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CONJUGATE Π-VARIATION AND PROCESS INVERSION

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The well-known concept of conjugate slowly varying functions is specialized to the subclass Π of the slowly varying functions. The concept is then used to connect convergence of certain increasing stochastic processes (suitably normalized) with convergence of their inverses.

1. Introduction. For a slowly varying function $L(x)$, it is well known that there always exists an asymptotically unique slowly varying function $L^*(x)$ satisfying

$$\lim_{x\to\infty} L(x)L^*(xL(x)) = \lim_{x\to\infty} L^*(x)L(xL^*(x)) = 1$$

(de Bruijn (1959), Seneta (1976), page 25). L and L^* are called conjugate slowly varying functions. Conjugate pairs arise when one attempts to find asymptotic inverses of regularly varying functions and a prescription for finding an L^* from L is as follows: given L form the 1-varying function $R(x) = xL(x)$ and then invert; i.e., find the asymptotically unique inverse R^{-1} with the property that $R \circ R^{-1}(x) \sim R^{-1} \circ R(x) \sim x$. R^{-1} is also 1-varying so that $R^{-1}(x)/x$ is slowly varying and this is the desired L^*.

In this paper the concept of conjugate slowly varying functions is specialized to a proper subclass of the slowly varying functions called Π. The class Π will now be defined. The definition differs slightly from de Haan (1970) but is appropriate for the present circumstances. All functions in this paper are assumed measurable and real valued. Suppose π: $R^+ \to R^+$ has the property that there exist functions g: $R^+ \to R^+$, b: $R^+ \to R^+$ such that for all $x > 0$

$$\lim_{t\to\infty} \frac{\pi(tx) - b(t)}{g(t)} = \log x. \tag{1.1}$$

Then we say π is Π^+-varying and write $\pi \in \Pi^+$. The class Π^- is defined similarly except for $\pi \in \Pi^-$ we require

$$\lim_{t\to\infty} \frac{\pi(tx) - b(t)}{g(t)} = -\log x. \tag{1.1'}$$

Finally $\Pi = \Pi^+ \cup \Pi^-$. The function g appearing in (1.1) and (1.1′) is called an auxiliary function of π and is known to be slowly varying as is π itself (cf. de Haan, 1970). Setting $x = 1$ in (1.1) and (1.1′) shows it is always possible to take $b = \pi$.

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A useful way to partition the class of slowly varying functions is to use the equivalence relation of asymptotic equivalence: L_1 and L_2 are asymptotically equivalent iff $L_1(x) \sim L_2(x)$ as $x \to \infty$. On Π a more useful partitioning is obtained as follows:

DEFINITION. Suppose $\pi_1, \pi_2 \in \Pi$ and g is an auxiliary function of π_1. We say π_1 and π_2 are Π-equivalent, written $\pi_1 \sim_\Pi \pi_2$, iff for some $c \in R$

$$\lim_{t\to\infty} \frac{\pi_1(t) - \pi_2(t)}{g(t)} = c. \tag{1.2}$$

In this case π_2 has auxiliary function g. Note if (1.2) holds and $\pi_1 \in \Pi^\pm$ then

$$\lim_{t\to\infty} \frac{\pi_1(t) - \pi_2(e^{\pm c}t)}{g(t)} = 0$$

so it is usually no loss of generality to suppose $c = 0$ in (1.2). It is easily verified that $\sim_\Pi$ is an equivalence relation which partitions Π into equivalence classes.

In Section 2 we catalogue some simple but essential properties of Π and prove that if $\pi \in \Pi^\pm$ then $\pi^* \in \Pi^\mp$. This is applied in Section 3 to an inversion problem in weak convergence of stochastic processes.

Suppose $X = \{X(t), t \geqslant 0\}$ is a stochastic process all of whose paths lie in $D(0, \infty)$, the space of right continuous functions with finite left limits on $(0, \infty)$. In applications X frequently arises from a sequence of random variables (rv's) $\{\xi_n, n \geqslant 0\}$ by setting $X(t) = \xi_{[t]}$. Call X a Π-varying process if there exist $\pi \in \Pi$ with auxiliary function g and a random element Y of $D(0, \infty)$ such that $Y(t)$ is a nondegenerate random variable for each t and

$$\frac{X(n.) - \pi(n)}{g(n)} \Rightarrow_{M_1} Y \tag{1.3}$$

where $\Rightarrow_{M_1}$ denotes weak convergence in the Skorohod M_1 topology on $D(0, \infty)$ (cf. Skorohod 1956, Vervaat 1973, Whitt 1971). M_1 convergence on $D(0, \infty)$ is M_1 convergence on $D[a, b]$ for all $0 < a < b$ which are continuity points of the limit. Weak convergence notation and usage are as in Billingsley (1968) except that "$\Rightarrow_T$" denotes weak convergence in the T-topology on $D(0, \infty)$. T is either M_1 or J_1 and is usually M_1. From Section 2 of Durrett and Resnick (1976) we have that Y is stochastically continuous and possesses a scaling property: for any $u > 0$

$$Y(u.) =_d Y(\cdot) \pm \log u;$$

the sign preceding $\log u$ is determined by whether $\pi \in \Pi^+$ or $\pi \in \Pi^-$.

We now form an inverse for the process X by analogy with the construction of a conjugate slowly varying function. Define $Z(t) = tX(t)$ and suppose Z has nondecreasing paths. Let Z^{-1} be the right continuous inverse and define the process X^* by $X^*(t) = Z^{-1}(t)/t$. Then in Section 3 we show

$$\frac{X^*(n.) - \pi^*(n)}{g_{\pi^*}(n)} \Rightarrow_{M_1} - Y \tag{1.4}$$

where π, π^* are a conjugate pair and g_{π^*} is an auxiliary function of π^*.

With the monotonicity assumption on Z, (1.3) and (1.4) are equivalent in the sense either implies the other and in fact a joint statement then ensues in the manner of Iglehart and Whitt (1971). Of course if X is monotone, Z is also and this is the case in the examples of Section 3 drawn from renewal and extreme value theory. If Z is not monotone but has unbounded paths, one can still invert and obtain (1.4) by passing first from X to $X^\uparrow$ where $X^\uparrow(t) = \sup_{0 \leqslant s \leqslant t} X(s)$. This point is thoroughly discussed in Vervaat (1972) and Whitt (1974).

The weak convergence results of Section 3 are complementary to those of several authors who have studied the relationship of weak convergence behavior of sums of i.i.d. rv's to the weak convergence of the associated first passage processes. See, for example, Vervaat (1972), Gut (1973, 1975), Mohan (1975), Whitt (1974), Bingham (1972, 1973), Lindberger (1978), Chow and Hsiung (1976) and, of course, Billingsley (1968). Whitt (1974) in particular is helpful in understanding the relationship between X, $X^\uparrow$ and the first passage processes and why the M_1 topology is used.

A final word on notational conventions: $U \in RV_\rho$ means U: $R^+ \to R^+$ and $\lim_{t\to\infty}(U(tx)/U(t)) = x^\rho$ for $x > 0$ and $\rho \in R$.

The auxiliary function of $\pi \in \Pi$ will always be denoted by g. If π appears with subscripts so will the auxiliary function. The auxiliary function of π^* will be denoted by g_{π^*}.

2. **Conjugate Π-variation.**

It is convenient to begin by collecting some preliminary results for easy reference:

PROPOSITION 1.

(2.1) *The limits* (1.1) *and* (1.1′) *hold uniformly on compact subsets of* $(0, \infty)$.

(2.2) $\pi \in \Pi$ iff *for every function* $r \in RV_1$ *we have* $\pi \circ r \in \Pi$. *The auxiliary function of* $\pi \circ r$ *is* $g \circ r$. *Moreover* $\pi \circ r \sim_\Pi \pi$ *if and only if* $r(x)/x \to c > 0 (x \to \infty)$.

(2.3) $\pi \in \Pi^+$ iff $1/\pi \in \Pi^-$. *The auxiliary function of* $1/\pi$ *is* g/π^2.

(2.4) $\lim_{x\to\infty} \pi(x)/g(x) = \infty$.

(2.5) *Suppose* $L \in RV_0$ *and* $\pi \in \Pi^\pm$. *Then* $L.\pi \in \Pi^\pm$ *with auxiliary function* $L.g$ iff

$$\lim_{t\to\infty}\left(\frac{L(tx)}{L(t)} - 1\right)\frac{\pi(t)}{g(t)} = 0 \quad \textit{for all } x > 0$$

(cf. *Bojanic and Seneta*, 1971).

PROOF. (2.1): See Balkema (1973), page 141.

(2.2): We have

$$\lim_{t\to\infty} \frac{\pi(r(tx)) - b(r(t))}{g(r(t))} = \lim_{t\to\infty} \frac{\pi\left(\dfrac{r(tx)}{r(t)} r(t)\right) - b(r(t))}{g(r(t))}$$

and by (2.1) the limit is $\pm \log x$ as required. Use (2.1) to see that $(\pi(t^{-1}r(t).t) - \pi(t))/g(t)$ converges if and only if $r(x)/x \to c > 0 (x \to \infty)$.

(2.3): If $\pi \in \Pi^{\pm}$

$$\lim_{t\to\infty} \frac{(1/(\pi(tx))) - (1/(\pi(t)))}{g(t)/\pi^2(t)} = \lim_{t\to\infty} \frac{\pi(t) - \pi(tx)}{\pi(tx)\pi(t)g(t)/\pi^2(t)} = \mp \log x$$

since $\Pi \subset RV_0$.

(2.4): If $\pi \in \Pi^+$ and π is nondecreasing, the result is well known, see, for example, de Haan (1976), page 539. If $\pi \in \Pi^+$ but is not monotone, use Proposition 2 below. If $\pi \in \Pi^-$, the result follows by (2.3).

(2.5): The result follows immediately from

$$\frac{L(tx)\pi(tx) - L(t)\pi(t)}{L(t)g(t)} = \left(\frac{L(tx) - L(t)}{L(t)}\right)\frac{\pi(tx)}{g(t)} + \frac{\pi(tx) - \pi(t)}{g(t)}$$

and in fact that $\pi \in RV_0$.

The definition of Π given in the introduction does not assume monotonicity. The reason this is not required is contained in the next result.

PROPOSITION 2.

(a) *If* $\pi \in \Pi^+$ *then* $\exists \pi_0 \in \Pi^+$, $\pi \sim_{\Pi} \pi_0$ *and* π_0 *is ultimately continuous and strictly increasing.*

(b) *If* $\pi \in \Pi^-$ *then* $\exists \pi_0 \in \Pi^-$, $\pi \sim_{\Pi} \pi_0$ *and* $x\pi_0(x)$ *is ultimately continuous and strictly increasing.*

PROOF. (a) from (1.1) and (2.1)

$$\lim_{t\to\infty} \frac{\int_1^e \frac{\pi(tx)}{x}\,dx - \int_1^e \frac{\pi(t)}{x}\,dx}{g(t)} = \int_1^e \frac{\log x}{x}\,dx = \tfrac{1}{2}.$$

The numerator of the left hand side equals

$$\int_{e^{-1}}^{1} \frac{\pi(se)}{s}\,ds + \int_1^t \frac{\pi(se) - \pi(s)}{s}\,ds - \pi(t).$$

Since $(\pi(te) - \pi(t))/g(t) \to \log e = 1$, there exists t_0 such that if $t \geqslant t_0$ we have $\pi(te) - \pi(t) > 0$. Define $\pi_0(t) = \int_{e^{-1}}^{1}\pi(se)/s\,ds + \int_1^t(\pi(se) - \pi(s))/s\,ds$ and the result follows.

(b) For $\pi_1 \in \Pi^+$ there is $\pi_2 \sim_{\Pi} \pi_1$ such that π_2' exists and $g_2(x) = x\pi_2'(x)$ (cf. de Haan, 1970, page 34 in case π_1 is monotone). To see this use the construction in (a) above to get a π_3 which is ultimately continuous and strictly increasing and Π-equivalent to π_1. Then repeat this procedure once more to get a π_2 which is differentiable and $\pi_2 \sim_{\Pi} \pi_3 \sim_{\Pi} \pi_1$.

Next note that $\pi_1 \sim_{\Pi} \pi_2$ iff $1/\pi_1 \sim_{\Pi} 1/\pi_2$ which can easily be checked by the method of (2.3). Therefore, if $\pi \in \Pi^-$, then $\pi_1 := 1/\pi \in \Pi^+$ and there exists π_2 as previously described with $1/\pi_2 \sim_{\Pi} \pi$ and $x/\pi_2(x)$ ultimately strictly increasing since

$$\left(\frac{x}{\pi_2(x)}\right)' = \frac{1}{\pi_2(x)}\left(1 - \frac{x\pi_2'(x)}{\pi_2(x)}\right)$$

is ultimately positive. This is because $x\pi_2'(x)/\pi_2(x) = g_2(x)/\pi_2(x) \to 0$ by (2.4). Finish by setting $\pi_0 = 1/\pi_2$.

DEFINITION. Conjugate- Π-function: suppose $\pi \in \Pi$ with auxiliary function g. Any function π^*: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\pi(x)\{g(x)\}^{-1}[\pi(x)\pi^*(x\pi(x)) - 1] = 0$$

is a conjugate Π-function for π.

We first give a general construction thus establishing the existence of a conjugate Π-function for any $\pi \in \Pi$. Let $\pi \in \Pi$ and suppose π_0 is the function associated with π by the construction of Proposition 2. A conjugate Π-function π^* is $\pi^*(x) = v_0^{-1}(x)/x$ where $v_0(x) = x\pi_0(x)$. To see this note that $\pi^*(x\pi(x)) \sim_\Pi \pi^*(x\pi_0(x)) = x/v_0(x) = 1/\pi_0(x) \sim_\Pi 1/\pi(x)$. Since $v_0(x)$ is ultimately continuous and strictly increasing there is no (ultimate) ambiguity in the definition of the inverse $v_0^{-1}(x)$.

THEOREM 1.

(a) *If* $\pi \in \Pi^\pm$ *then* $\pi^* \in \Pi^\mp$ *and its auxiliary function is given by* $g_{\pi^*}(x\pi(x)) \sim g(x)/\pi^2(x)(x \to \infty)$. *An explicit form for* g_{π^*} *is* $g_{\pi^*}(t) = g(v_0^{-1}(t))(\pi^*(t))^2$.

(b) $\pi_1 \sim_\Pi \pi_2$ iff $\pi_1^* \sim_\Pi \pi_2^*$. *In particular the conjugate* Π-*function is defined up to a* Π-*equivalence class.*

(c) $\pi^{**} \sim_\Pi \pi$.

PROOF. (a), (b): From the definition it follows $\pi^*(x.\pi(x)) \sim 1/\pi(x)$. Use (2.2).

(c): Referring to the construction above we have $\pi(x) \sim_\Pi v_0(x)/x$ hence $\pi(x\pi^*(x)) \sim_\Pi v_0(x\pi^*(x))/x\pi^*(x) \sim_\Pi 1/\pi^*(x)$, i.e.,

$$\pi^*(x)\{g_{\pi^*}(x)\}^{-1}[\pi^*(x)\pi(x\pi^*(x)) - 1] \to 0(x \to \infty).$$

REMARK. π^* is also a conjugate function for π according to de Bruijn's definition.

REMARK. $\pi^* \sim_\Pi 1/\pi$ if and only if $0 < \lim_{x\to\infty}\pi(x) < \infty$.

3. Weak convergence of first passage processes. Suppose X is a Π-varying process so that

$$\frac{X(u.) - \pi(u)}{g(u)} \Rightarrow_{M_1} Y \tag{3.1}$$

as $u \to \infty$ where $\pi \in \Pi$. Let $r \in RV_1$ such that

$$\lim_{u\to\infty}\left(\frac{r(ut)}{r(u)} - t\right)\frac{\pi(u)}{g(u)} = 0 \tag{3.2}$$

for all $t > 0$. If $r(u) = uL(u)$, $L \in RV_0$ we have by (2.5) that $\pi_1 := L.\pi \in \Pi$ with auxiliary function $g_1 = L.g$ and without loss of generality (by Proposition 2) we can suppose for our purposes that $v_1(u) = u\pi_1(u)$ is strictly increasing and continuous. Next define $Z(t) = r(t)X(t)$ and suppose the paths of Z are nonnegative and unbounded. The first passage process of Z is

$$Z^{-1}(t) = \inf\{u|Z(u) > t\}$$

and $X^*(t) = Z^{-1}(t)/t$.

THEOREM 2. *If* (3.1) *and* (3.2) *hold and the paths of* Z *are unbounded, then*

$$\frac{X^*(u.) - \pi_1^*(u)}{g_1^*(u)} \Rightarrow_{M_1} - Y \tag{3.3}$$

where π_1^* *is conjugate to* π_1 *and*

$$g_1^*(u) = L(v_1^{-1}(u))g(v_1^{-1}(u))(\pi_1^*(u))^2.$$

PROOF. For convenience regard t as the identity function on R^+. Multiply top and bottom of the left side of (3.1) by $r(ut)$ and use the fact that $r \in RV_1$ to obtain

$$\frac{Z(ut) - r(ut)\pi(u)}{r(u)g(u)} \Rightarrow_{M_1} tY(t).$$

Because of (3.2) this can be rewritten

$$\frac{\pi(u)}{g(u)}\left(\frac{Z(ut)}{r(u)\pi(u)} - t\right) \Rightarrow_{M_1} tY(t).$$

By (2.4) $\pi(u)/g(u) \to \infty$ and calling $Z(ut)/r(u)\pi(u) =: X_u(t)$ we obtain by Theorem 7.5 of Whitt (1974) that

$$\frac{\pi(u)}{g(u)}(X_u^{-1}(t) - t) \Rightarrow_{M_1} - tY(t)$$

and one checks that $X_u^{-1}(t) = (Z^{-}(r(u)\pi(u)t))/u = (Z^{-1}(v_1(u)t))/u$. Change variables $u \to v_1(u)$ and rearrange to obtain (3.3).

EXAMPLE 1. Extreme values: let $\{\xi_n, n \geqslant 1\}$ be i.i.d. rv's with common distribution $F(x)$ such that

$$P\left[\vee_{j=1}^{n}\xi_j \leqslant g(n)x + \pi(n)\right] \to \Lambda(x) = \exp\{-e^{-x}\}$$

for $x \in R$. It is known that $\pi(u) := F^{-1}(1 - 1/u) \in \Pi^+$ and $g(u) = F^{-1}(1 - (1/ue)) - F^{-1}(1 - (1/u))$ is an auxiliary function (de Haan, 1970). Futhermore setting $X(u) = \vee_{j=0}^{[u]}\xi_j$ we have

$$\frac{X(u.) - \pi(u)}{g(u)} \Rightarrow_{J_1} Y$$

where Y is the extremal process generated by $\Lambda(x)$ (Lamperti, 1964; Resnick, 1975). So (3.3) holds and a limit law for Z^{-1} ensues where

$$Z^{-1}(s) = \inf\left\{t | \vee_{j=1}^{[t]}\xi_j > \frac{s}{r(t)}\right\}.$$

EXAMPLE 2. Renewal theory: suppose $\{\xi_n, n \geqslant 1\}$ are i.i.d. nonnegative rv's in the domain of attraction of a stable law of index $\alpha = 1$ so that $F(x) = P[\xi_1 \leqslant x]$ has a regularly varying tail of index -1. Then it is known that

$$\frac{\sum_{j=1}^{[n.]}\xi_j - n\cdot\pi(n)}{a(n)} \Rightarrow_{J_1} X_1 \tag{3.4}$$

where X_1 is a stable process with only positive jumps of index $\alpha = 1$ and $a(u) = F^{-1}(1 - 1/u)$, $\pi(u) = \int_0^{a(u)}(1 - F(s))\, ds$ (Skorohod, 1957; Feller, 1971, page 315; Durrett and Resnick, 1978). Since $1 - F \in RV_{-1}$, $a(.) \in RV_1$ (de Haan, 1970, page 24) and $\int_0^x (1 - F(s))\, ds \in \Pi^+$ (de Haan, 1970, page 38) so by (2.2) $\pi(u) \in \Pi^+$. Define $X(t) = \sum_1^{[t]}\xi_j/t$ and from (3.4) we get

$$\frac{X(u.) - \pi(u)}{g(u)} \Rightarrow_{M_1} Y$$

where $Y(t) = X_1(t)/t$ and $g(u) = a(u)/u$. In this case we get a limit law for $Z^{-1}(t) = \inf\{s|\sum_1^{[s]}\xi_j > t/L(s)\}$. If $r(u) = u$ so $L \equiv 1$, then a limit emerges for $N(t) =$ number of renewals in $[0, t]$.

The norming constants for N have been computed by Bingham (1972, 1973) who also considered a related example of regenerative phenomena.

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