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AN ALMOST SURE INVARIANCE PRINCIPLE FOR THE PARTIAL SUMS OF INFIMA OF INDEPENDENT RANDOM VARIABLES

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Let $\{X_n, n > 1\}$ be a sequence of independent random variables uniformly distributed on the unit interval. Put $X_n^* = \inf(X_1, X_2, \cdots, X_n)$ and $S_n = X_1^* + X_2^* + \cdots + X_n^*$, $n > 2$, $S_1 = 0$. The aim of this note is to give an almost sure invariance principle for S_n. Next we extend the given results to the case when X_n, $n > 1$, are not uniformly distributed but bounded, and moreover, to sums $\hat{S}_n = X_m^{(m)} + X_{m+1}^{(m)} + \cdots + X_n^{(m)}$, where $X_j^{(m)}$ is the mth order statistic of $(X_1, X_2, \cdots, X_j)$.

1. Introduction. The limiting behaviour of the sequence $X_1, \inf(X_1, X_2), \cdots, \inf(X_1, X_2, \cdots, X_n), \cdots$ has been investigated, for example, in [4], [5] and [6]. The convergence in probability, almost sure and in law for sums $S_n = X_1^* + X_2^* + \cdots + X_n^*$, $n \geqslant 2$, $S_1 = 0$, where $X_n^* = \inf(X_1, X_2, \cdots, X_n)$ has been established in [1] and [3]. We here investigate the rate of almost sure convergence. Our goal is to give an almost sure invariance principle for the sums S_n which implies, among other things, the functional law of the iterated logarithm [11].

Namely, we are going to prove

THEOREM 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables uniformly distributed on* $[0, 1]$. *Define the process $\{S(t), t \geqslant 0\}$, where S is the random function obtained by interpolating $S_n - \log n$ at $2 \log n$ in such a way that $S(0) = 0$ and for each $n \geqslant 1$, S takes the constant value $S_n - \log n$ in the interval $\langle 2 \log n, 2 \log(n+1))$, or alternatively is linear in $\langle 2 \log n, 2 \log(n+1))$, i.e.,*

$$(1) \qquad S(t) = \left(1 - \frac{t - 2\log n}{2\log(1 + 1/n)}\right) S_n' + \frac{t - 2\log n}{2\log(1+1/n)} S_{n+1}',$$

if $2 \log n \leqslant t < 2 \log(n+1)$, where $S_n' = S_n - \log n$, $n \geqslant 1$.

Suppose that g is a positive real function on $\mathbb{R}^+$ such that

$$g(x)\uparrow, \qquad g(x)/x\downarrow \qquad \text{as } x \to \infty$$

and

$$(2) \qquad \sum_{n=2}^{\infty} 1/ng^2(\log n) < \infty.$$

Then without changing its distribution we can redefine the process $\{S(t), t \geqslant 0\}$ on a richer probability space, supporting a Brownian motion $\{X(t), t \geqslant 0\}$ in such a way that

$$(3) \qquad S(t) = X(t) + o\left((tg(t))^{\frac{1}{4}} \log t\right) \quad \text{a.s.} \qquad \text{as } t \to \infty.$$

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From Theorem 1 one can deduce an invariance principle for the law of the iterated logarithm and the law of the iterated logarithm for the partial sums of infima of independent uniformly distributed random variables.

THEOREM 2. *(An almost sure invariance principle). Let* $\{X_n, n \geq 1\}$ *be a sequence of independent random variables uniformly distributed on* [0, 1]. *Then the processes* $\{S(t), t \geq 0\}$ *and* $\{X(t), t \geq 0\}$ *of Theorem* 1 *can be redefined on a richer probability space without changing their respective laws, in such a way that for* $\beta > \frac{1}{2}$

$$S(t) = X(t) + o(t^{(1+\beta)/4}\log t) \quad \text{a.s.} \qquad \text{as } t \to \infty. \tag{4}$$

THEOREM 3. *(The law of the iterated logarithm). Under the assumptions of Theorem* 2

$$P\left[\limsup_{n\to\infty} \frac{S_n - \log n}{[2(2\log n)\log\log(2\log n)]^{\frac{1}{2}}} = 1\right] = 1. \tag{5}$$

Generalizations of the above assertions will be given in Section 4.

2. Preliminaries. Let $\{\varepsilon(n), n \geq 1\}$ be a sequence of positive real numbers strictly decreasing to zero. By $\{\tau(\varepsilon(n)), n \geq 1\}$ we denote the sequence of random variables such that

$$\tau(\varepsilon(n)) = \inf\{m : \inf(X_1, X_2, \cdots X_m) < \varepsilon(n)\}.$$

In what follows we need the following lemmas, for which proofs can be found in [3].

LEMMA 1. *The sequence* $\{\tau(\varepsilon(n)), n \geq 1\}$ *increases with probability* 1, *and* $\tau(\varepsilon(n)) \to \infty$ a.s. *as* $n \to \infty$.

LEMMA 2. *The random variables* $\tau(\varepsilon(n)) - \tau(\varepsilon(n-1)), n \geq 2$, *are independent, and*

$$E[\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))]^p \sim \varepsilon^{-1}(m)[\varepsilon(m) - \varepsilon(m-1)]p!\varepsilon^{-p}(m+1), \tag{6}$$

$$p \geq 1, m \geq 1;$$

$$\Sigma_{m=1}^n E[\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))][\varepsilon(m) - \varepsilon(m+1)] = 0(1); \tag{7}$$

and if $\varepsilon(n) = n^{-\alpha}$, $\alpha > 0$, *then*

$$\Sigma_{m=1}^n E[\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))]^p \varepsilon^p(m) \sim \tag{8}$$

$$\Sigma_{m=1}^n E[\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))]^p \varepsilon^p(m+1) \sim p! \log n,$$

where $a_n = 0(1)$ *denotes that a sequence* $\{a_n, n \geq 1\}$ *is bounded.*

LEMMA 3. *If*

$$S(\tau(\varepsilon(n))) = X_1 + \inf(X_1, X_2) + \cdots + \inf(X_1, X_2, \cdots, X_{\tau(\varepsilon(n))}),$$

then

(9) $[\tau(\varepsilon(n+1)) - \tau(\varepsilon(n))]\varepsilon(n+1) \leqslant S(\tau(\varepsilon(n+1)) - 1) - S(\tau(\varepsilon(n)) - 1);$

$$S(\tau(\varepsilon(n+1))) - S(\tau(\varepsilon(n))) \leqslant [\tau(\varepsilon(n+1)) - \tau(\varepsilon(n))]\varepsilon(n), \quad \text{a.s.}, n \geqslant 1;$$

(10) $-2 + \Sigma_{m=1}^{n-1}[\tau(\varepsilon(m+1)) - \tau(\varepsilon(n))]\varepsilon(m+1) \leqslant S(\tau(\varepsilon(n))) - S(\tau(\varepsilon(1)))$

$$\leqslant \Sigma_{m=1}^{n-1}[\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))]\varepsilon(m), \quad \text{a.s.}, n \geqslant 2;$$

(11) $$S(\tau(\varepsilon(n-1))) \leqslant S_m \leqslant S(\tau(\varepsilon(n))),$$

$$m \in [\tau(\varepsilon(n-1)), \tau(\varepsilon(n))).$$

LEMMA 4. *For* $\varepsilon(n) = n^{-\alpha}$, $\alpha > 0$,

(12) $$\lim_{n\to\infty} \log^{-1} n E\{\Sigma_{m=1}^{n-1}[\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))]\varepsilon(m)\} = \alpha;$$

(13) $$\lim_{n\to\infty} \log^{-1} n \sigma^2\{\Sigma_{m=1}^{n-1}[\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))]\varepsilon(m)\} = 2\alpha;$$

for all $A > 0$

(14) $\log(1/u) - (1+A)\log_2(1/u) \leqslant \log \tau(u)$

$$\leqslant \log(1/u) + (1+A)\log_3(1/u), \quad \text{a.s.},$$

for sufficiently small u, *where* $\log_p x = \log(\log_{p-1} x)$, $p \geqslant 2$, $\log_1 x = \log x$.

Moreover, one can easily prove

LEMMA 5. *Let* X_n, U_n, V_n, Z_n, *and* W_n, $n \geqslant 1$, *be random variables such that*

$$U_n + X_n \leqslant V_n \leqslant Z_n + W_n, \quad \text{a.s.}, \qquad n \geqslant 1.$$

If

$$P[\limsup_{n\to\infty} X_n = X] = P[\limsup_{n\to\infty} Z_n = X] = 1,$$

and

$$P[\lim_{n\to\infty} U_n = 0] = P[\lim_{n\to\infty} W_n = 0] = 1,$$

then

$$P[\limsup_{n\to\infty} W_n = X] = 1.$$

3. Proofs of results. Let us denote

$$U_n = \Sigma_{m=1}^{n-1}[\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))]\varepsilon(m),$$

$$Z_n = \Sigma_{m=1}^{n-1}[\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))]\varepsilon(m+1),$$

$$U_n' = U_n - EU_n, \quad Z_n' = Z_n - EZ_n, \quad n \geqslant 2,$$

and

$$Y_k = \{[\tau(\varepsilon(k+1)) - \tau(\varepsilon(k))] - E[\tau(\varepsilon(k+1)) - \tau(\varepsilon(k))]\}\varepsilon(k), \qquad k \geqslant 1.$$

We first prove an almost sure invariance principle for sums $U'_n = \Sigma_{k=1}^{n-1} Y_k$. To do that it is enough to note that the sequence $\{Y_k, k \geqslant 1\}$ satisfies the conditions of Theorem 4.4 [11]. Indeed, we have $EY_k = 0$, $k \geqslant 1$, and by (6) and (13)

$$s_n^2 = \Sigma_{k=1}^{n-1} \sigma^2 Y_k = \sigma^2 U_n \sim 2\alpha \log n \to \infty \qquad \text{as } n \to \infty.$$

Moreover, we see that

$$\begin{aligned}\Sigma_{n=2}^{\infty} 1/g(s_n^2) \textstyle\int_{y^2 > g(s_n^2)} y^2 \, dF_{Y_n}(y) &\leqslant \Sigma_{n=2}^{\infty} EY_n^4 / g^2(s_n^2) \\ &\leqslant 2^4 \Sigma_{n=2}^{\infty} \{ E[\tau(\varepsilon(n+1)) - \tau(\varepsilon(n))]^4 \\ &\quad + E^4[\tau(\varepsilon(n+1)) - \tau(\varepsilon(n))] \} \varepsilon^4(n)/g^2(s_n^2).\end{aligned}$$

But, for $\varepsilon(n) = n^{-\alpha}$, $\alpha > 0$, by (6) we have

$$E[\tau(\varepsilon(n+1)) - \tau(\varepsilon(n))]^4 \varepsilon^4(n) \sim \alpha n^{-1}$$

and

$$E^4[\tau(\varepsilon(n+1)) - \tau(\varepsilon(n))] \varepsilon^4(n) \sim \alpha^4 n^{-4}.$$

Therefore, by (2)

$$\Sigma_{n=2}^{\infty} 1/g(s_n^2) \textstyle\int_{y^2 > g(s_n^2)} y^2 \, dF_{Y_n}(y) \leqslant C \Sigma_{n=2}^{\infty} 1/n g^2(\log n) < \infty,$$

where C is a positive constant. Hence, we see that the sequence $\{Y_n, n \geqslant 1\}$ satisfies the assumptions of Theorem 4.4 [11].

Let now U' be the random function on $\mathbb{R}^+ \cup \{0\}$ obtained by interpolating U'_n at s_n^2 in such a way that $U'(0) = 0$ and U' takes the constant value U'_n in the interval $\langle s_n^2, s_{n+1}^2)$, $n \geqslant 1$. From the above mentioned theorem of [11], it follows that there exists a Brownian motion $\{X(t), t \geqslant 0\}$ such that

$$U'(t) = X(t) + o\big((tg(t))^{\frac{1}{4}} \log t\big) \quad \text{a.s.} \qquad \text{as } t \to \infty.$$

The same assertion is true for the random function U^* on $\mathbb{R}^+ \cup \{0\}$ obtained by interpolating $U_n - \alpha \log n$ at s_n^2 in a similar way as U'. To see that it is enough to note that by (8) and by (12)

$$U_n - \alpha \log n = U'_n + EU_n - \alpha \log n, \qquad \text{and} \quad EU_n - \alpha \log n = 0(1).$$

Furthermore, taking into account (13) and using the assumption (2) one can prove that

$$U(t) - U^*(t) = o\big((tg(t))^{\frac{1}{4}} \log t\big) \quad \text{a.s.} \qquad \text{as } t \to \infty,$$

where

$$U(t) = U_n - \alpha \log n \qquad \text{if} \quad 2\alpha \log n \leqslant t < 2\alpha \log(n+1),$$

$n \geqslant 1$. Hence, we conclude that

$$U(t) = X(t) + o\big((tg(t))^{\frac{1}{4}} \log t\big) = Z(t) \quad \text{a.s. } t \to \infty, \tag{15}$$

where $Z(t)$ is the random function on $\mathbb{R}^+ \cup \{0\}$ defined on the basis Z_n in the same way as $U(t)$.

Now note that (10) yields

$$Z_n + S(\tau(\varepsilon(1))) - 2 \leqslant S(\tau(\varepsilon(n))) \leqslant U_n + S(\tau(\varepsilon(1))) \quad \text{a.s.}$$

Hence

$$(16) \quad Z_n - \alpha \log n + S(\tau(\varepsilon(1))) - 2 \leqslant S(\tau(\varepsilon(n))) - \alpha \log n \leqslant U_n - \alpha \log n + S(\tau(\varepsilon(1))) \quad \text{a.s.}$$

Let now S_τ denote the random function on $\mathbb{R}^+ \cup \{0\}$ which takes the constant value $S(\tau(\varepsilon(n))) - \log n$ in the interval $\langle 2 \log n, 2\log(n+1))$, $n \geqslant 1$. Then, by (15) and (16) we conclude that

$$(17) \qquad S(t) = X(t) + o\big((tg(t))^{\frac{1}{4}}\log t\big) \quad \text{a.s.}, \qquad t \to \infty.$$

From (17), putting $\varepsilon(n) = n^{-1}$ and after using (11) and (14), we get (3).

The assertion of Theorem 2 follows immediately from Theorem 1 with $g(t) = t^\beta$, $\beta > \frac{1}{2}$.

To prove (5) we observe, analogously as previously, that

$$\Sigma_{n=1}^{\infty} P\left[\frac{|Y_n|}{s_n}(\log\log s_n^2)^{\frac{1}{2}} \geqslant \varepsilon\right] \leqslant C\Sigma_{n=1}^{\infty}(\log\log s_n^2)/ns_n^4 < \infty$$

where C is a positive constant. Hence, we conclude that $\{Y_n, n \geqslant 1\}$ satisfies Kolmogorov's law of the iterated logarithm ([6], page 260 or [8] page 376), i.e.

$$P\left[\limsup_{n\to\infty} \frac{U_n - EU_n}{(2s_n^2 \log\log s_n^2)^{\frac{1}{2}}} = 1\right] = 1.$$

Using the above quoted properties of U_n and Z_n we have

$$P\left[\limsup_{n\to\infty}\frac{U_n - a_n}{b_n^{\frac{1}{2}}} = 1\right] = P\left[\limsup_{n\to\infty}\frac{Z_n - a_n}{b_n^{\frac{1}{2}}} = 1\right] = 1,$$

where $a_n = \alpha \log n$, $b_n = 2(2\alpha \log n)\log\log(2\alpha\log n)$. Using now (10) and Lemma 5 one can get

$$(18) \qquad P\left[\limsup_{n\to\infty}\frac{S(\tau(\varepsilon(n))) - a_n}{b_n^{\frac{1}{2}}} = 1\right] = 1.$$

Putting $\varepsilon(n) = n^{-1}$ ($\alpha = 1$) and using (14), we have

$$\tau\big(\varepsilon\big(\big[n(\log_2 n)^{-(1+A)}\big]\big)\big) \leqslant n \leqslant \tau\big(\varepsilon\big(\big[n(\log n)^{1+A}\big]\big)\big) \quad \text{a.s.}$$

for all $A > 0$ and sufficiently large n. Hence,

$$\frac{S\big(\tau\big(\varepsilon\big(\big[n(\log_2 n)^{-(1+A)}\big]\big)\big)\big) - a_{[n(\log_2 n)^{-(1+A)}]}}{b^{\frac{1}{2}}_{[n(\log_2 n)^{-(1+A)}]}} c_n - d_n$$

$$\leqslant \frac{S_n - a_n}{b_n^{\frac{1}{2}}} \leqslant \frac{S\big(\tau\big(\varepsilon\big(\big[n(\log n)^{1+A}\big]\big)\big)\big) - a_{[n(\log n)^{1+A}]}}{b^{\frac{1}{2}}_{[n(\log n)^{1+A}]}} c_n' + d_n', \quad \text{a.s.},$$

where

$$c_n = \left(b_{[n(\log_2 n)^{-(1+A)}]}/b_n\right)^{\frac{1}{2}},$$

$$c_n' = \left(b_{[n(\log n)^{1+A}]}/b_n\right)^{\frac{1}{2}},$$

$$d_n = \left(a_n - a_{[n(\log_2 n)^{-(1+A)}]}\right)/b_n^{\frac{1}{2}},$$

$$d_n' = \left(a_{[n(\log n)^{1+A}]} - a_n\right)/b_n^{\frac{1}{2}}.$$

Substituting into (18) we obtain (5) as $c_n \to 1$, $c_n' \to 1$, $d_n \to 0$, $d_n' \to 0$ as $n \to \infty$.

4. A generalization. We now give generalizations of the results in Section 1 and at the same time reinforce some results of [2] and [3].

Let $\{Y_n, n \geqslant 1\}$ be a sequence of independent positive random variables with the same distribution function F. Suppose that for any $\varepsilon > 0$, $P[Y < \varepsilon] > 0$. Now we shall establish an almost sure invariance principle for the sums $Y_1 + \inf(Y_1, Y_2) + \cdots + \inf(Y_1, Y_2, \cdots, Y_n)$.

Let us set

$$G(t) = \inf\{x \geqslant 0 : F(x) \geqslant t\}.$$

It is obvious that G is a nondecreasing function. In what follows we put $\varepsilon(n) = 1/n$ and $G \leqslant C$, where C is a positive constant.

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent uniformly distributed random variables on [0, 1]. Then the sequences $\{G(X_n), n \geqslant 1\}$ and $\{Y_n, n \geqslant 1\}$ are the same in law, i.e., for $n \geqslant 1$, $G(X_n)$ and Y_n are identically distributed. Thus the asymptotic behaviour of

$$\bar{S}_n = Y_1 + \inf(Y_1, Y_2) + \cdots + \inf(Y_1, Y_2, \cdots, Y_n)$$

is equivalent to that of

$$S_n = G(X_1) + G(\inf(X_1, X_2)) + \cdots + G(\inf(X_1, X_2, \cdots, X_n)).$$

Let us put

$$h(u) = e^u G(e^{-u}), \quad H_p(u) = \textstyle\int_0^u h^p(v)\, dv, \qquad p \geqslant 1$$

and

$$V_n = \textstyle\sum_{m=1}^n [\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))] G(\varepsilon(m+1)),$$

$$W_n = \textstyle\sum_{m=1}^n [\tau(\varepsilon(m+1)) - \tau(\varepsilon(m))] G(\varepsilon(m)),$$

where $\varepsilon(n) = n^{-1}$, $n \geqslant 1$.

We shall use the following statements [3]

(19) $$EV_n = \textstyle\sum_{m=1}^n G(1/(m+1)), \quad EW_n = \sum_{m=1}^n G(1/m);$$

(20) $$\sigma^2 V_n = \textstyle\sum_{m=1}^n 2mG^2(1/(m+1)), \quad \sigma^2 W_n = \sum_{m=1}^n 2mG^2(1/m);$$

(21) $$EV_n - H(\log n) = 0(1), \quad EW_n - H(\log n) = 0(1);$$

(22) $$\sigma^2 V_n - 2H_2(\log n) = 0\left(\textstyle\sum_{m=1}^n G^2(1/m)\right),$$

$$\sigma^2 W_n - 2H_2(\log n) = 0\left(\textstyle\sum_{m=1}^n G^2(1/m)\right);$$

(23) $$-2 + V_{n-1} \leqslant S(\tau(\varepsilon(n))) - S(\tau(\varepsilon(1))) \leqslant W_{n-1}, \qquad n \geqslant 1,$$

a.s.

Let S now be the random function on $\mathbb{R}^+ \cup \{0\}$ obtained by interpolating $S_n - H(\log n)$ at $2H_2(\log n)$ in such a way that $S(0) = 0$ and S takes the constant value $S_n - H(\log n)$ in the interval $\langle 2H_2(\log n), 2H_2(\log(n+1))), n \geq 1$.

Under the above denotations, using the relations (19)–(23), and combining the methods and ideas from this note with the ones of the papers [3], [8], [9] and [11], it can be proved:

THEOREM 1′. *If g is a positive real function on $\mathbb{R}^+$ such that $g(x)\uparrow$, $g(x)/x\downarrow$ as $x \to \infty$, and*

$$\Sigma_{n=2}^{\infty} n^3 G^4(1/n)/g^2(2H_2(\log n)) < \infty,$$

and moreover, for any $A > 0$

$$(24)\quad H(\log n \pm (1+A)\log\log n) - H(\log n) = o\Big(\big[2H_2(\log n) g(2H_2(\log n))\big]^{\frac{1}{4}} \log 2H_2(\log n)\Big),$$

then we can redefine the process $\{S(t), t \geq 0\}$ on a richer probability space supporting a Brownian motion $\{X(t), t \geq 0\}$ in such a way that

$$S(t) = X(t) + o\big((tg(t))^{\frac{1}{4}} \log t\big)$$

a.s. *as $t \to \infty$.*

THEOREM 2′. *If for any $A > 0$, and $\beta > \frac{1}{2}$*

$$(25)\quad H(\log n \pm (1+A)\log\log n) - H(\log n) = o\Big(\big[2H_2(\log n)\big]^{(1+\beta)/4} \log 2H_2(\log n)\Big)$$

then the processes $\{S(t), t \geq 0\}$ and $\{X(t), t \geq 0\}$ of Theorem 1′ satisfy

$$S(t) = X(t) + o(t^{(1+\beta)/4} \log t) \quad \text{a.s.} \qquad \text{as } t \to \infty.$$

REMARK 1. From (19) and (22), it follows that $2H_2(\log n) \sim \Sigma_{m=1}^{n} 2mG^2(1/m)$. Therefore, one can have the conditions and the statements of Theorem 1′ and 2′ expressed in terms of the function G also.

The classical law of the iterated logarithm can be deduced from following propositions:

PROPOSITION 1. *If $s_n^2 = \sigma^2 W_n \to \infty$ as $n \to \infty$ and*

$$\Sigma_{n=1}^{\infty}\big[n^3 (\log\log s_n^2)^2 G^4(1/m)\big]/s_n^4 < \infty$$

then

$$P\left[\limsup_{n\to\infty} \frac{W_n - H(\log n)}{D_n} = 1\right] = P\left[\limsup_{n\to\infty} \frac{V_n - H(\log n)}{D_n} = 1\right]$$

$= 1$, *where* $D_n = [2(2H_2(\log n))\log\log(2H_2(\log n))]^{\frac{1}{2}}$.

PROPOSITION 2. *If* $s_n^2 = \sigma^2 W_n \to \infty$ *as* $n \to \infty$ *and*

$$\Sigma_{n=1}^{\infty}\left[n^3(\log\log s_n^2)^2 G^4(1/n)\right]/s_n^4 < \infty,$$

then

$$P\left[\limsup_{n\to\infty}\frac{S(\tau(\varepsilon(n))) - H(\log n)}{D_n} = 1\right] = 1,$$

where

$$S(\tau(\varepsilon(n))) = G(X_1) + G(\inf(X_1, X_2)) + \cdots + \\ + G(\inf(X_1, X_2, \cdots, X_{\tau(\varepsilon(n))})).$$

THEOREM 3'. *If* $\{u_n, n \geqslant 1\}$ *is an increasing sequence of real numbers such that*: $\lim_{n\to\infty} u_n = \infty$

$$\lim_{n\to\infty} H_2(\log n + u_n)/H_2(\log n) = 1,$$

$$\lim_{n\to\infty}\left[H(\log n + u_n) - H(\log n)\right]/\left[H_2(\log n)\right]^{\frac{1}{2}} = 0,$$

and

$$\Sigma_{n=1}^{\infty}\left[n^3(\log\log s_n^2)^2 G^4(1/n)\right]/s_n^4 < \infty,$$

then

$$P\left[\limsup_{n\to\infty}\frac{S_n - H(\log n)}{D_n} = 1\right] = 1.$$

REMARK 2. It can be observed that the assumption

$$\Sigma_{n=1}^{\infty}\left[n^3(\log\log s_n^2)^2 G^4(1/n)\right]/s_n^4 < \infty$$

can be replaced by

$$\Sigma_{n=1}^{\infty} n^3 G^4(1/n)/s_n^{4-\delta} < \infty$$

for some $\delta > 0$. Moreover, we note that instead of the above conditions we can use more general ones

$$\Sigma_{n=1}^{\infty}\left[n^{r-1}(\log\log s_n^2)^{r/2} G^r(1/n)\right]/s_n^r < \infty$$

or

$$\Sigma_{n=1}^{\infty}\left[n^{r-1} G^r(1/n)\right]/s_n^{r-\delta} < \infty,$$

where $r \geqslant 2$, $\delta > 0$.

5. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables uniformly distributed on [0, 1] and $\hat{S}_n = \Sigma_{j=m}^{n} X_j^{(m)}$, $n \geqslant 2$, $\hat{S}_1 = 0$, where $X_j^{(m)}$ is the mth order statistic of $(X_1, \cdots, X_j)$. If $m = 1$, then we have $\hat{S}_n = S_n = \Sigma_{j=1}^{n}\min(X_1, X_2, \cdots, X_j)$, the sum of Section 1.

Define the process $\{\hat{S}(t), t \geqslant 0\}$, where $\hat{S}$ is the random function obtained by interpolating $\hat{S}_n - m\log n$ at $2m\log n$ in such a way $\hat{S}(0) = 0$ and for each $n \geqslant 1$

$\hat{S}$ takes the constant value $\hat{S}_n - m \log n$ in the interval $\langle 2m \log n, 2m \log(n+1))$, or alternatively is linear there. Using the methods and ideas from this note with the ones of the papers [3] and [10], it can be proved:

THEOREM 1″. *If g is a positive real function on $\mathbb{R}^+$ such that*

$$g(x)\uparrow, \quad g(x)/x\downarrow \qquad \textit{as } x \to \infty,$$

and

$$\Sigma_{n=2}^{\infty} 1/ng^2(\log n) < \infty,$$

then the processes $\{\hat{S}(t),\ t \geqslant 0\}$ with a Brownian motion $\{X(t),\ t \geqslant 0\}$ can be redefined on a richer probability space without changing their respective laws, in such a way that

$$\hat{S}(t) = X(t) + o\big((tg(t))^{\frac{1}{4}}\log t\big), \quad \text{a.s.} \qquad \textit{as } t \to \infty.$$

THEOREM 2″. (*An almost sure invariance principle*). *For $\beta > \frac{1}{2}$, and the processes $\{\hat{S}(t),\ t \geqslant 0\}$ and $\{X(t),\ t \geqslant 0\}$ of Theorem 1″ we have*

$$\hat{S}(t) = X(t) + o(t^{(1+\beta)/4}\log t), \quad \text{a.s.} \qquad \textit{as } t \to \infty.$$

THEOREM 3″. *If $\{X_n,\ n \geqslant 1\}$ be a sequence of independent random variables uniformly distributed on* [0, 1] *and $X_j^{(m)}$ is the* mth *order statistic of $(X_1, X_2, \cdots, X_j)$, then*

$$P\left[\limsup_{n\to\infty} \frac{\hat{S}_n - m \log n}{\left[2(2m \log n)\log\log(2m \log n)\right]^{\frac{1}{2}}} = 1\right] = 1,$$

where $\hat{S}_n = \Sigma_{j=m}^{n} X_j^{(m)}$.

The above theorems generalize some results of [10] and their proofs can be obtained in the same way as the proofs of the statements given in the previous sections.

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