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SPECIAL INVITED PAPER

BROWNIAN MOTION AND ANALYTIC FUNCTIONS[1]

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This paper is mostly expository and is concerned with the connection between two dimensional Brownian motion and analytic functions provided by Lévy's result that, if Z_t, $0 \leqslant t < \infty$, is two dimensional Brownian motion, and if f is analytic and not constant, then $f(Z_t)$, $0 \leqslant t < \infty$, is also two dimensional Brownian motion, perhaps moving at a variable speed. This can be used to study Brownian motion via analytic functions and, conversely, to treat analytic functions probabilistically. Recently several open problems in analytic function theory have been solved in this manner. We will present some of Doob's earlier work on the range and boundary values of analytic functions, the probabilistic theory of H^p spaces due to Burkholder, Gundy and Silverstein, the author's results on conjugate function inequalities, and sketch probabilistic proofs of Picard's big and little theorems, and other theorems. There are some new results related to Hayman's generalization of Koebe's theorem.

1. Introduction. A fundamental connection between two dimensional Brownian motion $Z_t = X_t + iY_t$, $t \geqslant 0$, moving in the complex plane, and analytic function theory, is Paul Lévy's theorem that, if $f(z)$ is analytic and not constant, the process $f(Z_t)$, $t \geqslant 0$, is again Brownian motion, although perhaps moving at a variable speed. This conformal invariance can be used to study Brownian motion via analytic functions, by making a judicious choice of $f(z)$. For example, let $Z_0 \equiv 0$ and let $a \neq b$ be complex numbers. Then $(a - b)e^{Z_t} + b$, $t \geqslant 0$, is Brownian motion started at $(a - b)e^0 + b = a$. Clearly it never hits b, since e^z never vanishes. This proves the well-known result that the probability Brownian motion ever hits a fixed point other than its starting point is 0. (Here Brownian motion means two dimensional Brownian motion moving in the complex plane, the mathematical description by Norbert Wiener of a physical process observed by, among others, Brown).

Lévy's result can also be used to study analytic functions probabilistically, the principal subject of this paper. In this context it is usually applied to a collection of functions. For example, it is proved in the next section that Brownian motion hits each closed set of positive capacity with probability one. Thus, the range of each nonconstant entire function cannot omit a closed set of positive capacity, since the Brownian motion $f(Z_t)$ moves entirely in $f(\mathbb{C})$. This argument is Doob's.

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Except for new material in the last section, this paper is an expository presentation of some of the applications of the conformal invariance of Brownian motion to analytic function theory, including the recent solutions of several open problems. Often probabilistic expressions can be written in classical terms, and the theorems of probability can be brought to bear directly on nonprobabilistic problems. Other expressions arise which do not seem amenable to such a translation, and in this way intrinsically new tools arise, which, aside from their use in the study of classical function theory, are sometimes interesting enough to deserve attention in their own right, and are thus a source of new theorems and problems. Several of the latter are given at the end of this paper.

Brownian motion is a very intuitive thing to work with, because at the back of our mind is the corresponding physical process, or rather an idealized version of it, which is only very briefly described at the beginning of the next section. A much more detailed and very enjoyable physical and historical account of Brownian motion can be found at the beginning of Edward Nelson's book [33].

Although there is necessarily some overlap between the subject of this paper and the better known probabilistic potential theory, no attempt will be made to treat the latter systematically. See the book of E. F. Dynkin and A. A. Yushkevich [21] for a very readable elementary introduction to the use of Brownian motion in studying harmonic and subharmonic functions, and see [18] for more advanced treatment.

For simplicity, the treatment here is usually restricted to entire functions and functions defined on the open unit disc D. Extensions of the results here to more general situations will often be immediate. Proofs are usually just sketched except in the next section, where the foundations of the subject, up to and including some basic theorems of Kakutani and a version of Lévy's theorem, are rigorously presented.

In Section 3 the author's proof of Picard's little theorem is sketched. This theorem can also be proved using Brownian motion on Riemann surfaces and the modular function, together with ideas of Kakutani (see Section 8). Here Riemann surfaces are not used and the work of the modular function is done by a law of large numbers, which is used to derive a result about Brownian motion paths originally proved (via the modular function) by Ito and McKean. For another, shorter, exposition of some aspects of this proof see P. J. Kahane [28].

In Section 4 some of Doob's earlier work on the range and boundary values of analytic functions is sketched. One sample of this has already been given. The particularly close connection between Stoltz regions and Brownian paths is discussed here. See Burkholder [6] for related expository material.

The next topic is the theorems and techniques associated with the probabilistic treatment of H^p spaces of analytic functions, including the famous solution by D. L. Burkholder, R. F. Gundy, and M. L. Silverstein of a long standing problem in this area in [12], and some of Burkholder's later work. The original proof, which used martingales, is translated to enable use of Lévy's theorem. This is the approach taken by Burkholder more recently. Karl Peterson has written a book, [34], mostly devoted to presenting and explaining [12]. For another exposition of [12], as well as a thorough treatment of modern H^p theory, much of which was inspired by [12], see Charles Fefferman's paper [22]. Fefferman and E. Stein were the first to give a nonprobabilistic proof of the Burkholder, Gundy, and Silverstein result, in [23].

In Section 6 the author's method of studying conjugate function and Hilbert transform inequalities will be sketched. This provides a uniform approach to a number of these inequalities and often gives the extremal functions and best possible constants for them, some of which were found for the first time in this manner. There are expository accounts of this method, applied to different problems than the one considered here, in J. P. Kahane's paper [28] and D. L. Burkholder's paper [6]. Recently Albert Baernstein II has given nonprobabilistic proofs of most of these results, as well as some very nice extensions, in [2].

The next to last section contains a new proof of Picard's big theorem and new results related to Hayman's generalization of Koebe's theorem to multivalent functions. In the final sections some problems are posed and Brownian motion on Riemann surfaces, including an early paper of Kakutani on the type problem for Riemann surfaces, is briefly discussed.

2. Fundamentals. A very small particle suspended in a liquid can be seen to move rapidly about, due to the bombardment of the particle by the molecules of the liquid. If the position of the particle is projected onto a plane, a two dimensional motion results. This is what would be observed if the particle was watched through a microscope and it was impossible to discern the up and down component of its position. In the 1920's Norbert Wiener gave a mathematical description of an idealized version of this motion. We will distinguish between standard Brownian motion, supposed to describe the movement of a particle suspended in a liquid of a certain unchanging temperature, and Brownian motion, in which the temperature of the liquid, which influences the rapidity of the motion of the particle, is allowed to vary.

Wiener constructed a family of random variables $Z_t = X_t + iY_t$, $t \geq 0$, on a probability space Ω, Z_t representing the position of the particle at time t. For each $\varepsilon > 0$, let $\tau_0(\varepsilon) = \tau_0 = 0$, and if $i \geq 1$ define $\tau_i(\varepsilon) = \tau_i = \inf\{t > \tau_{i-1}: |Z_t - Z_{\tau_{i-1}}| = \varepsilon\}$, and let $\Delta_i(\varepsilon) = \Delta_i = Z_{\tau_i} - Z_{\tau_{i-1}}$. The process Z_t satisfies the following postulates.

(A) For almost every $\omega \in \Omega$, the path $Z_t(\omega)$, $0 \leq t < \infty$, is continuous and unbounded.

(B) For each $\varepsilon > 0$, $\Delta_1, \Delta_2, \cdots$ are independent and each is uniformly distributed with respect to linear Lebesgue measure on $\{|z| = \varepsilon\}$.

(C) For each $\varepsilon > 0$, $\tau_i - \tau_{i-1}$, $i \geq 1$, are independent and identically distributed. Furthermore $E(\tau_i(1) - \tau_{i-1}(1)) = \frac{1}{2}$ (a normalization).

Standard Brownian motion is defined to be a process satisfying these three postulates, while Brownian motion is defined to be a process satisfying the first two. It is not difficult to show that this definition of standard Brownian motion is equivalent to the usual one. What we call Brownian motion might be given other names elsewhere. Wiener's construction of standard Brownian motion may be found in [4], and there is a different construction in [32].

Although these definitions are new, it has been recognized for a long time that the property of hitting circles with a uniform distribution is very useful when using Brownian motion to study harmonic and analytic functions. Of course postulate (B) means that, for each integer n, the measure on $\mathbb{C}^n$ induced by the map $(\Delta_1, \cdots, \Delta_n)$ from Ω is product measure μ^n, where μ puts uniform measure on the circle of radius ε about 0.

In the rest of this paper Z_t will be standard Brownian motion, and P_z and E_z will denote probability and expectation associated with Z_t started at z, that is satisfying $Z_0 \equiv z$. Since Z_t will usually start at 0, P_0 and E_0 will be written P and E to avoid subscripts. Linear Lebesgue measure will be denoted by l, and $a \wedge b = \min(a, b)$.

Recall that a harmonic function defined on a region R is a continuous function which satisfies the averaging property. That is, if $\{|z - z_0| \leqslant \varepsilon\} \subset R$ then

$$u(z_0) = \textstyle\int_0^{2\pi} u(z_0 + \varepsilon e^{i\theta}) d\theta / 2\pi. \tag{2.1}$$

This connects nicely with postulate (B). The following fundamental theorem is due to Kakutani, in [29]. The proof is essentially Doob's ([18]). Although it is stated only for standard Brownian motion, since it is only a theorem about the paths of Z_t, it also holds for (nonstandard) Brownian motion. It will be seen that only postulates (A) and (B) are used in the proof. This also is true of Theorem 2.2 and Theorem 2.3.

THEOREM 2.1. *Let u be harmonic on a region R and continuous and bounded on $\overline{R}$ (closure of R). Let $\tau_R = \inf\{t > 0\colon Z_t \in \partial R$ (boundary of R)$\}$. Then if $z \in R$ and $P_z(\tau_R < \infty) = 1$, $E_z u(Z_{\tau_R}) = u(z)$.*

PROOF. Let $\varepsilon > 0$ be fixed for a while, and let $N = \min\{k : |Z_{\tau_k} - \partial R| \leqslant \varepsilon\}$, where $\tau_k = \tau_k(\varepsilon)$ is defined as before. Let I be the indicator function.

If $|z - \partial R| \leqslant \varepsilon$, $N = 0$, so assume $|z - \partial R| > \varepsilon$ which gives $P(N \geqslant 1) = 1$. Then

$$E_z u(Z_{\tau_1}) = \textstyle\int_0^{2\pi} u(z + \varepsilon e^{i\theta}) d\theta / 2\pi = u(z)$$

by (2.1) and (B).

Now $Z_{\tau_2} - Z_{\tau_1}$ is uniformly distributed on $\{|z| = \varepsilon\}$, and independent of $Z_{\tau_1} - Z_{\tau_0}$, so it is independent of $\{N > 1\}$. Thus, integrating first with respect to the second coordinate and then the first in $\mathbb{C} \times \mathbb{C}$, (2.1) and (B) again give

$$\begin{aligned} E_z\big(u(Z_{\tau_2}) - u(Z_{\tau_1})\big)I(N > 1) \\ &= E_z\left[\frac{1}{2\pi}\int_0^{2\pi} u\big(Z_{\tau_1} + \varepsilon e^{i\theta}\big)d\theta - u\big(Z_{\tau_1}\big)\right] I(N > 1) \\ &= E_z[0] \cdot I(N > 1) = 0, \qquad \text{so} \end{aligned}$$

$$E_z u\big(Z_{\tau_{2\wedge N}}\big) = E_z u\big(Z_{\tau_1}\big) + E_z\big(u(Z_{\tau_2}) - u(Z_{\tau_1})\big)I(N > 1) = u(z).$$

Continuing in this manner we get $E_z u(Z_{\tau_{N\wedge k}}) = u(z)$ for each k, and the bounded convergence theorem gives $E_z u(Z_{\tau_N}) = u(z)$. Now (A) gives $Z_{\tau_N} \to Z_{\tau_R}$ almost surely as $\varepsilon \to 0$, so $u(Z_{\tau_N}) \to u(Z_{\tau_R})$ a.s., and another application of the bounded convergence theorem completes the proof.

This theorem implies immediately that the distribution of Z_{τ_R} under P_z, and harmonic measure on ∂R with respect to the region R and the point z, are the same. Readers not familiar with the concept of harmonic measure may use this fact to give a probabilistic definition. Sometimes probabilistic arguments may be used to find or estimate harmonic measure when other methods do not work. Conversely, Brownian hitting probabilities can be found by using the standard methods to find harmonic measure, as in what follows. Note that, if R in Theorem 2.1 is bounded, property (A) guarantees $P_z(\tau_R < \infty) = 1$.

Let $0 < a < 1 < A < \infty$ and let R be the annulus $\{a < |z| < A\}$, and apply Theorem 2.1 to the function $u(z) = \ln|z|$. Let $p(a, A) = p = P_1(Z_{\tau_R} = a)$ so that $1 - p = P_1(Z_{\tau_R} = A)$. Theorem 2.1 gives

$$0 = \ln(1) = E_1 \ln|Z_{\tau_R}| = p\ln(a) + (1 - p)\ln(A). \tag{2.2}$$

If A is held fixed and $a \to 0$, (2.2) gives $p \to 0$, which proves, since A was arbitrary and Z_t has continuous paths a.s., that the probability Z_t ever hits $\{0\}$ is 0. This proof goes through similarly if Z_t starts at any point except 0, and uses only the existence of a harmonic function on $\mathbb{C} - \{0\}$ which goes to ∞ at 0 and to $-\infty$ at ∞. An analogous function exists for every compact set of capacity 0 ([35], page 77), so the probability that Brownian motion, started at a point outside such a set, ever hits it is 0.

If we hold a fixed and let $A \to \infty$ in (2.2), then $p \to 1$. Thus Brownian motion started at 1 hits $\{|z| = a\}$ with probability 1, for each $a > 0$, and it is easy to argue (using the strong Markov property, which will soon be discussed) that $\{|z| = a\}$ is almost surely visited at arbitrarily large times t, or, to put it another way, the probability that Z_t, $t \geqslant n$, visits $\{|z| = a\}$ is one for each integer n. Any compact set of positive capacity may be handled similarly, giving the following theorem of Kakutani, ([29]), which can be used to give a definition of capacity 0.

THEOREM 2.2. *Let K be a compact subset of $\mathbb{C}$.*
If $\operatorname{Cap}(K) = 0$, $P_z(Z_t \in K$ *for some* $t \geqslant 0) = 0$, $z \notin K$.
If $\operatorname{Cap}(K) > 0$, $P_z(Z_t \in K$ *for arbitrarily large* $t) = 1$, *for all z in* $\mathbb{C}$.

A nonnegative random variable T will be called a Markov time for Z_t if $Z_{T+t} - Z_T$, $t \geqslant 0$ is a standard Brownian motion independent of the σ-field $\sigma(Z_{t\wedge T}, 0 \leqslant t < \infty)$ (the past up to time T). A nonnegative random variable τ is called a stopping time for Z_t if, for each $\lambda > 0$, $\{\tau \leqslant \lambda\}$ is in $\sigma(Z_t, t \leqslant \lambda)$. Thus, if τ is a stopping time, whether stopping has occurred by time λ can be determined by observing Z_t up to time λ. For instance, the first exit time from a region is a stopping time. It can be proved that all stopping times for standard Brownian motion are Markov times (see [4]). Another way to say this is that standard Brownian motion has the strong Markov property.

Now a version of Lévy's theorem will be proved. In its simplest form, which is sufficient for many applications, this theorem says the following.

THEOREM 2.3. *If $f(z)$ is a nonconstant entire function, and Z_t starts at z_0, then $f(Z_t)$, $0 \leqslant t < \infty$, is Brownian motion starting at $f(z_0)$.*

PROOF. Clearly, $f(Z_t)$ has continuous paths a.s., since Z_t does, and they are almost surely unbounded since, by Theorem 2.2, Z_t visits $\{|f(z)| \geqslant \lambda\}$ with probability one. This takes care of (A).

Let $\gamma_1 = \inf\{t > 0 : |f(Z_t) - f(z_0)| = \varepsilon\}$. To show that $f(Z_{\gamma_1})$ is uniformly distributed on $\{|z - f(z_0)| = \varepsilon\}$ it is sufficient to show that, for all functions u harmonic on $S = \{z : |z - f(z_0)| < \varepsilon\}$, and continuous on $\bar{S}$, $E_{z_0}u(f(Z_{\gamma_1})) = u(f(z_0))$. This is an immediate consequence of Theorem 2.1, since, if R is the component of $\{z : |f(z) - f(z_0)| < \varepsilon\}$ containing z_0, $u(f(z))$ is harmonic on R and bounded and continuous on $\bar{R}$, while $\gamma_1 = \inf\{t > 0 : Z_t \in \partial R\}$.

Now if, $\gamma_2 = \inf\{t > \gamma_1 : |f(Z_t) - f(Z_{\gamma_1})| = \varepsilon\}$, $f(Z_{\gamma_2}) - f(Z_{\gamma_1})$ is uniformly distributed on $\{|z| = \varepsilon\}$ and independent of $f(Z_{\gamma_1}) - f(Z_0) = f(Z_{\gamma_1}) - f(z_0)$. This follows immediately from the strong Markov property, the fact that γ_1 is a stopping time for Z_t, and the result of the previous paragraph. An iteration of this argument gives property (B) for $f(Z_t)$.

Several things may now be evident. We have only made use of property (B) in the limit as $\varepsilon \to 0$. In fact, (B) for only very small ε can be shown to imply (B) for all ε. This has the corollary that whether a process is a Brownian motion is largely a local property. Thus, an heuristic proof of Lévy's theorem could be given as follows. Locally, analytic functions are almost like $az + b$. The functions $az + b$ clearly take Brownian motion to Brownian motion, Q.E.D. This argument can be made rigorous, too (see McKean's book [32], page 109).

Now a more complicated version of Lévy's theorem will be stated but not proved. See [32] for a proof. The following notation will be used throughout the paper. If $f(z)$ is analytic and not constant in the unit disc D, and $Z_0 \equiv 0$, define

$$\rho_f(s) = \rho(s) = \textstyle\int_0^s |f'(Z_t)|^2 \, dt, \qquad 0 \leqslant s < \tau_D. \tag{2.3}$$

Since Z_t misses the (countable) zeros of f', ρ is almost surely strictly increasing. Now $f(Z_t)$ is Brownian motion, but perhaps moves locally too fast or slow to be standard Brownian motion. We speed it up or slow it down by changing the time scale. Let

$$W_t = f(Z_{\rho^{-1}(t)}), \qquad 0 \leqslant t < \tau_D. \tag{2.4}$$

Then W_t is locally standard Brownian motion, but it is only defined up to $t = \rho(\tau_D)$, which may or may not be infinite. The process W_t could be talked about as standard Brownian motion up to time $\rho(\tau_D)$, but it is more convenient to define W_t for $t \geqslant \rho(\tau_D)$ so that the whole process W_t, $0 \leqslant t < \infty$, is standard Brownian motion. This is done by defining, on $\{\rho(\tau_D) < \infty\}$,

$$W_{\rho(\tau_D)} = \operatorname{limit}_{t \to \tau_D} W_{\rho(t)}, \tag{2.5}$$

and

$$W_{\rho(\tau_D)+t} = W_{\rho(\tau_D)} + (Z_{\tau_D + t} - Z_{\tau_D}), \qquad t > 0.$$

The limit can be shown to exist almost surely on $\{\rho(\tau_D) < \infty\}$. One form of Lévy's theorem states:

THEOREM 2.4. *The process W_t, $0 \leqslant t < \infty$, is standard Brownian motion.*

Because of the central role of the time $\rho_f(\tau_D)$ in what follows we designate

$$\rho_f(\tau_D) = \nu(f)$$

which will be further shortened to ν if the function f is clear from the context. The distribution of ν is a measure of the size of $f(D)$. For example, if f is univalent and maps D onto R and 0 to 0, then $P(\nu > a)$ is the probability that standard Brownian motion started at 0 takes more time than a to exit from R.

3. Picard's little theorem. This theorem, which states that if f is a nonconstant entire function then the range of f contains every complex number, except perhaps one, is, of course, equivalent to the statement that if a and b are distinct complex numbers then either a or b is in $f(\mathbb{C})$. It involves no loss of generality to assume $a = 1$ and $b = -1$, and we make the further simplifying assumption, which can be easily circumvented, that $f(0) = 0$. What will be shown, then, is that if $f(z)$ is a nonconstant entire function satisfying $f(0) = 0$, then one of the points ± 1 is in $f(\mathbb{C})$. Assume, to the contrary, that $f(\mathbb{C})$ contains neither of these points. Identify the points of $\{z : |z| \leqslant .1\}$, call this set $\hat{0}$, and let $\hat{\mathbb{C}}$ be $\mathbb{C} - \{+1, -1\}$ with the points of $\hat{0}$ identified. Let $\tilde{0}$ be the component of $f^{-1}(\hat{0})$ containing 0, and let $\tilde{\mathbb{C}}$ be $\mathbb{C}$ with the points of $\tilde{0}$ identified. Then f gives a continuous map from $\tilde{\mathbb{C}}$ to $\hat{\mathbb{C}}$, in the usual manner, and a closed curve in $\tilde{\mathbb{C}}$ is mapped to a closed curve in $\hat{\mathbb{C}}$. Clearly any closed curve in $\tilde{\mathbb{C}}$ can be continuously shrunk, while remaining a closed curve, down to a single point, while this is not true of all closed curves in $\hat{\mathbb{C}}$, only those which are not tangled around ± 1 (i.e., those homotopic to 0). Thus the image under f of any closed curve K in $\tilde{\mathbb{C}}$ must be a closed curve in $\hat{\mathbb{C}}$ which is homotopic to 0, since, as K shrinks to a point, so must $f(K)$. A contradiction will be gotten by exhibiting a closed curve in $\tilde{\mathbb{C}}$ with an image not homotopic to 0 in $\hat{\mathbb{C}}$. This curve will be (the projection of) a Brownian motion path.

First we need the following lemma, equivalent to a theorem of Itô and McKean (see [32]), who used the modular function in their proof. A different proof is indicated.

THEOREM 3.1. *There is a time* τ, $P(\tau < \infty) = 1$, *such that* $s \geqslant \tau$ *and* $Z_s \in \hat{0}$ *implies* Z_t, $0 \leqslant t \leqslant s$, *is not homotopic to* 0 *in* $\hat{\mathbf{C}}$.

The tangling of a curve in $\hat{\mathbf{C}}$ can be represented by a word written using the four "letters" a, a^{-1}, b, and b^{-1}, where a and a^{-1} stand respectively for clockwise and counterclockwise loops around 1 and b and b^{-1} serve similarly for -1. The curve in Figure 1 would have $a^{-1}b$ as its word. If this curve is a Brownian motion path, it will get less tangled in the future only if it loops counterclockwise around -1, i.e., unwinds before it does any of the other three possibilities, which would tangle it more. Each of the four possibilities is about equally likely, by symmetry, so the Brownian motion is about three times as likely to become more tangled up as less. By considering times $\gamma_1 < \gamma_2 < \gamma_3 \cdots \to \infty$ such that $Z_{\gamma_i} \in \hat{0}$, and such that Z_t, $0 \leqslant t \leqslant \gamma_{i+1}$, is about three times as likely to be more tangled up in $\hat{\mathbf{C}}$ than Z_t, $0 \leqslant t \leqslant \gamma_i$, and by using a law of large numbers related to the one which says that, if a coin with probability $\frac{3}{4}$ heads and $\frac{1}{4}$ tails is tossed repeatedly, eventually heads become and stay more numerous than tails, Theorem 3.1 can be proved.

Note that although Theorem 3.1 is stated for standard Brownian motion, since it involves only path properties it is also true for Brownian motion.

Now the proof of Picard's theorem can be completed. Let $\tau(f)$ be the τ guaranteed by Theorem 3.1 for the Brownian motion $f(Z_t)$. Since $Z_t \in \tilde{0}$ for arbitrarily large times t by Theorem 2.2, there is a time $\eta > \tau(f)$, $P(\eta < \infty) = 1$, such that $Z_\eta \in \tilde{0}$, which implies $f(Z_\eta) \in \hat{0}$. Any of the paths Z_t, $0 \leqslant t \leqslant \eta$, gives the contradiction. Details of the above can be found in [16].

4. Stoltz angles. The idea to use probability to help prove theorems of the type proved in this section is Doob's, and he worked in a very general setting. His proofs, which were based in part on classical theorems in potential theory, especially those involving the fine topology, have been altered here to make use of recent techniques of Burkholder, Gundy, and Silverstein for dealing probabilistically with Stoltz domains. First, two more or less immediate theorems concerning Brownian motion and analytic functions will be proved, and then the older classical analogues will be stated, together with a sketch of how to get from the former to the latter. If f is nonconstant and analytic in D and H is a closed set then, using the notation introduced at the end of Section 2, $\{W_\nu \in H\} = \{\lim_{t\to\tau_D} f(Z_t) \in H\}$, using (2.4) and (2.5). Theorem 2.2 now immediately gives the following.

THEOREM 4.1. *If f is nonconstant and analytic in D, and K is a compact set of capacity* 0, *then*

$$P\big(\lim_{t\to\tau_D} f(Z_t) \in K\big) = 0.$$

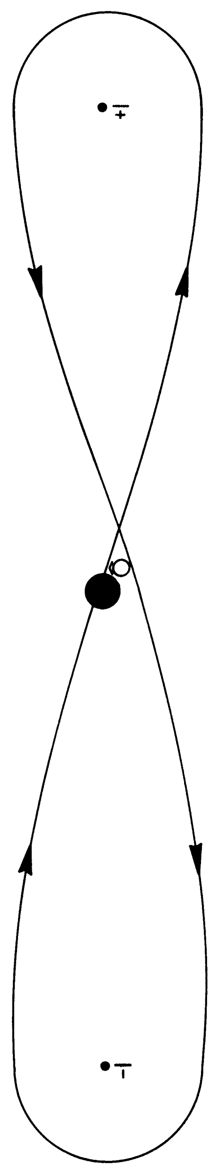

FIG. 1

The other half of Theorem 2.2 implies that, for each number a, almost every path W_t, $a \leqslant t < \infty$, is dense in the plane, since the probability it visits each disc of rational center and radius is one. Thus, the probability W_t, $a \leqslant t < \infty$, is dense in $\mathbb{C}$ for all a is one, which implies that on $\{\nu = \infty\}$, almost all the paths $f(Z_t)$, $\tau_D - (\varepsilon \wedge \tau_D) \leqslant t < \tau_D$, are dense in $\mathbb{C}$ for all $\varepsilon > 0$, since W_t, $0 \leqslant t < \nu$, and $f(Z_t)$, $0 \leqslant t < \tau_D$, traverse the same paths at different speed. On $\{\nu < \infty\}$, $\lim_{t\to\tau_D} f(Z_t)$ exists and equals W_ν almost surely. This gives the following dichotomy.

THEOREM 4.2. *With probability* 1, *either* $\lim_{t\to\tau_D} f(Z_t)$ *exists or* $f(Z_t)$, $\tau_D - (\varepsilon \wedge \tau_D) < t < \tau_D$, *is dense in the plane for each* $\varepsilon > 0$.

The classical analogues of these theorems involve Stoltz domains, which will now be defined. For each $e^{i\theta}$, and each α between 0 and 1, define the Stoltz domain $S_\alpha(\theta)$ to be the interior of the smallest convex set containing the disc $\{|z| \leqslant \alpha\}$ and the point $e^{i\theta}$ (see Figure 2). If A is a subset of ∂D, define $S_\alpha(A) = \cup_{e^{i\theta}\in A} S_\alpha(\theta)$. These Stoltz domains often have sawtooth-like boundaries. The following theorem is implicit in [20], although the proof is based on a result of Naím about fine limits. A probabilistic proof is implicit in the last half of Burkholder's paper, [6].

THEOREM 4.3. *For each fixed* α, $0 < \alpha < 1$, *and each Borel set* A, $P(Z_t \varepsilon S_\alpha(A)$, $\tau_D - (\varepsilon \wedge \tau_D) < t < \tau_D$, *and* $Z_{\tau_D} \in A) \to P(Z_{\tau_D} \in A)$ *as* $\varepsilon \to 0$.

This theorem is, of course, vacuous if $l(A) = 2\pi P(Z_{\tau_D} \in A) = 0$, but, if $P(A) > 0$, it says that almost all Brownian paths which hit points in A get in and stay in $S_\alpha(A)$ before they hit. It will not be proved, but note that it is immediate if A is open.

A function g, analytic or not, defined in D, is said to have a nontangental limit at $e^{i\theta}$ if

$$\lim_{z\to e^{i\theta};\ z\in S_\alpha(\theta)} g(z) \qquad \text{exists for all} \quad \alpha, \quad 0 < \alpha < 1.$$

Using Theorem 4.3 it is easy to show that, if g is any function with a nontangental limit $\hat{g}(e^{i\theta})$ on a Borel set $B \subset \partial D$,

$$P\big(\lim_{t\to\tau_D} g(Z_t) = \hat{g}(Z_{\tau_D}), Z_{\tau_D} \in B\big) = P(Z_{\tau_D} \in B). \tag{4.1}$$

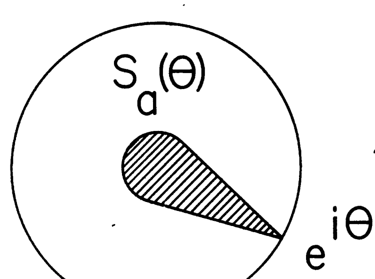

FIG. 2

This equality and Theorem 4.1 now give the following theorem of Privalov.

THEOREM 4.4. *If f is nonconstant and analytic in D, and K is a compact set of capacity 0, then the Lebesgue measure of the set of ∂D where f has nontangental limit in K is 0.*

Now let θ be fixed. The concept of Brownian motion Z_t conditioned so that $Z_{\tau_D} = e^{i\theta}$ is intuitive, but difficult to make rigorous since $P(Z_{\tau_D} = e^{i\theta}) = 0$. Nonetheless this was done by Doob in [19] and the resulting process will be denoted Z_t^θ, $0 \leqslant t \leqslant \tau_D$. Here, Z_0^θ will always be 0. As you would expect, if E a Borel set in D such that $e^{i\theta} \notin \overline{E}$, and $\Gamma_\varepsilon = \{e^{i\varphi} : \theta - \varepsilon \leqslant \varphi \leqslant \theta + \varepsilon\}$,

$$(4.2)\quad P(Z_t^\theta \in E \quad \text{for some} \quad t, 0 \leqslant t \leqslant \tau_D) = \lim_{\varepsilon\to 0} P(Z_t \in E \text{ for some } \quad t, 0 \leqslant t \leqslant \tau_D, \text{ and } \quad Z_{\tau_D} \in \Gamma_\varepsilon)/P(Z_{\tau_D} \in \Gamma_\varepsilon),$$

and the probabilities of other events for Z_t^θ are similarly computed. (If $e^{i\theta} \in \overline{E}$, the limit may not exist.) Conditional Brownian motion is nice enough to be used in the following way. If E is a Borel subset of D,

$$(4.3)\quad P(Z_t \in E \quad \text{for some} \quad t, 0 \leqslant t \leqslant \tau_D) = \textstyle\int_0^{2\pi} P(Z_t^\theta \in E \quad \text{for some} \quad t, 0 \leqslant t \leqslant \tau_D)\, d\theta/2\pi.$$

(If $\overline{E} \cap \partial D = \varnothing$, (4.3) can be derived directly from (4.2) without much difficulty).

A fundamental connection between conditional Brownian motion and Stoltz angles is the following theorem, essentially proved in [12].

THEOREM 4.5. *For each σ, $0 < \sigma < 1$, there is a constant $a(\sigma) > 0$ such that, for all $z \in S_\sigma(\theta)$,*

$$P(Z_t^\theta, 0 \leqslant t < \tau_D, \text{ contains a closed loop around } z) \geqslant a(\sigma).$$

If F is a set such that $D - F$ is simply connected, a closed loop in D which wraps around a point in F intersects F, so if $F \cap S_\sigma(\theta)$ contains points $z_n \to e^{i\theta}$, and if A_n is the event that Z_t^θ, $0 \leqslant t < \tau_D$, contains a closed loop around z_n, then $P(\limsup A_n) \geqslant \limsup P(A_n) \geqslant a_\sigma$. From this it can be concluded, since Z_t^θ, $0 \leqslant t \leqslant \tau_D$, is a curve which first hits ∂D at τ_D, that

$$(4.4)\quad P(Z_t^\theta, 0 \leqslant t < \tau_D, \text{ hits } F \text{ at times arbitrarily close to } \tau_D) \geqslant a(\sigma),$$

and a $0 - 1$ law ([3], page 30), applied to the reversed process, gives

$$(4.5)\quad P(Z_t^\theta, 0 \leqslant t < \tau_D, \text{ hits } F \text{ at times arbitrarily close to } \tau_D) = 1.$$

Now let u be a harmonic function, and define $u_\sigma^+(e^{i\theta}) = \limsup_{z\to e^{i\theta};\, z\in S_\sigma(\theta)} u(z)$. Applying (4.5) and the maximum principle to the sets $\{u > \lambda\}$ gives

$$P(\limsup_{t\to\tau_D} u(Z_t^\theta) \geqslant u_\sigma^+(e^{i\theta})) = 1,$$

and integrating with respect to $d\theta$ gives

$$(4.6)\quad P(\limsup_{t\to\tau_D} u(Z_t) \geqslant u_\sigma^+(Z_{\tau_D})) = 1.$$

There is a corresponding result for lim inf.

Now let $T_\sigma = \{e^{i\theta} : \lim_{z\to e^{i\theta};\, z\in S_\sigma(\theta)} f(z)$ does not exist$\}$. Either the limit of the real or imaginary part of f fails to exist as $z \to e^{i\theta}$ in $S_\sigma(\theta)$, $e^{i\theta} \in T_\sigma$, so that (4.6) and its analogue for lim inf give

$$P\left(\lim_{t\to\tau_D} f(Z_t) \quad \text{exists}, Z_{\tau_D} \in T_\sigma\right) = 0,$$

which implies

$$P\left(\nu = \infty, Z_{\tau_D} \in T_\sigma\right) = P\left(Z_{\tau_D} \in T_\sigma\right) = l(T_\sigma)/2\pi, \tag{4.7}$$

while (4.1) gives

$$P\left(\nu < \infty, Z_{\tau_D} \notin T_\sigma\right) = P\left(Z_{\tau_D} \notin T_\sigma\right). \tag{4.8}$$

Equation (4.7), Theorem 4.3, and the second sentence before the statement of Theorem 4.2, give that, if A is any Borel subset of T_σ, $l(A) > 0$, then $f(S_\sigma(A))$ is dense in $\mathbb{C}$, which implies that $f(S_\sigma(\theta))$ is dense in $\mathbb{C}$ for almost every $\theta \in T_\sigma$. Furthermore, (4.7) and (4.8) imply $l(T_\sigma)$ is the same $(2\pi P(\nu = \infty))$ for all σ. Thus the following theorem of Plessner holds.

THEOREM 4.6. *Except for a set of θ of Lebesgue measure* 0, *either f has a nontangental limit at $e^{i\theta}$ or $f(S_\alpha(\theta))$ is dense in $\mathbb{C}$ for each α, $0 < \alpha < 1$.*

5. Hardy spaces. The probabilistic treatment of H^p spaces, by D. L. Burkholder, R. F. Gundy, and M. L. Silverstein, was made possible by Burkholder and Gundy's proof, a few years earlier, of a theorem about standard *one* dimensional Brownian motion. If $Z_t = X_t + iY_t$ is standard two dimensional Brownian motion then X_t and Y_t are independent standard one dimensional Brownian motions. This connection is usually used to define two dimensional Brownian motion, in fact. The definition of Markov time for standard one dimensional Brownian motion parallels the definition of Markov time given in Section 2, and if τ is a Markov time for Z_t then it is for X_t and Y_t also. When $P(\tau = \infty) > 0$, we say that τ is a Markov time if $\tau \wedge n$ is a Markov time for each integer n. It follows from the construction of Section 2 that ν is a Markov time for W_t, and thus for Re W_t and Im W_t. If A_t, $0 < t < \infty$, is any stochastic process, define the new process A_t^*, $0 \leqslant t < \infty$, by $A_t^* = \sup_{0\leqslant s\leqslant t}|A_t|$. In [11], Burkholder and Gundy prove that, if Γ_t, $0 \leqslant t < \infty$, is standard one dimensional Brownian motion satisfying $\Gamma_0 = 0$, then, for each $p > 0$, there are positive constants c_p and C_p such that

$$c_p E\tau^{p/2} \leqslant E\Gamma_\tau^{*p} \leqslant C_p E\tau^{p/2} \tag{5.1}$$

for all Markov times τ.

Let $\mathfrak{F}$ be the class of all functions $f = u + iv$, analytic in D, which satisfy $f(0) = 0$. If $A_p(f)$ and $B_p(f)$ are two quantities associated with the functions $f \in \mathfrak{F}$, we write $A_p \approx B_p$ if these exist positive constants k_p and K_p, not depending on f, such that $k_pA_p \leqslant B_p \leqslant K_pA_p$, $0 < p < \infty$. The relation $\approx$ is clearly an equivalence relation. An application of (5.1) to Re W_t, $t \geqslant 0$, gives

$$E\nu^{p/2} \approx E\,\text{Re}\, W_\nu^{*p} = Eu\left(Z_{\tau_D}\right)^{*p}, \tag{5.2}$$

and similarly we have

(5.3) $$Ev^{p/2} \approx Ev(Z_{\tau_D})^{*p},$$

which together give

(5.4) $$Eu(Z_{\tau_D})^{*p} \approx Ev(Z_{\tau_D})^{*p}.$$

Now define, for a set $F \subset D$, the sets $A(F) = \{Z_t \in F \text{ for some } t < \tau_D\}$ and $N_\sigma(F) = \{\theta : S_\sigma(\theta) \cap F\} \neq \varnothing$. It is proved in [12], in a slightly different form, that for each σ, $0 < \sigma < 1$, there are positive constants c_σ and C_σ such that, if F is a Borel set and $D - F$ is simply connected,

(5.5) $$c_\sigma P(A(F)) \leqslant l(N_\sigma(F)) \leqslant C_\sigma P(A(F)).$$

(We may take C_σ^{-1} to be the $a(\sigma)$ of Theorem 4.5.) Note that if F is closed, $P(A(F))$ is the harmonic measure of F relative to $D - F$. An application of (5.5) to the sets $\{|u| > \lambda\}$ yields the beautiful inequalities

(5.6) $$c_\sigma P(u^* > \lambda) \leqslant l(N_\sigma(u) > \lambda) \leqslant C_\sigma P(u^* > \lambda), \qquad \lambda > 0,$$

where $N_\sigma(u)(\theta) = \sup_{z \in S_\sigma(\theta)} |u(z)|$ is the nontangental maximal function of u. Integrating (5.6) times λ^{p-1} gives

(5.7) $$Eu(Z_{\tau_D})^{*p} \approx \textstyle\int_0^{2\pi} N_\sigma(u)(\theta)^p d\theta, \qquad \sigma \text{ fixed}.$$

The equivalent result holds for v, which, when combined with (5.7) and (5.4) yields

(5.8) $$\textstyle\int_0^{2\pi} N_\sigma(u)^p(\theta) d\theta \approx \int_0^{2\pi} N_\sigma(v)^p(\theta) d\theta, \qquad \sigma \text{ fixed}.$$

This is a major theorem of [12]. Together with previously existing results, (5.8) made it possible to prove

(5.9) $$\textstyle\int_0^{2\pi} N_\sigma(u)^p(\theta) d\theta \approx \|f\|_{H^p}^p = \lim_{r \to 1} (2\pi)^{-1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta, \qquad \sigma \text{ fixed},$$

where $\|f\|_{H^p}$ is the H^p norm of f.

The relationship (5.9) can be used to give a number of quantities that are equivalent to the H^p norm for $f \in \mathfrak{F}$. Perhaps the most useful is to combine (5.9) with (5.2) and (5.7) to get

$$\|f\|_{H^p}^p \approx Ev^{p/2}.$$

In particular, $f \in H^p$ if and only if $Ev^{p/2} < \infty$. For many purposes $Ev^{p/2}$ is a simpler quantity to work with than $\|f\|_{H^p}$, as D. L. Burkholder has recently shown. We illustrate by sketching a proof of the following theorem, first proved by Burkholder in [8], although implicit in the work of Lowell Hansen, [26], if f is univalent.

THEOREM 5.1. *Let f and g be analytic in D with $f(0) = g(0) = 0$. Suppose $g \in H^p$ and that $\lim_{r \to 1} g(re^{i\theta})$ exists and belongs to $\mathbb{C} - f(D)$ for almost every θ. Then $f \in H^p$.*

The hypotheses of this theorem, together with equation (4.5) (here F will be a segment $\{re^{i\theta}, R < r < 1\}$), imply there is a random variable $s < \tau_D$, such that

(5.10) $$P(g(Z_s) \notin f(D)) = 1.$$

Let $\eta = \inf\{t > 0 : W_t(g) \notin f(D)\}$. Then (5.10) implies $P(\eta < \nu(g)) = 1$. Let $\gamma = \inf\{t > 0 : W_t(f) \notin f(D)\}$. Clearly $\gamma \geqslant \nu(f)$. Now γ and η have the same distribution, since both are the first exit time of a standard Brownian motion started at 0 from the set $f(D)$. Thus

$$E\nu(f)^{p/2} \leqslant E\gamma^{p/2} = E\eta^{p/2} \leqslant E\nu(g)^{p/2}.$$

Since $g \in H^p$, $E\nu(g)^{p/2} < \infty$, so $E\nu(f)^{p/2} < \infty$ and thus $f \in H^p$, completing the proof.

Albert Baernstein II has recently shown, without probability, that the hypotheses of Theorem 5.1 imply $\|f\|_{H^p} \leqslant \|g\|_{H^p}$ (see [8]). There are versions of the above formulas for functions which do not vanish at 0, as well as for functions analytic in a half plane, in [12]. For related recent applications of probability to analytic functions, see [5], [7], [9] and [10].

6. Conjugate function inequalities. Let $f = u + iv$ be analytic in D, continuous in $\bar{D}$, and satisfy $f(0) = 0$. If U and $\tilde{U}$ denote, respectively, the restrictions of u and v to the boundary of D, it is easily seen that U determines $\tilde{U}$ completely, and $\tilde{U}$ is called the conjugate function of U. This map can be extended so that the conjugate function of any totally finite signed measure is defined in a way compatible with the definition just given, but here we will work with U which are as just above, that is, boundary values of functions analytic in D, vanishing at 0, and continuous in $\bar{D}$. The collection of all functions on ∂D of this type will be denoted $\mathcal{Q}$. In a number of senses, the distribution of $\tilde{U}$ cannot be too much greater than that of U. An example is the following theorem of Kolmogorov. For a number of related theorems, see [36].

THEOREM 6.1. *For each p, $0 < p < 1$, there are positive constants K_p such that*

$$\left(\textstyle\int_0^{2\pi} |\tilde{U}(e^{i\theta})|^p d\theta\right)^{1/p} \leqslant K_p \textstyle\int_0^{2\pi} |U(e^{i\theta})| d\theta, \qquad U \in \mathcal{Q}. \tag{6.1}$$

Note that if we define

$$\sup_{U \not\equiv 0;\, U \in \mathcal{Q}} \left(\textstyle\int_0^{2\pi} |\tilde{U}(e^{i\theta})|^p d\theta\right)^{1/p} \Big/ \textstyle\int_0^{2\pi} |U(e^{i\theta})| d\theta = C_p, \tag{6.2}$$

then Theorem 6.1 is equivalent to $C_p < \infty$.

Since Z_{τ_D} is uniformly distributed on ∂D,

$$\frac{1}{2\pi} \textstyle\int_0^{2\pi} |\tilde{U}(e^{i\theta})|^p d\theta = E|v(Z_{\tau_D})|^p = E|\mathrm{Im}\, W_\nu|^p, \tag{6.3}$$

and

$$\frac{1}{2\pi} \textstyle\int_0^{2\pi} |U(e^{i\theta})| d\theta = E|\mathrm{Re}\, W_\nu| = \lim_{t\to\infty} E|\mathrm{Re}\, W_{\nu\wedge t}|, \tag{6.4}$$

the last equality by the bounded convergence theorem. Also $W_0 = 0$, by our simplifying assumption.

Let $\mathfrak{M}$ be the class of all Markov times for W_t. In view of (6.3) and (6.4), if

$$\sup_{\eta \in \mathfrak{M};\, \eta \not\equiv 0} \frac{(E|\mathrm{Im}\, W_\eta|^p)^{1/p}}{\lim_{t\to\infty} E|\mathrm{Re}\, W_{\eta\wedge t}|} = C_p', \tag{6.5}$$

then $C_p' \geqslant C_p$, so if C_p' is finite then C_p is. (Since W_t is connected in some way with f, it is perhaps conceivable that C_p' depends on f. In reality, C_p' depends only on the fact that W_t is Brownian motion started at 0). A natural way to prove $C_p' < \infty$ is to find the $\eta = \mu$ which maximizes the ratio (6.5), and then to calculate

$$(E|\mathrm{Im}\ W_\mu|^p)^{1/p}/\lim_{t\to\infty} E|\mathrm{Re}\ W_{\mu\wedge t}|, \tag{6.6}$$

which will, of course, be C_p'. A solution is $\mu = \inf\{t > 0 : |\mathrm{Im}\ W_t| \geqslant 1, \mathrm{Re}\ W_t = 0\}$. This will not be proved here. It happens that C_p' is not only finite, but also $C_p' = C_p$ which is, of course, the best possible value for K_p in Theorem 6.1. The examples necessary to show this are associated with the standard analytic function $g(z) = 2z/(1 - z^2)$ mapping D onto $\mathbb{C} - \{z : |\mathrm{Im}\ z| \geqslant 1, \mathrm{Re}\ z = 0\}$, which is not surprising, since $\nu(g) = \mu$. The constant is $C_p = ((2\pi)^{-1}\int_0^{2\pi}|\mathrm{Im}\ g(e^{i\theta})|^p d\theta)^{1/p}$.

With a little more work it can be shown that this is also the best constant in the analogue of Theorem 6.1 for conjugate functions of arbitrary signed measures (see [17]). This method is applied to other problems in [14] and [15].

7. Covering properties of analytic functions. The first part of this section will be concerned with theorems related to Hayman's generalization of Koebe's theorem (see [35], page 85). If f is analytic in D let E_f be the set of those real numbers $r \geqslant 0$ such that the circle of radius r around 0 lies entirely in $f(D)$. Hayman proved that $f(0) = 0$ and $f'(0) = 1$ imply $l(E_f) \geqslant \frac{1}{4}$. The Koebe function shows that $\frac{1}{4}$ is the best possible constant. We are not able to prove this sharp result with the method given here, but can prove it with a smaller constant in place of $\frac{1}{4}$. Furthermore, the condition $f'(0) = 1$ can be relaxed. In what follows, C will stand for an absolute constant. The value of C may be different in different theorems. Numerical values for C could be found only with considerable complication of the arguments, so this is not done. Let $A_{\sigma,f}(\theta) = A(\theta) = \int\int_{S_\sigma(\theta)}|f'(z)|^2$ be the classical area function of f. Here $S_\sigma(\theta)$ is the Stoltz domain defined in Section 4, and integration is with respect to area, so that $A(\theta)$ is the area of $f(S_\sigma(\theta))$ counting multiplicity (see [36], page 290). The area function of z^n approaches a positive constant as $n \to \infty$. We prove

THEOREM 7.1. *There is a constant $C(\varepsilon, \sigma) = C$ such that if f is analytic in D, vanishes at 0, and satisfies $l\{\theta : A(\theta) > \varepsilon\} > \varepsilon$, then $l(E_f) > C$.*

If $f'(0) = 1$ then $A(\theta) \geqslant \int\int_{\{|z|<\sigma\}}|f'(z)|^2 \geqslant \pi\sigma^2$, so that Theorem 7.1 implies a weak version of Hayman's theorem with any of the constants $\mathbb{C}(\pi\sigma^2, \sigma)$ in place of $\frac{1}{4}$.

Theorem 7.1 follows from the following two theorems.

THEOREM 7.2. *There is a constant $C(\varepsilon) = C > 0$ such that, if f is analytic in D and satisfies $f(0) = 0$ and $P(\nu > \varepsilon) > \varepsilon$, then $l(E_f) > C$.*

THEOREM 7.3. *There is a constant $C(\sigma, \varepsilon) = C > 0$ such that, if f is as above and $l(\theta : A(\theta) > \varepsilon) > \varepsilon$, then $P(\nu > C) > C$.*

Theorem 7.3. follows from Theorem 2 of [13]. (The area function in [13] is wrongly defined, by the way.)

Now Theorem 7.2 will be proved. Since W_t is standard Brownian motion and ν is $\inf\{t > 0 : W_t \notin f(D)\}$, Theorem 7.2 is a consequence of the following lemma. If F is a closed subset of $\mathbb{C}$ let $E(F)$ be the set of all those $r \geqslant 0$ such that the circle of radius r is completely contained in $E(F)$.

LEMMA 7.4. *There is a $C(\varepsilon) = C > 0$ such that, if F is a closed subset of $\mathbb{C}$ and $l(E(F)) < C$, then*

$$P(Z_t \in F \quad \text{for some} \quad t, 0 \leqslant t \leqslant \varepsilon) > 1 - \varepsilon.$$

PROOF. Let $\tau_r = \inf\{t > 0 : |Z_t| = r\}$. Pick s so small that

$$P(\tau_s < \varepsilon) > 1 - (\varepsilon/2). \tag{7.1}$$

Let $A = F \cap \{z \leqslant s\}$. It will be shown that, for some $\delta > 0$,

$$P(Z_t \in A \quad \text{for some} \quad t, 0 \leqslant t < \tau_s) > 1 - (\varepsilon/2) \qquad \text{if} \quad l(E(A)) < \delta. \tag{7.2}$$

This δ will suffice for C in Lemma 7.4, by (7.1). Now

$$P(Z_t \in A \quad \text{for some} \quad t < \tau_s) \geqslant P(Z_t \in E(A) \quad \text{for some} \quad t < \tau_s). \tag{7.3}$$

This equation follows from probabilistic arguments of Haliste [25]. For a different proof see Baernstein [1]. Note that the probabilities in (7.3) are the harmonic measures of A and $E(A)$ with respect to the point 0 and the regions $\{|z| < s\} - A$ and $\{|z| < s\} - E(A)$ respectively. Thus it suffices to prove (7.2) with $E(A)$ in place of A. Let Q be the region $\{|z| < s\} - [0, s)$. Let b be so small that, if $\eta = \inf\{t > \tau_b : Z_t \in \partial Q\}$, then

$$P(Z_\eta \in [0, s)) > 1 - (\varepsilon/4). \tag{7.4}$$

That such an η exists follows from the statement at the top of the page 60 of [21]. It is easily shown that the distribution of Z_η on ∂Q is absolutely continuous with respect to Lebesgue measure, either probabilistically or using the interpretation of this distribution as an average of harmonic measure. Thus, since $P(Z_t \in E(A)$ for some $t < \tau_s) \leqslant P(Z_\eta \in E(A))$, the existence of a δ such that (7.2) holds with $E(A)$ in place of A follows, completing the proof of Lemma 7.4 and thereby that of Theorem 7.2.

To conclude this section the following theorem will be proved.

THEOREM 7.5. *Let $h(z)$ be analytic in $|z| > A$ for some number $A > 0$. Suppose*

(1) $\lim_{r\to\infty}\{\max_{|z|=r}|h'(z)|\} = \infty$

and

(2) $\limsup_{|z|\to\infty}|1/h(z)| = \infty$.

Then either $h(z) = +1$ for z of arbitrarily large magnitude or $h(z) = -1$ for z of arbitrarily large magnitude, or both.

Picard's big theorem (see [1]) is easily derived from Theorem 7.5, since if (1) does not hold the maximum principle gives that $h'(z)$ is bounded in $|z| > 2A$ and thus has a removable singularity at infinity, while, if (2) doesn't hold, $1/h(z)$ is bounded in $\{|z| > B\}$ for some B, and thus has a removable singularity at ∞, so if either (1) or (2) fails to hold, h has a nonessential singularity at ∞.

First several lemmas will be stated. The first of these is an extension of Theorem 7.1, and is a little more general than equation 10 of [13]. It can be proved in a similar way.

Let rD be the disc of radius r around 0, let $\tau(r) = \inf\{t : |Z_t| = r\}$, (so that $\tau(1) = \tau_D$), let f be analytic in rD, define ρ as in (2.3), and let $M(r) = \max_{|z|\leqslant r}|f'(z)|$. Then

LEMMA 7.6. *For each s, $0 < s < 1$, there is a positive constant $C(s)$, which decreases as s increases, such that*

$$p\varepsilon^{\frac{1}{2}}\rho(\tau(\alpha)) > C(s)M(s\alpha)^2\alpha^2) > C(s).$$

The statement of the next lemma is related to Theorem 3.1. Although Z_t started at 0 in that theorem this was done only to simplify notation, and, in fact, Z_t could have started at any point in $\hat{0}$. The proof of the following lemma follows from the proof of Theorem 3.1 of [16].

LEMMA 7.7. *If $Z_0 = z \in \hat{0}$, there is a time τ, $P_z(\tau < \infty) = 1$, such that $s > \tau$ and $Z_s \in \hat{0}$ implies Z_s, $0 \leqslant t \leqslant s$, is not homotopic to 0 in $\hat{\mathbb{C}}$. Furthermore, there is a function $\lambda(x)$, which does not depend on $z \in \hat{0}$, satisfying $\lambda(x) \to 1$ as $x \to \infty$, such that*

$$P_z(\tau \leqslant x) > \lambda(x).$$

Next the following weak form of the Picard-Schottky theorem (see [1]) is proved.

LEMMA 7.8. *For each r, $0 < r < 1$, there is a positive constant $K(r)$, such that $M(r) > K(r)$ and $|f(0)| \leqslant \frac{1}{20}$ implies $f(D)$ contains either $+1$ or -1 or both.*

PROOF. Let r be fixed. Either Lemma 7.6 or Schwarz's lemma guarantees that, if $M(r)$ is large enough, then $f(D_{(r)^{\frac{1}{2}}})$ is not contained in $\hat{0}$, and we assume that $M(r)$ is this large. Let D_x be the largest disc with center 0 such that $f(D_x) \subset \hat{0}$.

Let $T = \inf\{t > x^{\frac{1}{2}}: Z_t = 1 \text{ or } x\}$, and let $S = \inf\{t > \tau(r^{\frac{1}{4}}) : Z_t = 1 \text{ or } x\}$. The strong Markov property and an equation like (2.2) give

$$P\big(|Z_T| = x|Z_t, 0 \leqslant t \leqslant \tau(x^{\frac{1}{2}})\big) = \tfrac{1}{2}. \tag{7.5}$$

Since $x < r^{\frac{1}{2}}$, $x^{\frac{1}{2}} < r^{\frac{1}{4}}$, so $C(\quad) \leqslant C(r^{\frac{1}{4}})$, and Lemma 7.3 with $s = \alpha = x^{\frac{1}{2}}$ implies

$$P\Big(\rho\big(\tau(x^{\frac{1}{2}})\big) > C(r^{\frac{1}{4}})M(x)^2x\Big) > C(r^{\frac{1}{4}}). \tag{7.6}$$

By the mean value theorem, $xM(x) \leqslant (\frac{1}{10}) - (\frac{1}{20})$, so, since $\rho(T) \geqslant \rho(\tau(x^{\frac{1}{2}}\ ))$, (7.5) and (7.6) give

$$P\big(\rho(T) > C(r^{\frac{1}{4}})(400\,x)^{-1} \quad \text{and} \quad Z_T \in D_x\big) > C(r^{\frac{1}{4}})/2. \tag{7.7}$$

Also,

$$P\big(|Z_S| = x|Z_t, 0 \leqslant t \leqslant \tau(r^{\frac{1}{4}})\big) = \ln(r)/4\ln(x), \tag{7.8}$$

and an argument similar to the one which gave (7.7) gives

(7.9) $P\big(\rho(S) > C(r^{\frac{3}{4}})M(r)^2 r^{\frac{1}{2}} \quad \text{and} \quad Z_S \in D_x\big) > C(r^{\frac{3}{4}})\ln(r)/4\ln(x).$

From (7.7) and (7.9) it can be deduced that, if $M(r)$ is large enough, then either

(7.10) $P(\rho(S) > y \quad \text{and} \quad Z_S \in D_x) > 1 - \lambda(y), \quad \text{for some } y,$

or

(7.11) $P(\rho(T) > y \quad \text{and} \quad Z_T \in D_x) > 1 - \lambda(y), \quad \text{for some } y,$

where $\lambda(y)$ is as in Lemma 7.7. (If x is very small, (7.10) holds, while if x and $M(r)$ are large then (7.11) will hold).

Now (7.10) guarantees $P(\rho(S) > \tau, Z_S \in D_x) > 0$, and (7.11) guarantees $P(\rho(T) > \tau, Z_T \in D_x) > 0$, τ as in Lemma 7.7, so, if $M(r)$ is large enough, either $+1$ or -1 or both are in $f(D)$, by the argument of Section 3.

Now let Γ be the region $\{z : \frac{1}{2} < |z| < 2 \text{ and } -\pi/4 < \operatorname{Arg} z < 3\pi/2\}$. The following lemma can be proved in a manner similar to the proof of the last lemma, or can be proved using this lemma and the existence of a univalent analytic function mapping Γ onto D.

LEMMA 7.9. *Let g be analytic in Γ and let $|g(1)| \leqslant \frac{1}{20}$. Let $\Gamma_\varepsilon = \{z \in \Gamma :$ distance $(z, \partial\Gamma) > \varepsilon\}$. There is a constant $\Theta(\varepsilon) > 0$, such that $|g'(\varepsilon)| > \Theta(\varepsilon)$ for $z \in \Gamma_\varepsilon$ implies that $g(\Gamma)$ contains either $+1$ or -1 or both.*

Now the proof of Theorem 7.5 can be completed. Let z_1 and z_2 be two points on $\{z : |z| = R\}$, $R/2 \geqslant A$, such that $|h(z_1)| \leqslant \frac{1}{20}$ and $|h'(z_2)| > \Theta(.01)$. Such z_1, z_2 occur for arbitrarily large R. Both $h(z_1 z) = g_1(z)$ and $h(z_1/z) = g_2(z)$ are analytic in Γ and satisfy $|g_i(1)| \leqslant \frac{1}{20}$. Furthermore, $|g_1'(z_2/z_1)| = R|h'(z_2)| = |g_2'(z_1/z_2)|$, and one of z_1/z_2, z_2/z_1 is in Γ and at least a distance of .01 from the boundary of Γ. Thus Lemma 7.9 guarantees that one of $g_1(\Gamma)$, $g_2(\Gamma)$ contains either -1 or 1, which implies that h takes on either -1 or 1 in $R/2 < |z| < 2R$. Since this is true for arbitrarily large R, h takes on either -1 or 1 infinitely often.

8. Concluding remarks. As was mentioned in the introduction, some probabilistic expressions have arisen in the preceding sections which are directly translatable to more conventional expressions, for example, $E|W_\nu| = \int_0^{2\pi} |f(e^{i\theta})| d\theta/2\pi$, while others cannot be so handled. The most interesting of these, to me, is the time $\nu(f)$. The distribution of $\nu(f)$ is an intuitively appealing measure of the size of $f(D)$, since if f is univalent then ν is the first exit time of standard Brownian motion from $f(D)$ and, even if f is not univalent, a similar interpretation can be given using Riemann surfaces. In the preceding sections there are a number of instances where information about the size of $\nu(f)$ translates easily into more conventional information about the size of $f(D)$. However, little of a precise nature is known about ν. For example, it is not even known in what sense, with regard to $\nu(f)$, z is the smallest analytic function in D satisfying $|f'(0)| = 1$. It would be nice if

(8.1) $$P(\nu(z) \geqslant \lambda) \leqslant P(\nu(f) \geqslant \lambda), \qquad \lambda > 0,$$

for all such f, and it is probably true that

$$E\Phi(\nu(z)) \leqslant E\Phi(\nu(f)), \tag{8.2}$$

for all increasing and convex functions Φ on $[0, \infty)$, but neither of these results is known (although it is known that (8.2) holds if $\Phi(x) = x$; see equation (2) on page 309 of [27]). In the other direction, it can be asked in what sense, with regard to ν, the Koebe function $z/(1-z)^2$ is the largest univalent function in D such that $|f'(0)| = 1$. The reader interested in these questions should see Burkholder, [7].

The subject of Brownian motion on Riemann surface is outside the scope of this paper, but it has been successfully defined, and, since the analogue of Lévy's theorem holds, can be used to study analytic functions on Riemann surfaces. Many of the arguments in Section 4 were adapted from proofs for such functions. Now on some surfaces, for example the plane, Brownian motion is recurrent, that is, returns to each neighborhood of its starting point at arbitrarily large times, while Brownian motion on other surfaces, for example an open disc, does not have this property. If there is an analytic function mapping one Riemann surface onto another, then, by Lévy's theorem, Brownian motion is recurrent on both or neither. Kakutani [29] used such ideas to study when there exist such mappings. See McKean's book [32] for more about Brownian motion on Riemann surfaces. It is in this context that a number of interesting applications of function theory to Brownian motion occur.

The whole subject of conformal invariance has been studied in a more abstract setting in [24]. Finally, we remark that there are uses of Brownian motion in analysis other than those mentioned here. See, for examples, J. -P. Kahane's paper [28].

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