The Annals of Probability
1980, Vol. 8, No. 2, 298–313

CONVERGENCE RATES FOR PROBABILITIES OF MODERATE DEVIATIONS FOR SUMS OF RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES

BY ALLAN GUT

Uppsala University

For a set of i.i.d. random variables indexed by Z_+^d, $d \geqslant 1$, the positive integer d-dimensional lattice points, convergence rates for moderate deviations are derived, i.e., the rate of convergence to zero of, for example, certain tail probabilities of the partial sums, are determined. As an application we obtain results on the integrability of last exit times (in a certain sense) and the number of boundary crossings of the partial sums.

1. Introduction. Let Z_+^d, $d \geqslant 1$, be the positive integer d-dimensional lattice points with coordinate-wise partial ordering, $\prec$. Points in Z_+^d are denoted by $\mathbf{m}$, $\mathbf{n}$ etc. Also, for $\mathbf{n} = (n_1, n_2, \cdots, n_d)$, we define $|\mathbf{n}| = \prod_{i=1}^d n_i$ and $\mathbf{n} \to \infty$ is interpreted as $n_i \to \infty$, $i = 1, 2, \cdots, d$.

Throughout the paper X and $\{X_\mathbf{n}; \mathbf{n} \in Z_+^d\}$ are i.i.d. random variables and $S_\mathbf{n} = \sum_{\mathbf{k} \prec \mathbf{n}} X_\mathbf{k}$. In [9] we extended results of Baum and Katz [1] to $d \geqslant 2$. As a typical example we proved that, for $\alpha r \geqslant 1$, $\alpha > \frac{1}{2}$, $\sum_\mathbf{n} |\mathbf{n}|^{\alpha r - 2} \cdot P(|S_\mathbf{n}| \geqslant |\mathbf{n}|^\alpha \cdot \varepsilon) < \infty$ if and only if $E|X|^r \cdot (\lg|X|)^{d-1} < \infty$ and, if $r \geqslant 1$, $EX = 0$.

For $\alpha = \frac{1}{2}$ the sum cannot converge in view of the central limit theorem and for this case $|\mathbf{n}|^\alpha$ has to be replaced by $(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}$, where $\lg x = \max\{1, \log x\}$. (Similarly, $\lg_2 x = \max\{1, \lg \lg x\}$ etc.) For $d = 1$, Davis [3] and Lai [12] have obtained results for that case.

One aim of this paper is to study these problems for $d \geqslant 2$, thus generalizing the results of [3] and [12]. In Section 3 we state some results, proofs of which are given in Sections 4 and 5. In Section 6 we give the corresponding theorems for $P(|S_\mathbf{n}| \geqslant \varepsilon(|\mathbf{n}|\lg_2|\mathbf{n}|)^{\frac{1}{2}})$ which give connections to the law of the iterated logarithm. Some of the theorems of Section 6 seem to be new also for $d = 1$, although results along those lines have earlier been given by Baum and Katz [1] and Davis [2].

In Section 8 we use the above theorems and results from [9] to give an application to last exit times and the number of boundary crossings defined as $L_d = \sup\{|\mathbf{n}|; |S_\mathbf{n}| \geqslant \varepsilon \cdot a_\mathbf{n}\}$ and $N_d = \sum_\mathbf{n} I\{|S_\mathbf{n}| \geqslant \varepsilon \cdot a_\mathbf{n}\}$ respectively, where $a_\mathbf{n} = |\mathbf{n}|^\alpha$, $\alpha > \frac{1}{2}$ or $(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}$ or $(|\mathbf{n}|\lg_2|\mathbf{n}|)^{\frac{1}{2}}$. The preceding results are used in order to investigate the existence of moments of L_d and N_d, thereby extending earlier results from $d = 1$, see, e.g., [19], [16], [17], [20], [13] and [14].

The main tool in earlier papers dealing with moderate deviations, except for [3],

Received September 13, 1977; revised August 23, 1978.
AMS 1970 *subject classifications.* Primary 60F15, 60G50.
Key words and phrases. i.i.d. random variables, multidimensional index, convergence rate, law of the iterated logarithm, last exit time, the number of boundary crossings.

Theorem 1 and [12], are remainder term estimates for the central limit theorem, whereas our approach is a direct estimation of the tail probabilities involved, and so, in some instances, our approach provides new proofs for the case $d = 1$.

2. Auxiliary results. In this section we collect some auxiliary results needed later.

Let $d(x) = \text{Card}\{\mathbf{n} \in Z_+^d;\ |\mathbf{n}| = [x]\}$ and $M(x) = \text{Card}\{\mathbf{n} \in Z_+^d;\ |\mathbf{n}| \leqslant [x]\}$. Then, as $x \to \infty$, we have $M(x) = O(x(\log x)^{d-1})$ and $d(x) = o(x^\delta)\forall\delta > 0$.

The sign $\simeq$ is used to indicate that the quantities on either side converge simultaneously. $I\{\cdot\}$ denotes the indicator function of the set in braces. We also use $\pi(j)$, $j \geqslant 1$, to denote the point $(j, 1, \cdots, 1)$ and $\mathbf{n}(i)$, $i \geqslant 1$, for the point $(n_1, n_2, \cdots, n_{d-1}, i)$. The first result corresponds to [9], Lemma 2.1.

LEMMA 2.1. *Let $r > 0$ and $m = 0, 1, 2, \cdots$. For any random variable X the following are equivalent:*

$$E|X|^r \cdot (\lg|X|)^{d-1+m-(r/2)} < \infty$$

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-1} \cdot (\log|\mathbf{n}|)^m \cdot P\big(|X| \geqslant (|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) < \infty$$

$$\Sigma_{j=1}^\infty j^{(r/2)-1} \cdot (\log j)^{d-1+m} \cdot P\big(|X| \geqslant (j\lg j)^{\frac{1}{2}}\big) < \infty$$

$$\Sigma_{j=1}^\infty j^{(r/2)-1} \cdot (\log j)^m \cdot d(j) \cdot P\big(|X| \geqslant (j\lg j)^{\frac{1}{2}}\big) < \infty.$$

We also need the classical exponential bounds, see [15], page 254, more precisely the following version.

LEMMA 2.2. *Let $Y_1, Y_2, \cdots Y_n$ be* i.i.d. *random variables, $|Y_1| \leqslant b$, $EY_1 = 0$, $\text{Var}\, Y_1 = \sigma^2 < \infty$. Then, for $0 < t \leqslant b^{-1}$,*

$$P(|\Sigma_1^n Y_k| > x) \leqslant 2\exp\left\{-tx + \frac{nt^2\sigma^2}{2}\left(1 + \frac{tb}{2}\right)\right\}.$$

PROOF. From [15], page 255 we have $E\exp(tY_1) \leqslant 1 + (t^2\sigma^2/2)(1 + (tb/2))$. The rest is immediate.

The next tool is a variation of the Lévy inequalities. We denote a median of X by $\text{med}(X)$.

LEMMA 2.3. *Let $\{X_{\mathbf{k}};\ \mathbf{k} \in Z_+^d\}$ be* i.i.d. *random variables with mean 0 and variance σ^2. Then*

(i) $P(\max_{\mathbf{k}<\mathbf{n}} S_{\mathbf{k}} \geqslant \lambda) \leqslant 2^d \cdot P(S_{\mathbf{n}} \geqslant \lambda - d\sigma(2|\mathbf{n}|)^{\frac{1}{2}})$.

(ii) $P(\max_{\mathbf{k}<\mathbf{n}}|S_{\mathbf{k}}| \geqslant \lambda) \leqslant 2^d \cdot P(|S_{\mathbf{n}}| \geqslant \lambda - d\sigma(2|\mathbf{n}|)^{\frac{1}{2}})$.

PROOF OF (i). The proof is based on induction. For $d = 1$, see [15], page 248. Suppose the result is true for $d - 1$ dimensions. Define $m_d(\mathbf{k}) = \max_{1\leqslant i\leqslant n_d}|\text{med}(S_{\mathbf{k}(i)} - S_{\mathbf{k}(n_d)})|$ and $m_d = \max_{\mathbf{k}<\mathbf{n}} m_d(\mathbf{k})$.

By Tjebyshev's inequality $|\text{med}(S_{\mathbf{k}(i)} - S_{\mathbf{k}(n_d)})|^2 \leqslant 2\,\text{Var}(S_{\mathbf{k}(i)} - S_{\mathbf{k}(n_d)}) = 2|\mathbf{k}(1)| \cdot (n_d - i)\sigma^2$, (cf., [15], page 244), and so $m_d(\mathbf{k}) \leqslant \sigma(2\mathbf{k}(n_d))^{\frac{1}{2}}$ and $m_d \leqslant \sigma(2|\mathbf{n}|)^{\frac{1}{2}}$.

Next we note that

$$\{S_{\mathbf{k}(i)} \geqslant \lambda\} \subset \{S_{\mathbf{k}(i)} - \operatorname{med}(S_{\mathbf{k}(i)} - S_{\mathbf{k}(n_d)}) \geqslant \lambda - m_d(\mathbf{k})\}$$

and

$$\{\{S_{\mathbf{k}(i)} \geqslant \lambda\} \cap \{S_{\mathbf{k}(n_d)} - S_{\mathbf{k}(i)} - \operatorname{med}(S_{\mathbf{k}(n_d)} - S_{\mathbf{k}(i)}) \geqslant 0\}\} \subset \{S_{\mathbf{k}(n_d)} \geqslant \lambda - m_d(\mathbf{k})\}$$

and so, by minor changes in an argument of Gabriel, [7], Ch. II, Section 1 or [8], pages 9–10, we obtain

$$P(\max_{\mathbf{k}\prec\mathbf{n}} S_{\mathbf{k}} \geqslant \lambda) \leqslant 2P(\max_{\mathbf{k}(n_d)\prec\mathbf{n}} S_{\mathbf{k}(n_d)} \geqslant \lambda - m_d) \leqslant 2P\left(\max_{\mathbf{k}(n_d)\prec\mathbf{n}} S_{\mathbf{k}(n_d)} \geqslant \lambda - \sigma(2|\mathbf{n}|)^{\frac{1}{2}}\right).$$

Since, for $\mathbf{n}$ fixed, $\{\mathbf{k}(n_d) \prec \mathbf{n}\}$ is a set of indices of $d - 1$ dimensions, the induction hypothesis is applicable and the conclusion follows.

The proof of (ii) is immediate.

REMARK. The proof actually shows that we may use $\lambda - \sigma(2)^{\frac{1}{2}}\Sigma_{i=1}^{d}(n_1 \cdot n_2 \cdot \cdots \cdot n_i)^{\frac{1}{2}}$ instead of $\lambda - d\sigma(2|\mathbf{n}|)^{\frac{1}{2}}$ in the right-hand side.

3. The log-case, $\alpha = \frac{1}{2}$. Let $\{X_{\mathbf{n}};\ \mathbf{n} \in Z_+^d\}$ be i.i.d. random variables. In [9], Theorems 4.1 and 4.2 the following results were established.

THEOREM 3.1. *For $\alpha r \geqslant 1$, $\alpha > \frac{1}{2}$, the following are equivalent:*

(3.1) $$E|X|^r(\lg|X|)^{d-1} < \infty \quad \textit{and, if} \quad r \geqslant 1,\ EX = 0.$$

(3.2) $$\Sigma_{\mathbf{n}}|\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}| \geqslant |\mathbf{n}|^{\alpha} \cdot \varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

(3.3) $$\Sigma_{\mathbf{n}}|\mathbf{n}|^{\alpha r-2} \cdot P(\max_{\mathbf{k}\prec\mathbf{n}}|S_{\mathbf{k}}| \geqslant |\mathbf{n}|^{\alpha} \cdot \varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

If $\alpha r > 1$, $\alpha > \frac{1}{2}$, then the above statements are also equivalent to

(3.4) $$\Sigma_{j=1}^{\infty} j^{\alpha r-2} \cdot P\left(\sup_{j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/|\mathbf{k}|^{\alpha}| \geqslant \varepsilon\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

THEOREM 3.2. *For $\alpha r = 1$, $\alpha > \frac{1}{2}$, the following are equivalent:*

(3.5) $$E|X|^r \cdot (\lg|X|)^d < \infty \quad \textit{and, if} \quad r \geqslant 1,\ EX = 0.$$

(3.6) $$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot \log|\mathbf{n}| \cdot P\left(|S_{\mathbf{n}}| \geqslant |\mathbf{n}|^{1/r} \cdot \varepsilon\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

(3.7) $$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot \log|\mathbf{n}| \cdot P\left(\max_{\mathbf{k}\prec\mathbf{n}}|S_{\mathbf{k}}| \geqslant |\mathbf{n}|^{1/r} \cdot \varepsilon\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

(3.8) $$\Sigma_{j=1}^{\infty} j^{-1} \cdot P\left(\sup_{j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/|\mathbf{k}|^{1/r}| \geqslant \varepsilon\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

The theorems generalize results of Baum and Katz [1].

In this section we state the results corresponding to $r\alpha \geqslant 1$, $\alpha = \frac{1}{2}$.

THEOREM 3.3. *Let $r > 2$. If $EX = 0$, $EX^2 = \sigma^2$ and*

$$E|X|^r \cdot (\lg|X|)^{d-1-(r/2)} < \infty, \text{ then} \tag{3.9}$$

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-2} \cdot P\big(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) < \infty, \qquad \varepsilon > \sigma(r-2)^{\frac{1}{2}}. \tag{3.10}$$

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-2} \cdot P\big(\max_{\mathbf{k}<\mathbf{n}}|S_{\mathbf{k}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) < \infty, \quad \varepsilon > \sigma(r-2)^{\frac{1}{2}}. \tag{3.11}$$

$$\Sigma_{j=1}^{\infty} j^{(r/2)-2} \cdot P\big(\sup_{j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/ (|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}| \geqslant \varepsilon\big) < \infty, \quad \varepsilon > \sigma(r-2)^{\frac{1}{2}}. \tag{3.12}$$

Conversely, if one of the sums is finite for some ε, then so are the others, $EX = 0$ and (3.9) *holds.*

The case $d = 1$ has been studied in [3] and [12].

THEOREM 3.4. *The following are equivalent*:

$$EX^2 \cdot (\lg|X|)^{d-1} < \infty \quad \text{and} \quad EX = 0. \tag{3.13}$$

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot \log|\mathbf{n}| \cdot P\big(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3.14}$$

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot \log|\mathbf{n}| \cdot P\big(\max_{\mathbf{k}<\mathbf{n}}|S_{\mathbf{k}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3.15}$$

$$\Sigma_{j=1}^{\infty} j^{-1} \cdot P\big(\sup_{j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/ (|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}| \geqslant \varepsilon\big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3.16}$$

For $d = 1$, see [3], Theorem 1.

Before we proceed with the proofs of these results we make some remarks.

(i) In full accordance with the results of [9], i.e., with Theorems 3.1 and 3.2 above, (3.9)–(3.11) should also be equivalent when $r = 2$.

By checking the proof of Theorem 3.4 below it is easily seen that the same estimates also yield (3.9) $\Rightarrow$ (3.10) $\Leftrightarrow$ (3.11) provided the variance is finite, something which is guaranteed as soon as $d \geqslant 2$. In fact, we have

THEOREM 3.5. *For $d \geqslant 2$, the following are equivalent*:

$$EX^2 \cdot (\lg|X|)^{d-2} < \infty \quad \text{and} \quad EX = 0. \tag{3.17}$$

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot P\big(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3.18}$$

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot P\big(\max_{\mathbf{k}<\mathbf{n}}|S_{\mathbf{k}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3.19}$$

Furthermore, if $d = 1$, (3.19) $\Rightarrow$ (3.18) $\Rightarrow$ (3.17).

This leaves the case $d = 1$ without a complete solution. By modifying the proof of Theorem 3.4 it is possible to show that $EX^2 \cdot (\lg|X|)^{-1+\eta} < \infty$ for some $\eta > 0$ implies (3.18) and (3.19) for symmetric random variables, i.e., finite variance is not necessary. (Recall that if the variance is finite we have (3.14) by Theorem 3.4, i.e., (3.18) is trivially satisfied.) Since the variance may be infinite it is likely that one has to impose conditions on the tail behaviour of the distribution to obtain the best result, (just as in the central limit theorem and the law of the iterated logarithm, cf. [5], vol. II, and [6]).

(ii) It is natural to ask whether the constraint $\varepsilon > \sigma(r-2)^{\frac{1}{2}}$ can be removed in Theorem 3.3. To see that this is not the case, let $X \in N(0, \sigma^2)$. By using the well-known lower bounds for normal probabilities (see [5], vol. I, page 175) simple calculations show that, for large j,

$$P\big(|S_{\pi(j)}| \geqslant \varepsilon(j \log j)^{\frac{1}{2}}\big) \geqslant \text{const.}\cdot (\log j)^{-\frac{1}{2}} \cdot j^{-\varepsilon^2/2\sigma^2}. \tag{3.20}$$

Thus,

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-2}\cdot P\big(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big)$$
$$= \Sigma_{j=1}^{\infty} j^{(r/2)-2}\cdot d(j)\cdot P\big(|S_{\pi(j)}| \geqslant \varepsilon(j \lg j)^{\frac{1}{2}}\big) = \infty$$

if $(r/2) - 2 - (\varepsilon^2/2\sigma^2) \geqslant -1$, i.e., for $\varepsilon \leqslant \sigma(r-2)^{\frac{1}{2}}$.

4. Proof of Theorem 3.3. The methods of [9] do not apply in order to prove Theorem 3.3, so we have to use a different approach, which is partly based on the method of Erdös [4] and Katz [11].

(i) *(3.9) ⇒ (3.10).* Recall that $r > 2$ and thus $\sigma^2 < \infty$. Choose δ, $0 < \delta < \frac{1}{2}$, arbitrarily small and set $b_{\mathbf{n}} = 2\delta\sigma^2\varepsilon^{-1}(|\mathbf{n}|\cdot(\lg|\mathbf{n}|)^{-1})^{\frac{1}{2}}$ and $c_{\mathbf{n}} = (\varepsilon/2)(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}$.

Define $X'_{\mathbf{k}} = X_{\mathbf{k}}\cdot I\{|X_{\mathbf{k}}| \leqslant b_{\mathbf{n}}\}$, $X''_{\mathbf{k}} = X_{\mathbf{k}}\cdot I\{|X_{\mathbf{k}}| \geqslant c_{\mathbf{n}}\}$ and $X'''_{\mathbf{k}} = X_{\mathbf{k}}\cdot I\{b_{\mathbf{n}} < |X_{\mathbf{k}}| < c_{\mathbf{n}}\} = X_{\mathbf{k}} - X'_{\mathbf{k}} - X''_{\mathbf{k}}$ for $\mathbf{k} \prec \mathbf{n}$, $|\mathbf{n}|$ large (to ensure that $b_{\mathbf{n}} < c_{\mathbf{n}}$).

Set $A_{\mathbf{n}} = \{|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\}$, $A'_{\mathbf{n}} = \{|X_{\mathbf{k}}| \leqslant b_{\mathbf{n}}$ for all $\mathbf{k} \prec \mathbf{n}\}$, $A''_{\mathbf{n}} = \{$at least one $X''_{\mathbf{k}} \neq 0, \mathbf{k} \prec \mathbf{n}\}$ and $A'''_{\mathbf{n}} = \{$at least two $X'''_{\mathbf{k}} \neq 0, \mathbf{k} \prec \mathbf{n}\}$. With this notation we have

$$A_{\mathbf{n}} \subset \{A_{\mathbf{n}} \cap A'_{\mathbf{n}}\} \cup A''_{\mathbf{n}} \cup A'''_{\mathbf{n}}. \tag{4.1}$$

We now proceed to estimate the probabilities of the events on the right-hand side of (4.1).

Set $S'_{\mathbf{n}} = \Sigma_{\mathbf{k}\prec\mathbf{n}} X'_{\mathbf{k}}$. Then $P(A_{\mathbf{n}} \cap A'_{\mathbf{n}}) = P(|S'_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}})$. From Lemma 2.2 with $b = b_{\mathbf{n}}$, $t = 2\delta b_{\mathbf{n}}^{-1}$ and $(\sigma')^2 = \operatorname{Var}(X'_1)$ we obtain after elementary computations

$$P\big(|S'_{\mathbf{n}} - ES'_{\mathbf{n}}| \geqslant \varepsilon(\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) \leqslant 2\exp\left\{-\frac{\varepsilon^2(1-\delta)}{2\sigma^2}\lg|\mathbf{n}|\right\}. \tag{4.2}$$

Furthermore, since $EX_1 = 0$, it follows that

$$|ES'_{\mathbf{n}}| = |\mathbf{n}|\cdot|EX'_1| = |\mathbf{n}|\cdot\left|\textstyle\int_{|x|>b_{\mathbf{n}}} x\,dF\right| \leqslant |\mathbf{n}|\cdot\textstyle\int_{|x|>b_{\mathbf{n}}}|x|dF$$
$$\leqslant \text{const.}\ (|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\textstyle\int_{|x|\geqslant b_{\mathbf{n}}} x^2\,dF = o\big((|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) \qquad \text{as} \quad \mathbf{n}\to\infty.$$

Thus

$$P\big(|S'_{\mathbf{n}}| \geqslant \varepsilon(1+\delta)(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) \leqslant 2\cdot|\mathbf{n}|^{-\varepsilon^2(1-\delta)/2\sigma^2} \quad \text{for large} \quad |\mathbf{n}| \tag{4.3}$$

and consequently

$$\Sigma_{|\mathbf{n}|>n_0}|\mathbf{n}|^{(r/2)-2}\cdot P(A_{\mathbf{n}} \cap A'_{\mathbf{n}}) \leqslant \text{const.}\ \Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-2-\beta}, \tag{4.4}$$

where $\beta = \varepsilon^2(1-\delta)/2\sigma^2(1+\delta)^2$.

The last sum is finite if $(r/2) - 2 - \beta < -1$, i.e., for $\varepsilon > \sigma(r-2)^{\frac{1}{2}} \cdot (1+\delta)^2/(1-\delta)$.

Secondly, $P(A_{\mathbf{n}}'') \leqslant |\mathbf{n}| \cdot P(|X| \geqslant c_{\mathbf{n}})$ and hence

(4.5) $$\Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-2} \cdot P(A_{\mathbf{n}}'') \leqslant \Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-1} \cdot P(|X| \geqslant c_{\mathbf{n}}) < \infty$$

because of Lemma 2.1.

Finally, for large $|\mathbf{n}|$, $P(A_{\mathbf{n}}''') \leqslant |\mathbf{n}|^2 \cdot ((P|X| \geqslant b_{\mathbf{n}}))^2 \leqslant C_1 \cdot |\mathbf{n}|^2 \cdot [P(|X|^r \cdot (\lg|X|)^{d-1-(r/2)} \geqslant C(\sigma, \varepsilon) \cdot \delta^r \cdot |\mathbf{n}|^{r/2}(\lg|\mathbf{n}|)^{d-1-r})]^2 \leqslant C^{-2}(\sigma, \varepsilon) \cdot \delta^{-2r} \cdot |\mathbf{n}|^{2-r} \cdot (\lg|\mathbf{n}|)^{2(r+1-d)}$ and thus, since $r > 2$,

(4.6) $$\Sigma_{|\mathbf{n}|>n_0}|\mathbf{n}|^{(r/2)-2} \cdot P(A_{\mathbf{n}}''') \leqslant \text{const.} \cdot \delta^{-r} \cdot \Sigma_{\mathbf{n}}|\mathbf{n}|^{-r/2} \cdot (\lg|\mathbf{n}|)^{2(r+1-d)} < \infty.$$

By combining (4.1), (4.4), (4.5) and (4.6) we obtain

(4.7) $$\Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-2} \cdot P\big(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) < \infty \qquad \text{if} \quad \varepsilon > \sigma(r-2)^{\frac{1}{2}}\frac{(1+\delta)^2}{1-\delta}$$

and since δ was arbitrarily chosen, (3.10) follows.

(ii) *(3.10) ⇒ (3.11).* Choose s, $\frac{1}{2} < s < r/4$. Then,

$$\infty > \Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-2} \cdot P\big(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big)$$
$$\geqslant \text{const.} \cdot \Sigma_{\mathbf{n}}|\mathbf{n}|^{(r/2)-2} \cdot P(|S_{\mathbf{n}}| \geqslant \varepsilon \cdot |\mathbf{n}|^s)$$

and so, by [9], Theorem 4.1 (i.e., Theorem 3.1 above) we have $E|X|^{r/2s} \cdot (\lg|X|)^{d-1} < \infty$. Since $r/2s > 2$ we also have $EX = 0$ and $EX^2 = \sigma^2 < \infty$.

From Lemma 2.3 we obtain, for large $|\mathbf{n}|$,

(4.8) $$P\big(\max_{\mathbf{k}<\mathbf{n}}|S_{\mathbf{k}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) \leqslant 2^d \cdot P\big(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}} - d\sigma(2|\mathbf{n}|)^{\frac{1}{2}}\big)$$
$$\leqslant 2^d \cdot P\big(|S_{\mathbf{n}}| \geqslant (\varepsilon - o(1))(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big)$$

and (3.11) follows.

(iii) *(3.11) ⇒ (3.10).* Obvious.

(iv) *(3.10) ⇒ (3.12).* For $d = 1$, see [3], page 2022 and [12], Theorem 3.

Now, suppose that $d \geqslant 2$. The validity of (3.12) for "large" ε follows as in [9]. To obtain the sharper result we have to refine the method (cf. also [2], page 1481 and [3], page 2022.)

We first note from (ii) that $EX = 0$ and that $EX^2 = \sigma^2 < \infty$.

Next, choose $c > 1$ and define $c_j = [c^j], j \geqslant 1$ and $S_{\pi(x)} = S_{\pi([x])}, x \geqslant 1$.

For later use we note that

(4.9)
$$\varepsilon(c^{j-1} \cdot \lg c^{j-1})^{\frac{1}{2}} - d\sigma(2 \cdot c^{j-1+2d})^{\frac{1}{2}} - \sigma(2 \cdot c_{j+2d})^{\frac{1}{2}} \geqslant \varepsilon \cdot c^{-d-3}(c^{j+2d} \cdot \lg c^{j+2d})^{\frac{1}{2}}$$

for $j \geqslant j_0 = j_0(c)$.

Let m_0 be the smallest integer such that $c_{m_0+1} \geqslant c^{m_0}$ and define

$$E_i = E_i(c) = \{\mathbf{k} = (k_1, \cdots, k_d);\ \text{exactly } i \text{ of the indices } k_1, \cdots, k_d \leqslant c_{m_0}\},$$
$$i = 1, 2, \cdots, d$$

and

$$E_0 = E_0(c) = \{\mathbf{k} = (k_1, \cdots, k_d);\ k_i > c_{m_0},\ i = 1, 2, \cdots, d\}.$$

It follows that

$$(4.10)\quad P\left(\sup_{j \leqslant |\mathbf{k}|} |S_\mathbf{k}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}| \geqslant \varepsilon\right) \leqslant P\left(\sup_{\mathbf{k}\in E_0, j\leqslant|\mathbf{k}|} |S_\mathbf{k}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}| \geqslant \varepsilon\right) + P\left(\sup_{\mathbf{k}\in \bigcup_{i=1}^d E_i, j\leqslant|\mathbf{k}|} |S_\mathbf{k}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}| \geqslant \varepsilon\right).$$

The last two probabilities will be treated separately and we begin with the first one.

Choose j_0 large enough to ensure that $\{\mathbf{k};\ c^{j_0-1} \leqslant |\mathbf{k}| < c^{j_0}\}$ is not empty.

$$\begin{aligned}
&\Sigma_{j=j_0}^{\infty} j^{(r/2)-2} \cdot P\left(\sup_{\mathbf{k}\in E_0, j\leqslant|\mathbf{k}|} |S_\mathbf{k}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}| \geqslant \varepsilon\right)\\
&\leqslant \Sigma_{i=i_0}^{\infty} \Sigma_{c^i \leqslant j < c^{i+1}} j^{(r/2)-2} \cdot P\left(\sup_{\mathbf{k}\in E_0, j\leqslant|\mathbf{k}|} |S_\mathbf{k}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}| \geqslant \varepsilon\right)\\
&\leqslant \text{const.}\Sigma_{i=i_0}^{\infty} c^{i((r/2)-1)} \cdot \Sigma_{j=i+1}^{\infty} P\left(\sup_{\mathbf{k}\in E_0,\ c^{j-1}\leqslant|\mathbf{k}|<c^j} |S_\mathbf{k}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}| \geqslant \varepsilon\right)\\
&\leqslant \text{const.}\Sigma_{j=j_0}^{\infty} c^{j((r/2)-1)} \cdot P\left(\sup_{\mathbf{k}\in E_0,\ c^{j-1}\leqslant|\mathbf{k}|<c^j} |S_\mathbf{k}| \geqslant \varepsilon(c^{j-1} \cdot \lg c^{j-1})^{\frac{1}{2}}\right).
\end{aligned}$$

The problem at this point is to derive a Lévy-type inequality to replace (5.2) of [9].

Let A_j, $j = 0, 1, 2, \cdots$, be the set of points in R_+^d, (the set of d-tuples with positive coordinates), of the form $(c^{i_1}, c^{i_2}, \cdots, c^{i_d})$, where $i_1 + i_2 + \cdots + i_d = j$. These sets divide R_+^d into d-dimensional cubes, where each cube has its smallest corner (in the natural ordering) in some A_j and the largest corner in A_{j+d}. Further, the curves $|\mathbf{x}| = j$, where $\mathbf{x} \in R_+^d$, $j = 1, 2, \cdots$ split the cubes into d curved "slices".

Now, choose an arbitrary but fixed point $\mathbf{k}$, $c^{j-1} \leqslant |\mathbf{k}| < c^j$. This point belongs to some cube and it is dominated by the largest corner, which belongs to some A_i, where the value of $i = j, j+1, \cdots, j+d-1$, depends on the slice of the cube to which $\mathbf{k}$ belongs. However, by increasing some of the coordinates of the dominating point (if necessary) we have

(4.11) For each $\mathbf{k}$, $c^{j-1} \leqslant |\mathbf{k}| < c^j$, there exists $\mathbf{t} \in A_{j+d-1}$ such that $\mathbf{k} \prec \mathbf{t}$.

Since $\mathbf{t}$ does not in general have integer valued coordinates we have to find $\mathbf{k}^*$, $\mathbf{k} \prec \mathbf{t} \prec \mathbf{k}^*$, such that $\mathbf{k}^*$ has integer valued coordinates.

Now, if $\mathbf{t} = (c^{i_1+1}, c^{i_2+1}, \cdots, c^{i_d+1})$ it follows from the fact that $\mathbf{k} \in E_0$ that $\mathbf{k}^* = (c_{i_1+2}, c_{i_2+2}, \cdots, c_{i_d+2})$ has the desired property. Also, $c^{j-1+d} \leqslant |\mathbf{k}^*| \leqslant c^{j-1+2d}$.

This together with Lemma 2.3 yields

$$P\Big(\sup_{\mathbf{k}\in E_0,\, c^{j-1}\leqslant|\mathbf{k}|<c^j}|S_{\mathbf{k}}| \geqslant \varepsilon\big(c^{j-1}\cdot \lg c^{j-1}\big)^{\frac12}\Big)$$

$$\leqslant \Sigma'\cdot P\Big(\sup_{\mathbf{k}\prec(c_{i_1+2},\cdots,c_{i_d+2})}|S_{\mathbf{k}}| \geqslant \varepsilon\big(c^{j-1}\cdot \lg c^{j-1}\big)^{\frac12}\Big)$$

$$\leqslant 2^d\cdot\Sigma'\cdot P\Big(|S_{(c_{i_1+2},\cdots,c_{i_d+2})}| \geqslant \varepsilon\big(c^{j-1}\cdot \lg c^{j-1}\big)^{\frac12} - d\sigma\big(2\Pi_{n=1}^d c_{i_n+2}\big)^{\frac12}\Big)$$

$$\leqslant 2^d\cdot\Sigma'\cdot P\Big(|S_{(c_{i_1+2},\cdots,c_{i_d+2})}| \geqslant \varepsilon\big(c^{j-1}\cdot \lg c^{j-1}\big)^{\frac12} - d\sigma\big(2\cdot c^{j-1+2d}\big)^{\frac12}\Big).$$

Here Σ' denotes summation over all points $(i_1,\cdots,i_d)$ such that $i_1+\cdots+i_d = j-1$. This means that the number of terms in the sum is majorized by j^{d-1}.

Because of the equal distribution, Lévy's inequality and (4.9) the last expression is majorized by

$$2^d\cdot\Sigma'\cdot P\Big(|S_{\pi(\Pi_{n=1}^d c_{i_n+2})}| \geqslant \varepsilon\big(c^{j-1}\cdot \lg c^{j-1}\big)^{\frac12} - d\sigma\big(2\cdot c^{j-1+2d}\big)^{\frac12}\Big)$$

$$\leqslant 2^d\cdot\Sigma'\cdot P\Big(\max_{i\leqslant c_{j+2d}}|S_{\pi(i)}| \geqslant \varepsilon\big(c^{j-1}\cdot \lg c^{j-1}\big)^{\frac12} - d\sigma\big(2\cdot c^{j-1+2d}\big)^{\frac12}\Big)$$

$$\leqslant 2^{d+1}\cdot j^{d-1}\cdot P\Big(|S_{\pi(c_{j+2d})}| \geqslant \varepsilon\cdot c^{-d-3}\big(c^{j+2d}\cdot \lg c^{j+2d}\big)^{\frac12}\Big).$$

This proves the following inequality of Lévy-type:

$$\text{(4.12)}\quad P\Big(\sup_{\mathbf{k}\in E_0,\, c^{j-1}\leqslant|\mathbf{k}|<c^j}|S_{\mathbf{k}}| \geqslant \varepsilon\big(c^{j-1}\cdot \lg c^{j-1}\big)^{\frac12}\Big) \leqslant 2^{d+1}\cdot j^{d-1}\cdot P\Big(|S_{\pi(c_{j+2d})}| \geqslant \varepsilon\cdot c^{-d-3}\big(c^{j+2d}\cdot \lg c^{j+2d}\big)^{\frac12}\Big),$$

for large j.

We thus conclude that

$$\Sigma_{j=j_0}^{\infty} j^{(r/2)-2}\cdot P\Big(\sup_{\mathbf{k}\in E_0,\, j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/\,(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac12}| \geqslant \varepsilon\Big)$$

$$\leqslant \text{const. } \Sigma_{j=j_0}^{\infty} c^{j((r/2)-1)}\cdot j^{d-1}\cdot P\Big(|S_{\pi(c_{j+2d})}| \geqslant \varepsilon\cdot c^{-d-3}\big(c^{j+2d}\cdot \lg c^{j+2d}\big)^{\frac12}\Big)$$

$$\leqslant \text{const.} \Sigma_{i=i_0}^{\infty} i^{((r/2)-2)}\cdot (\lg i)^{d-1}\cdot P\Big(|S_{\pi(i)}| \geqslant \varepsilon\cdot c^{-d-3}(i\cdot \lg i)^{\frac12}\Big).$$

The last inequality is obtained by a change of variable (cf., [12], page 439).

By using the estimates from step (i) it follows that the last sum is finite for $\varepsilon > c^{d+3}\cdot\sigma(r-2)^{\frac12}$ (cf. (5.4) of [9]). (This can also be seen by the change of variable $i\to 2^i\to i$ together with the fact that $d(2^i)\sim(\lg 2^i)^{d-1}\sim i^{d-1}$).

This proves that

$$\text{(4.13)}\qquad \Sigma_{j=j_0}^{\infty} j^{(r/2)-2}\cdot P\Big(\sup_{\mathbf{k}\in E_0(c),\, j<|\mathbf{k}|}|S_{\mathbf{k}}/\,(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac12}| \geqslant \varepsilon\Big) < \infty \quad \text{if } \varepsilon > c^{d+3}\cdot\sigma(r-2)^{\frac12}.$$

It remains to study the convergence of

$$\Sigma_{j=j_0}^{\infty} j^{(r/2)-2}\cdot P\Big(\sup_{\mathbf{k}\in \cup_{i=1}^d E_i,\, j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/\,(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac12}| \geqslant \varepsilon\Big).$$

Obviously,

(4.14)

$$P\big(\sup_{\mathbf{k}\in\cup_{i=1}^{d}E_i, j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}|\geqslant\varepsilon\big)\leqslant\Sigma_{i=1}^{d}P\big(\sup_{\mathbf{k}\in E_i, j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}|\geqslant\varepsilon\big).$$

The probabilities of the right-hand side are treated very much as before. First, let $i = 1$.

By symmetry,

$$P\big(\sup_{\mathbf{k}\in E_1, j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}|\geqslant\varepsilon\big)\leqslant\Sigma_{n=1}^{d}\Sigma_{m=1}^{c_{m_0}}P\big(\sup_{k_n=m, j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}|\geqslant\varepsilon\big)$$

$$= d\cdot\Sigma_{m=1}^{c_{m_0}}P\big(\sup_{k_1=m, j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}|\geqslant\varepsilon\big).$$

For the case $i = 1$ it therefore suffices to prove that

$$\Sigma_{j=j_0}^{\infty}j^{(r/2)-2}\cdot P\big(\sup_{k_1=m, j\leqslant|\mathbf{k}|}|S_{\mathbf{k}}/(|\mathbf{k}|\lg|\mathbf{k}|)^{\frac{1}{2}}|\geqslant\varepsilon\big)<\infty,\quad m=1,2,\cdots,c_{m_0}. \tag{4.15}$$

Since the first coordinate is kept fixed in the sum, the problem is essentially the same as before but reduced to $d - 1$ dimensions. We therefore do not give all the details.

For $j = 0, 1, 2, \cdots$, let A_j' be the set of points in R_+^d of the form $(m, c^{i_2}, c^{i_3}, \cdots, c^{i_d})$ with $i_2 + i_3 + \cdots + i_d = j$. For points $\mathbf{k}$, where $k_1 = m$ and $c^{j-1} \leqslant |\mathbf{k}| < c^j$, the dominating points are of the form $(m, c_{i_2+2}, \cdots, c_{i_d+2})$.

Thus, by Lemma 2.3

$$P\big(\sup_{k_1=m, m\cdot c^{j-1}\leqslant|\mathbf{k}|<m\cdot c^j}|S_{\mathbf{k}}|\geqslant\varepsilon(m\cdot c^{j-1}\cdot\lg m\cdot c^{j-1})^{\frac{1}{2}}\big)$$

$$\leqslant\Sigma'\cdot P\big(\sup_{\mathbf{k}\prec(m, c_{i_2+2},\cdots,c_{i_d+2})}|S_{\mathbf{k}}|\geqslant\varepsilon(m\cdot c^{j-1}\cdot\lg m\cdot c^{j-1})^{\frac{1}{2}}\big)$$

$$\leqslant 2^d\cdot\Sigma'\cdot P\big(|S_{(m, c_{i_2+2},\cdots,c_{i_d+2})}|\geqslant\varepsilon(m\cdot c^{j-1}\cdot\lg m\cdot c^{j-1})^{\frac{1}{2}}\dot{-}d\sigma(2mc^{j-1+2d-2})^{\frac{1}{2}}\big).$$

Since only the coordinates $k_2, \cdots, k_d$ vary, the number of terms in the sum is majorized by j^{d-2}.

By proceeding just as before we find that (4.15) holds for all $\varepsilon > c^{d+1}\cdot\sigma(r - 2)^{\frac{1}{2}}$.

This completes the proof of the case $i = 1$. The estimates for $i = 2, \cdots, d$ are similar and therefore omitted.

We thus conclude that (3.12) holds for all $\varepsilon > c^{d+3}\cdot\sigma(r - 2)^{\frac{1}{2}}$. Since c may be chosen arbitrarily close to 1 the proof is complete.

(v) Finally, suppose that (3.12) holds for some $\varepsilon_0 > 0$. By [12], Theorem 3, we know that $EX = 0$ and that (3.9) holds if $d = 1$. For the case $d \geqslant 2$ the conclusion follows by arguments like those of [9], Section 5, (see also [3], page 2020). The details are omitted.

5. Proof of Theorems 3.4 and 3.5. The proofs follow the ideas of [9] and are, therefore, only sketched.

LEMMA 5.1. $\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1}\cdot(\log|\mathbf{n}|)^{-\beta}$ *converges if* $\beta > d$ *and diverges if* $\beta \leqslant d$.

PROOF. Follows from integration by parts together with the fact that $M(x) = O(x(\log x)^{d-1})$ as $x \to \infty$.

PROOF OF THEOREM 3.4. First, assume that the random variables have a symmetric distribution.

(3.13) $\Rightarrow$ *(3.14).* Pick j such that $2^j > d + 1$. Var $X = \sigma^2 < \infty$. By an inequality of Hoffmann-Jørgensen ([10], page 164, see also [9], Lemma 2.4) together with Lemma 2.1 with $r = 2$, $m = 1$ and Lemma 5.1 we obtain

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1}\cdot \lg|\mathbf{n}|\cdot P\big(|S_{\mathbf{n}}| \geqslant \varepsilon\cdot(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big)$$
$$\leqslant C_j\cdot\Sigma_{\mathbf{n}}\lg|\mathbf{n}|\cdot P\big(|X| \geqslant \varepsilon\cdot(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) + D_j(\varepsilon,\sigma)\cdot\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1}\cdot(\lg|\mathbf{n}|)^{1-2^j} < \infty.$$

This proves (3.14). The proofs of (3.15) and (3.16) (for symmetric random variables) and the desymmetrization follow as in [9] (except for obvious modifications) and are therefore omitted.

(3.16) $\Rightarrow$ *(3.13).* For $d = 1$, see [3], Theorem 1. For $d \geqslant 2$ the conclusion follows by the induction procedure used in [9], Section 6.

PROOF OF THEOREM 3.5. Suppose that $d \geqslant 2$. The estimates used in the preceding proof also yield (3.17) $\Rightarrow$ (3.18) $\Leftrightarrow$ (3.19) for symmetric random variables and the desymmetrization is by now standard.

Now, $d \geqslant 1$. It remains to show (3.18) $\Rightarrow$ (3.17).

Pick s, $\frac{1}{2} < s < 1$.

$$\infty > \Sigma_{\mathbf{n}}|\mathbf{n}|^{-1}\cdot P\big(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\big) \geqslant \text{const.}\cdot\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1}\cdot P(|S_{\mathbf{n}}| \geqslant \varepsilon|\mathbf{n}|^{s})$$

and thus, by [9], Theorem 4.1 (i.e., Theorem 3.1 above) we have $E|X|^{1/s}\cdot(\lg|X|)^{d-1} < \infty$ and, since $1/s > 1$, $EX = 0$.

Next, suppose that $d = 1$ and suppose further, that the variables have a symmetric distribution. By a change of variable (cf. above) and the Lévy-inequalities we obtain

$$\begin{aligned}\infty &> \Sigma_{n=1}^{\infty} n^{-1}\cdot P\big(|S_n| \geqslant \varepsilon(n\cdot\lg n)^{\frac{1}{2}}\big)\\ &\geqslant \text{const.}\cdot\Sigma_{n=1}^{\infty}P\big(|S_{2^n}| \geqslant \varepsilon(2^n\cdot\lg 2^n)^{\frac{1}{2}}\big)\\ &\geqslant \text{const.}\cdot\Sigma_{n=1}^{\infty}P\big(\max_{k\leqslant 2^n}|S_k| \geqslant \varepsilon(2^n\cdot\lg 2^n)^{\frac{1}{2}}\big)\\ &\geqslant \text{const.}\cdot\Sigma_{n=1}^{\infty}P\big(\max_{2^{n-1}<k\leqslant 2^n}|S_k| \geqslant \varepsilon(2^n\cdot\lg 2^n)^{\frac{1}{2}}\big)\\ &\geqslant \text{const.}\cdot\Sigma_{n=1}^{\infty}P\big(\max_{2^{n-1}<k\leqslant 2^n}|S_k/(k\cdot\lg k)^{\frac{1}{2}}| \geqslant 4\varepsilon\big).\end{aligned}$$

Thus, we conclude that

$$P\big(|S_n| \geqslant 4\varepsilon(n\lg n)^{\frac{1}{2}}\text{i.o.}\big) = 0. \tag{5.1}$$

Since $|X_n| \leqslant |S_n| + |S_{n-1}|$, (5.1) implies that $P(|X_n| \geqslant 8\varepsilon(n \lg n)^{\frac{1}{2}}$ i.o.$) = 0$, i.e., by independence and Borel-Cantelli,

$$\infty > \Sigma_{n=1}^{\infty} P\left(|X_n| \geqslant 8\varepsilon(n\ lg\ n)^{\frac{1}{2}}\right) = \Sigma_{n=1}^{\infty} P\left(|X| \geqslant 8\varepsilon(n \lg n)^{\frac{1}{2}}\right),$$

which is equivalent to $EX^2 \cdot (\lg|X|)^{-1} < \infty$.

To desymmetrize we note that if (3.18) holds, the weak symmetrization inequalities [15], page 245, imply that (3.18) also holds for symmetrized variables. Hence $E|X^s|^2 \cdot (\lg|X^s|)^{-1} < \infty$, and thus also $E|X|^2 \cdot (\lg|X|)^{-1} < \infty$.

Now, let $d \geqslant 2$.

$$\begin{aligned}\infty &> \Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot P\left(\max_{\mathbf{k}<\mathbf{n}}|S_{\mathbf{k}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\right)\\ &\geqslant \Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot P\left(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\right) \geqslant \Sigma_{j=1}^{\infty} j^{-1} \cdot P\left(|S_{\pi(j)}| \geqslant \varepsilon(j \lg j)^{\frac{1}{2}}\right)\end{aligned}$$

and thus, since the proof for $d = 1$ is complete, it follows that $EX^2 \cdot (\lg|X|)^{-1} < \infty$. By the Erdös-Katz method [4], [11] we have

$$P\left(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\right) \geqslant |\mathbf{n}| \cdot P\left(|X| \geqslant c(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\right) \tag{5.2}$$

for some $c > 0$ and consequently,

$$\begin{aligned}\infty &> \Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot P\left(\max_{\mathbf{k}<\mathbf{n}}|S_{\mathbf{k}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\right)\\ &\geqslant \Sigma_{\mathbf{n}} P\left(|X| \geqslant c(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\right),\end{aligned}$$

which, by Lemma 2.1, is equivalent to $E|X|^2 \cdot (\lg|X|)^{d-2} < \infty$.

The proof is complete.

6. The loglog-case. In this section we state some results related to the law of the iterated logarithm.

THEOREM 6.1. *Let $EX = 0$ and $EX^2 = \sigma^2$. If*

$$EX^2 \cdot (\lg_2|X|)^{-1}(\lg|X|)^d < \infty \tag{6.1}$$

then

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot \log|\mathbf{n}| \cdot P\left(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg_2|\mathbf{n}|)^{\frac{1}{2}}\right) < \infty, \quad \varepsilon > \sigma(2(d+1))^{\frac{1}{2}} \tag{6.2}$$

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot \log|\mathbf{n}| \cdot P\left(\max_{\mathbf{k}<\mathbf{n}}|S_{\mathbf{k}}| \geqslant \varepsilon(|\mathbf{n}|\lg_2|\mathbf{n}|)^{\frac{1}{2}}\right) < \infty, \quad \varepsilon > \sigma(2(d+1))^{\frac{1}{2}} \tag{6.3}$$

$$\Sigma_{j=1}^{\infty} j^{-1} \cdot P\left(\sup_{j \leqslant |\mathbf{k}|}|S_{\mathbf{k}}/(|\mathbf{k}|\lg_2|\mathbf{k}|)^{\frac{1}{2}}| \geqslant \varepsilon\right) < \infty, \quad \varepsilon > \sigma(2(d+1))^{\frac{1}{2}}. \tag{6.4}$$

Conversely, if one of the sums is finite for some ε, then so are the others, $EX = 0$ and (6.1) *holds.*

This result seems to be new also for the case $d = 1$.

THEOREM 6.2. *Let $EX = 0$ and $EX^2 = \sigma^2 < \infty$. If*

$$EX^2 \cdot (\lg_2|X|)^{-1} \cdot (\lg|X|)^{d-1} < \infty, \tag{6.5}$$

then

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot P\left(|S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg_2|\mathbf{n}|)^{\frac{1}{2}}\right) < \infty, \qquad \varepsilon > \sigma(2d)^{\frac{1}{2}} \tag{6.6}$$

$$\Sigma_{\mathbf{n}}|\mathbf{n}|^{-1} \cdot P\left(\max_{\mathbf{k}\prec\mathbf{n}}|S_{\mathbf{k}}| \geqslant \varepsilon(|\mathbf{n}|\lg_2|\mathbf{n}|)^{\frac{1}{2}}\right) < \infty, \qquad \varepsilon > \sigma(2d)^{\frac{1}{2}}. \tag{6.7}$$

Conversely, if one of the sums is finite for some ε, *then so is the other,* $EX = 0$, $EX^2 < \infty$ *and* (6.5) *holds.*

When $d = 1$ it follows from [2], Theorem 4, that (6.6) holds if $EX = 0$ and $EX^2 = 1$.

Note that $EX^2 < \infty$ implies (6.5) for $d = 1$, so in that case the moment requirements reduce to $EX = 0$ and $EX^2 < \infty$. For $d \geqslant 2$, (6.5) implies $EX^2 < \infty$. In this connection we also refer to the law of the iterated logarithm, see [21], page 280.

Again it is easy to see that the ε-bounds cannot be improved upon and, further, that the results cannot be generalized to higher moments.

7. On the proofs of Theorems 6.1 and 6.2.

The proof of Theorem 6.1 follows closely that of Theorem 3.3. We omit the details. For $d = 1$, cf. also [2] and [3].

PROOF OF THEOREM 6.2. (i) For $d = 1$ it follows from [2], Theorem 4, that (6.6) holds if $EX = 0$ and $EX^2 = \sigma^2 < \infty$ and for $d \geqslant 2$ the above methods yield (6.6) provided $EX = 0$, $EX^2 = \sigma^2 < \infty$ and (6.5) holds.

(ii) *(6.6)* $\Rightarrow$ $EX = 0$ *and* $EX^2 < \infty$. Since (6.6) $\Rightarrow$ (3.18) it follows from Theorem 3.5 that $EX = 0$ and that $EX^2 \cdot (\lg|X|)^{d-2} < \infty$. If $d \geqslant 2$ we are done, so suppose that $d = 1$. Assume first that the variables are symmetric. By arguments as those preceding (5.1) we obtain

$$P\left(|S_n| \geqslant 4\varepsilon(n \cdot \lg_2 n)^{\frac{1}{2}}\text{i.o.}\right) = 0. \tag{7.1}$$

By the converse of the law of the iterated logarithm [18] we conclude that $EX^2 < \infty$. The desymmetrization follows as in Section 5 and thus $EX^2 < \infty$ also in the general case.

(iii) *(6.6)* $\Leftrightarrow$ *(6.7).* From (ii) and Lemma 2.3 we have (6.6) $\Rightarrow$ (6.7) by the usual method. The converse is obvious.

(iv) Finally, suppose that (6.6) holds. From step (ii) we know that $EX = 0$ and $EX^2 < \infty$. Hence nothing remains to prove if $d = 1$. For $d \geqslant 2$ we obtain (6.5) by the method of [4] and [11] (cf. also (5.2)).

We conclude this section with a remark about the law of the iterated logarithm for $d = 1$.

Suppose that $EX = 0$ and $EX^2 = \sigma^2 < \infty$. By Theorem 6.2 we know that (6.6) holds and by modifying the arguments leading up to (5.1) we have, (no symmetry assumed),

$$\infty > \Sigma_{n=1}^{\infty} n^{-1} \cdot P\big(|S_n| \geqslant \varepsilon(n \cdot \lg_2 n)^{\frac{1}{2}}\big)$$
$$\geqslant \text{const.} \cdot \Sigma_{j=j_0}^{\infty} P\big(|S_{c_j}| \geqslant \varepsilon(c_j \lg_2 c_j)^{\frac{1}{2}}\big)$$
$$\geqslant \text{const.} \cdot \Sigma_{j=j_0}^{\infty} P\big(|S_{c_j}| \geqslant \varepsilon \cdot c^2 \cdot (c_{j-1} \cdot \lg_2 c_{j-1})^{\frac{1}{2}} - \sigma(2c_j)^{\frac{1}{2}}\big)$$
$$\geqslant \text{const.} \cdot \Sigma_{j=j_0}^{\infty} P\big(\max_{c_{j-1} < k \leqslant c_j} |S_k| \geqslant \varepsilon \cdot c^2 \cdot (c_{j-1} \cdot \lg_2 c_{j-1})^{\frac{1}{2}}\big)$$
$$\geqslant \text{const.} \cdot \Sigma_{j=j_0}^{\infty} P\big(\max_{c_{j-1} < k \leqslant c_j} |S_k / (k \cdot \lg_2 k)^{\frac{1}{2}}| \geqslant \varepsilon c^2\big).$$

Thus,

(7.2) $$P\big(|S_n| \geqslant \varepsilon \cdot c^2 \cdot (n \cdot \lg_2 n)^{\frac{1}{2}} \text{i.o.}\big) = 0 \qquad \text{for} \quad \varepsilon > \sigma(2)^{\frac{1}{2}}.$$

Since c may be chosen arbitrarily close to 1 this proves the upper class result for the law of the iterated logarithm.

COROLLARY. *Let $d = 1$. Suppose that $EX = 0$ and $EX^2 = \sigma^2 < \infty$. Then*

$$P\big(|S_n| > \varepsilon(n \cdot \lg_2 n)^{\frac{1}{2}} \text{i.o.}\big) = 0, \qquad \varepsilon > \sigma(2)^{\frac{1}{2}}.$$

8. Last exit times and the number of boundary crossings. For $d = 1$ several authors have investigated the last exit times $L = \sup\{n; |S_n| \geqslant \varepsilon \cdot a_n\}$ and the counting variables $N = \Sigma_{n=1}^{\infty} I\{|S_n| \geqslant \varepsilon \cdot a_n\}$, i.e., the number of boundary crossings of the random walk, where $a_n = n^\alpha$, $\alpha > \frac{1}{2}$ and $a_n = (n \lg n)^{\frac{1}{2}}$ and $a_n = (n \lg_2 n)^{\frac{1}{2}}$, with regard to existence of moments. See, e.g., [19], [16], [17], [20], [13], [14]. (Throughout, $\sup \varnothing = 0$.)

Note that $N \leqslant L$ and that if, e.g., $a_n = n$, the statement $P(N < \infty) = 1$ for every $\varepsilon > 0$ is equivalent to the strong law of large numbers (see, e.g., [17]).

Since the index set is no longer totally ordered when $d \geqslant 2$, these results do not generalize immediately. In this section we shall present some results for $L_d = \sup\{|\mathbf{n}|; |S_\mathbf{n}| \geqslant \varepsilon \cdot a_\mathbf{n}\}$ and $N_d = \Sigma_\mathbf{n} I\{|S_\mathbf{n}| \geqslant \varepsilon \cdot a_\mathbf{n}\}$, where $a_\mathbf{n} = |\mathbf{n}|^\alpha$, $\alpha > \frac{1}{2}$ and $a_\mathbf{n} = (|\mathbf{n}| \lg |\mathbf{n}|)^{\frac{1}{2}}$ and $a_\mathbf{n} = (|\mathbf{n}| \lg_2 |\mathbf{n}|)^{\frac{1}{2}}$, by applying results from earlier sections and from [9].

We first observe that for all choices of boundary we have

(8.1) $$P(L_d \geqslant j) = P(\sup_{j \leqslant |\mathbf{k}|} |S_\mathbf{k} / a_\mathbf{k}| \geqslant \varepsilon)$$

and consequently it follows that

(8.2) $$E(L_d)^p \simeq \Sigma_{j=1}^{\infty} j^{p-1} \cdot P(\sup_{j \leqslant |\mathbf{k}|} |S_\mathbf{k} / a_\mathbf{k}| \geqslant \varepsilon), \qquad p > 0.$$

(8.3) $$E \lg L_d \simeq \Sigma_{j=1}^{\infty} j^{-1} \cdot P(\sup_{j \leqslant |\mathbf{k}|} |S_\mathbf{k} / a_\mathbf{k}| \geqslant \varepsilon).$$

THEOREM 8.1. *Let* $L_d = \sup\{|\mathbf{n}|; |S_{\mathbf{n}}| \geqslant \varepsilon|\mathbf{n}|^{\alpha}\}$, $\alpha > \frac{1}{2}$. *For* $r > 1/\alpha$, *the following are equivalent*:

$$(8.4) \qquad E|X|^r \cdot (\lg|X|)^{d-1} < \infty \qquad \text{and, if} \quad r \geqslant 1, EX = 0.$$

$$(8.5) \qquad E(L_d)^{\alpha r - 1} < \infty \qquad \text{for all} \quad \varepsilon > 0.$$

For $d = 1$ see [13], page 625 and [14], page 60.

PROOF. Immediate from (8.2) with $p = \alpha r - 1$ together with [9], Theorem 4.1 (i.e., Theorem 3.1 above).

THEOREM 8.2. *Let* $L_d = \sup\{|\mathbf{n}|; |S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg|\mathbf{n}|)^{\frac{1}{2}}\}$. *If* $EX = 0$, $EX^2 = \sigma^2$ *and* $E|X|^r \cdot (\lg|X|)^{d-1-(r/2)} < \infty$, $r > 2$, *then*

$$(8.6) \qquad E(L_d)^{(r/2)-1} < \infty, \qquad \varepsilon > \sigma(r-2)^{\frac{1}{2}}$$

$$(8.7) \qquad E(L_d)^{(r/2)-1} = \infty, \qquad \varepsilon < \sigma(r-2)^{\frac{1}{2}}.$$

Conversely, if $E(L_d)^{(r/2)-1} < \infty$ *for some* $\varepsilon > 0$, *then* $EX = 0$ *and* $E|X|^r \cdot (\lg|X|)^{d-1-(r/2)} < \infty$.

For $d = 1$ see [13], Theorem 3 and [14] Theorem 3.

PROOF. Let $\varepsilon < \sigma(r-2)^{\frac{1}{2}}$. Since $L_d \geqslant L = \sup\{n; |S_{\pi(n)}| \geqslant \varepsilon(n \lg n)^{\frac{1}{2}}\}$ it follows from [13], page 615, that $EL_d^{(r/2)-2} \geqslant EL^{(r/2)-2} = \infty$. (Note that if $X \in N(0, \sigma^2)$ the conclusion follows from (8.2) and the remark at the end of Section 3). The rest is immediate from (8.2) together with Theorem 3.3.

Finally, let $L_d = \sup\{|\mathbf{n}|; |S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg_2|\mathbf{n}|)^{\frac{1}{2}}\}$. For $d = 1$, Slivka [16] proves that the corresponding counting variable, N_1, possesses no moments of positive order. Since $L_d \geqslant L_1 \geqslant N_1$ it follows that no moments of L_d can exist. For $d = 1$, see also [19], page 315. However, with the aid of Theorem 6.1 we can give conditions for the finiteness of a logarithmic moment of L_d, a result which seems to be new also for $d = 1$.

THEOREM 8.3. *Let* $L_d = \sup\{|\mathbf{n}|; |S_{\mathbf{n}}| \geqslant \varepsilon(|\mathbf{n}|\lg_2|\mathbf{n}|)^{\frac{1}{2}}\}$ *and suppose that* $EX = 0$ *and* $EX^2 = \sigma^2 < \infty$.

(i) *For no* $r > 0$ *and no* $\varepsilon > 0$ *does* $E(L_d)^r$ *exist*.

(ii) *If in addition* $EX^2 \cdot (\lg_2|X|)^{-1} \cdot (\lg|X|)^d < \infty$, *then* $E \lg L_d < \infty$ *for* $\varepsilon > \sigma(2(d+1))^{\frac{1}{2}}$.

PROOF. (i) has already been demonstrated and (ii) follows from (8.3) and Theorem 6.1.

Finally we give some results for the counting variable N.

COROLLARY 8.1. *If the assumptions of Theorem* 8.1 *are fulfilled, then, for* $N_d = \Sigma_{\mathbf{n}} I\{|S_{\mathbf{n}}| \geqslant \varepsilon|\mathbf{n}|^{\alpha}\}$, $\alpha > \frac{1}{2}$, *we have*

$$(8.8) \qquad EN_d^{\alpha r - 1} \cdot (\lg N_d)^{-(d-1)(\alpha r - 1)} < \infty \qquad \text{for all} \quad \varepsilon > 0.$$

For $d = 1$ the corollary reduces to the sufficiency part of [14], Theorem 1, ($1 < \alpha r < 2$) and [20], ($\alpha r > 2$), and for $d = 1$, $\alpha = 1$ we have Theorem 1 of [17].

COROLLARY 8.2. *If the assumptions of Theorem* 8.2 *are fulfilled, then, for* $N_d = \Sigma_{\mathbf{n}} I\{|S_{\mathbf{n}}| \geq \varepsilon(|\mathbf{n}| \lg|\mathbf{n}|)^{\frac{1}{2}}\}$, *we have*

$$EN_d^{(r/2)-1} \cdot (\lg N_d)^{-(d-1)((r/2)-1)} < \infty, \qquad \varepsilon > \sigma(r-2)^{\frac{1}{2}}. \tag{8.9}$$

For $d = 1$ this reduces to (1.18) $\Rightarrow$ (1.20) of [14], Theorem 3.

COROLLARY 8.3. *Suppose that* $EX = 0$ *and* $EX^2 = \sigma^2 < \infty$ *and set* $N_d = \Sigma_{\mathbf{n}} I\{|S_{\mathbf{n}}| \geq \varepsilon(|\mathbf{n}| \lg_2|\mathbf{n}|)^{\frac{1}{2}}\}$.

(i) *For no* $r > 0$ *and no* $\varepsilon > 0$ *does* $E(N_d)^r$ *exist.*

(ii) *If, in addition,* $EX^2 \cdot (\lg_2|X|)^{-1} \cdot (\lg|X|)^d < \infty$, *then* $E \lg N_d < \infty$, $\varepsilon > (2(d+1))^{\frac{1}{2}}$.

For $d = 1$ (i) reduces to the theorem of [16].

PROOFS. Recall that $M(j) = \operatorname{card}\{\mathbf{n} \in Z_+^d;\ |\mathbf{n}| \leq j\} = O(j(\log j)^{d-1})$ as $j \to \infty$. Thus, for all cases,

$$P(L_d \geq j) \geq P(N_d \geq M(j)) \geq P\big(N_d \geq cj(\log j)^{d-1}\big), \tag{8.10}$$

where c is some constant.

Consequently, if $a_{\mathbf{n}} = |\mathbf{n}|^\alpha$, $\alpha > \frac{1}{2}$, (8.2), (8.10) and Theorem 8.1 yield

$$\begin{aligned} \infty > E(L_d)^{\alpha r - 1} &\geq \text{const.} \Sigma_{j=1}^{\infty} j^{\alpha r - 2} \cdot P(L_d \geq j) \\ &\geq \text{const.} \cdot \Sigma_{j=1}^{\infty} j^{\alpha r - 2} \cdot P\big(N_d \geq cj(\lg j)^{d-1}\big), \end{aligned}$$

which implies that $EN_d^{\alpha r - 1} \cdot (\lg N_d)^{-(d-1)(\alpha r - 1)} < \infty$.

The other cases follow similarly.

Finally, (i) of Corollary 8.3 follows from [16] since $N_d \geq N_1$.

REFERENCES

[1] BAUM, L. E. and KATZ, M. (1965). Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.* **120** 108–123.

[2] DAVIS, J. A. (1968). Convergence rates for the law of the iterated logarithm. *Ann. Math. Statist.* **39** 1479–1485.

[3] DAVIS, J. A. (1968). Convergence rates for probabilities of moderate deviations. *Ann. Math. Statist.* **39** 2016–2028.

[4] ERDÖS, P. (1949). On a theorem of Hsu and Robbins. *Ann. Math. Statist.* **20** 286–291.

[5] FELLER, W. (1966), (1968). *An Introduction to Probability Theory and its Applications, Vol. I, 3rd ed. and Vol II.* Wiley, New York.

[6] FELLER, W. (1968). An extension of the law of the iterated logarithm to variables without variance. *J. Math. Mech.* **18** 343–355.

[7] GABRIEL, J. P. (1975). Loi des grands nombres, séries et martingales indexées par un ensemble filtrant. Thèse de doctorat, EPF Lausanne.

[8] GABRIEL, J. P. (1976). An inequality for sums of random variables indexed by finite dimensional filtering sets and its applications to the convergence of series. Preprint, Courant Institute of Math. Sciences, New York.

[9] GUT, A. (1978). Marcinkiewicz laws and convergence rates in the law of large numbers for random variables with multidimensional indices. *Ann. Probability* **6** 469–482.

[10] HOFFMANN-JØRGENSEN, J. (1974). Sums of independent Banach space valued random variables. *Studia Math.* **L II** 159–186.

[11] KATZ, M. (1963). The probability in the tail of a distribution. *Ann. Math. Statist.* **34** 312–318.

[12] LAI, T. L. (1974). Limit theorems for delayed sums. *Ann. Probability* **2** 432–440.

[13] LAI, T. L. (1976). On r-quick convergence and a conjecture of Strassen. *Ann. Probability* **4** 612–627.

[14] LAI, T. L. and LAN, K. K. (1976). On the last time and the number of boundary crossings related to the strong law of large numbers and the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **34** 59–71.

[15] LOÈVE, M. (1963). *Probability Theory, 3rd ed.* Van Nostrand, Princeton.

[16] SLIVKA, J. (1969). On the law of the iterated logarithm. *Proc. Nat. Acad. Sci. U.S.A.* **63** 289–291.

[17] SLIVKA, J. and SEVERO, N. C. (1970). On the strong law of large numbers. *Proc. Amer. Math. Soc.* **24** 729–734.

[18] STRASSEN, V. (1966). A converse to the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **4** 265–268.

[19] STRASSEN, V. (1967). Almost sure behaviour of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statist. Probability* **2** 315–343.

[20] STRATTON, H. (1972). Moments of oscillations and ruled sums. *Ann. Math. Statist.* **43** 1012–1016.

[21] WICHURA, M. J. (1973). Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments. *Ann. Probability* **1** 272–296.

UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
THUNBERGSVÄGEN 3
S-752 38 UPPSALA, SWEDEN