=1}^{\infty} \Sigma_{k=1}^{\infty} n_k P(|X| > c \cdot n_k m) = c \Sigma_{j=1}^{\infty} (\Sigma_{n_k m = j} n_k) P(|X| > cj) \\ &\geqslant c \Sigma_{j=1}^{\infty} (\Sigma_{n_k m = j} n_k) \Sigma_{i=j}^{\infty} P(ci < |X| \leqslant c(i+1)) \\ &= c \Sigma_{i=1}^{\infty} \Sigma_{j=1}^{i} (\Sigma_{n_k m = j} n_k) P(ci < |X| \leqslant c(i+1)) \\ &\geqslant c \Sigma_{i=1}^{\infty} (\Sigma_{n_k m \leqslant i} n_k) P(ci < |X| \leqslant c(i+1)) \geqslant cE|X|\psi(|X|/c) \sim E|X|\psi(|X|). \end{aligned}$$

The last inequality follows because

$$\begin{aligned} \Sigma_{n_k m \leqslant i} n_k &= \Sigma_{n_k \leqslant i} (\Sigma_{m \leqslant i/n_k} 1) n_k = \Sigma_{n_k \leqslant i} [i/n_k] \cdot n_k \sim \Sigma_{n_k \leqslant i} i = i \cdot \text{card}\{k; n_k \leqslant i\} \\ &= i \cdot \psi(i), \end{aligned}$$

and the final relation is simply a matter of scaling.

This completes the proof.

5. Complements. In his paper [1] Burkholder also shows that (1.1) and (1.2) are equivalent to

$$E \sup_n |X_n/n| < \infty. \tag{5.1}$$

This means that the largest summand is of essentially the same magnitude as the largest partial sum. In the case of subsequences we have two quantities which could be of interest, namely $W = \sup_k |Y_k/n_k| = \sup_k |(\Sigma_{i=n_{k-1}+1}^{n_k} X_i)/n_k|$ and $\tilde{W} = \sup_k |X_{n_k}/n_k|$.

(5.2) The proofs in Sections 3 and 4 also reveal that (2.3) and (2.4) are equivalent to $EW < \infty$.

(5.3) For $E\tilde{W}$ this is not the case. Suppose that $\sum_{k=1}^{\infty} n_k^{-1} < \infty$. Then $E\tilde{W} \leqslant \sum_{k=1}^{\infty} n_k^{-1} \cdot E|X| < \infty$ if $E|X| < \infty$. Conversely, $E\tilde{W} < \infty$ trivially implies that $E|X| < \infty$ and so

$$E \sup_k |X_{n_k}/n_k| < \infty \Leftrightarrow E|X| < \infty, \tag{5.4}$$

provided $\sum_{k=1}^{\infty} n_k^{-1} < \infty$.

In particular, this is already the case if, e.g., $n_k = k \cdot (\lg k)^{\alpha}$, $\alpha > 1$.

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