A RENEWAL MODEL WITH RANDOMLY SELECTED PARAMETERS

BY FREDERICK SOLOMON

State University of New York, Purchase

Let \( \{ \mu_1, \mu_2, \cdots \} \) be chosen from a strictly stationary, ergodic sequence of random variables each with distribution concentrated on \((0, \infty)\). Let \( S_n = T_1 + \cdots + T_n \) be a sum of independent random variables where \( T_j \) is exponential with mean \( \mu_j \). Limiting properties of \( S_n \) are considered. More limiting properties are derived under the assumption that \( \{ \mu_1, \mu_2, \cdots \} \) is strongly mixing and then under the assumption of independence.

1. The model. Let \( T_1, T_2, \cdots \) be independent, exponential random variables with parameters (means) respectively \( \mu_1, \mu_2, \cdots \). The sequence \( \Lambda = \{ \mu_1, \mu_2, \cdots \} \) constitutes the parameter sequence for the renewal process \( \{ S_n = T_1 + \cdots + T_n \}_{n=0}^{\infty} (S_0 = 0) \). The \( \mu_j \)'s are chosen previous to the renewal process; they form a sample from a strictly stationary sequence of random variables each with distribution \( G \) concentrated on \((0, \infty)\). This paper is concerned with limit behaviors of the renewal process \( \{ S_n \} \) given a "typical" parameter sequence \( \Lambda \).

Notation. Set \( \lambda_i = \mu_i^{-1} \) for all \( i \). Let \( F_i \) be the exponential distribution with mean \( \mu_i \) and let \( f_i \) be the corresponding density. As usual count time 0 as renewal number 1. The convolution of distribution functions \( H_1 \) and \( H_2 \) is

\[
H_1 \ast H_2(t) = \int_{-\infty}^{\infty} h_1(t - x)H_2(dx)
\]

whereas the convolution of densities \( h_1 \) and \( h_2 \) is

\[
h_1 \ast h_2(t) = \int_{-\infty}^{\infty} h_1(t - x)h_2(x) \, dx.
\]

\( N_t \) denotes the number of renewals in \((0, t] \) so that

\[
P(N_t = n) = P(T_1 + \cdots + T_n < t, T_1 + \cdots + T_{n+1} > t)
\]

\[
= F_1 \ast \cdots \ast F_n(t) - F_1 \ast \cdots \ast F_{n+1}(t)
\]

\[
= \mu_{n+1} \cdot f_1 \ast \cdots \ast f_{n+1}(t)
\]

as can easily be verified for exponential distributions. Finally, \( U(t) \) is the expected number of renewals in \([0, t] \)

\[
U(t) = \sum_{n=0}^{\infty} F_1 \ast \cdots \ast F_n(t)
\]

—the addend for index \( n = 0 \) being the atom at the origin (evaluated at \( t \)).

The main results in this paper are contained in these two theorems:

Received September 1978; revised March 1979.

AMS 1970 subject classifications. Primary 60K05; secondary 60J75.

Key words and phrases. Renewal process, jump process, exponential distribution.
Theorem 1. Suppose \( \{\mu_1, \mu_2, \cdots\} \) is chosen from a strictly stationary, ergodic sequence. Then for a.e. parameter sequence

(a) \( U(t) = 1 + E(N_t) < \infty \) for all \( t \),
(b) \( t^{-1}U(t) \to (E(\mu))^{-1} \) as \( t \to \infty \).

If \( \{\mu_1, \mu_2, \cdots\} \) is strongly mixing, then for a.e. parameter sequence

(c) \( n^{-1}S_n \to E(\mu_1), t^{-1}N_t \to (E(\mu_1))^{-1} \) a.e. respectively as \( n, t \to \infty \).

Theorem 2. In addition if \( \{\mu_1, \mu_2, \cdots\} \) are independent, identically distributed

(d) \( t^{-1} \cdot \int_0^t P(\mu(s) < x) \, ds \to (E(\mu_1))^{-1} \cdot \int_0^\infty yG(dy) \)

where \( \mu(t) = \mu_{N_t+1} \) is parameter of the component in operation at time \( t \),

(e) \( t^{-1} \cdot \int_0^t P(H_t > \xi) \, dt \to (E(\mu_1))^{-1} \cdot \int_0^\infty y \exp(-\xi y^{-1})G(dy) \)

where \( H_t \) is the residual waiting time \( S_{N_t+1} - t \), the spent waiting time \( t - S_N \), or the interarrival time containing \( t \) : \( S_{N_t+1} - S_N \).

2. Proofs. The proof of Theorem 1 is straightforward enough; the proof of Theorem 2 relies on Lemma 4 below.

Feller [2], page 452, shows \( N_t \to \infty \) for all \( t \) (this is a pure birth process) if and only if \( \sum_{n=1}^\infty \mu_t = \infty \). Since the ergodic theorem implies that \( n^{-1}\sum_{i=1}^n \mu_i \to E(\mu) \) a.e., \( N_t \) is finite for all \( t \) for most every parameter sequence.

Proof of Theorem 1. (a)

\[
U(t) = 1 + \sum_{n=1}^\infty P(S_n < t) \\
  \leq 1 + \sum_{n=1}^\infty P(T_1 < t) \cdots P(T_n < t) \\
  = 1 + \sum_{n=1}^\infty (1 - \exp(-\lambda_1 t)) \cdots (1 - \exp(-\lambda_n t)).
\]

But for a.e. fixed parameter sequence

\[
\prod_{j=1}^n (1 - \exp(-\lambda_j t)) = \exp\left[ \sum_{j=1}^n \log(1 - \exp(-\lambda_j t)) \right] \\
  \leq \exp\left[ n(E(\log(1 - \exp(-\lambda_1 t))) + \varepsilon) \right]
\]

for \( n \) sufficiently large by the ergodic theorem. Choosing \( \varepsilon \) so that \( E(\log(1 - \exp(-\lambda_1 t))) + \varepsilon < 0 \) implies the tail of the series \( U(t) \) is bounded above by the tail of a convergent geometric series.

(b) Assume \( n^{-1}(\mu_1 + \cdots + \mu_n) \to E(\mu) \). Taking Laplace transforms

\[
\Phi(s) = \int_0^\infty e^{-st}U(dt) = \sum_{n=0}^\infty \phi_1(s) \cdots \phi_n(s)
\]

by monotone convergence where the addend for \( n = 0 \) is 1 and

\[
\phi_1(s) = \int_0^\infty e^{-st}F_1(dt) = (1 + s\mu_1)^{-1}
\]

for exponential distribution \( F_1 \). Since \( (1 + s\mu_1)^{-1} \geq e^{-s\mu_1} \),

\[
\Phi(s) > \sum_{n=0}^\infty \exp\left[ -s(\mu_1 + \cdots + \mu_n) \right].
\]

Given \( \varepsilon > 0 \), choose \( N = N(\varepsilon) \) so large that \( n > N \) implies \( \mu_1 + \cdots + \mu_n \leq n(E(\mu_1) + \varepsilon) \). Hence

\[
\Phi(s) > \sum_{n=0}^N \exp\left[ -s(\mu_1 + \cdots + \mu_n) \right] + \sum_{n=N}^\infty \exp\left[ -sn(E(\mu_1) + \varepsilon) \right].
\]
Thus
\[
\liminf_{\epsilon \downarrow 0} \Phi(s) > \liminf_{\epsilon \downarrow 0} \left( s \exp[-sN(\epsilon)(E(\mu_1) + \epsilon)] \right) / \left( 1 - \exp[-s(\epsilon + \epsilon)] \right)
\]
\[= (E(\mu_1) + \epsilon)^{-1}.
\]
So \( \liminf s\Phi(s) \geq (E(\mu_1))^{-1} \). On the other hand, let \( \tau_j^* = \mu_j \) if \( \mu_j < A \) and \( \tau_j^* = A \) if \( \mu_j > A \). For \( a < 1 \), choose \( \delta \) so that \( 0 < x < \delta \) implies \( 1 + x > ae^x \). Then for \( s < \delta/A \), \( 1 + s\mu_j \geq 1 + s\tau_j^* = ae^{s\tau_j^*} \). Thus as before, given \( \epsilon > 0 \) so that \( \tau_1 + \cdots + \tau_n > n(\epsilon(\tau_1) - \epsilon) \) for \( n > N = N(\epsilon) \),
\[
\Phi(s) \leq \sum_{n=0}^{N-1} (1 + s\mu_j)^{-1} + \sum_{n=N}^\infty a^{-n} \exp[-ns(\epsilon(\tau_1) - \epsilon)]
\]
and
\[
\limsup_{\epsilon \downarrow 0} s\Phi(s) \leq \limsup_{\epsilon \downarrow 0} a^{-N(\epsilon)} \left( \exp[-sN(\epsilon)(\epsilon(\tau_1) - \epsilon)] \right) / \left( 1 - \exp[-s(\epsilon(\tau_1) - \epsilon)] \right)
\]
\[= a^{-N(\epsilon)} / (\epsilon(\tau_1) - \epsilon) \quad \text{(at least for } A \text{ large enough).}
\]
Now letting \( a \uparrow 1 \) (it is independent of \( \epsilon \), \( \epsilon \downarrow 0 \) and \( A \uparrow \infty \) implies \( \lim s\Phi(s) = (E(\mu_1))^{-1} \) as \( s \downarrow 0 \). Thus a Tauberian theorem [3], page 421, implies \( t^{-1}U(t) \rightarrow (E(\mu_1))^{-1} \) as \( t \uparrow \infty \).

(c) We embedded the process in the larger one consisting of the Cartesian product of the set of parameter sequences \( R = (0, \infty)^N \) and the set of component lifetimes \( T = (0, \infty)^N \) where \( N \) denotes the set of positive integers. To define a probability measure on \( (R \times T, F) \) where \( F \) is the \( \sigma \)-field generated by the cylinder sets, begin by letting \( Q_\Lambda \) denote the product space measure on \( T \) where the \( j \)th slot has exponential distribution with mean \( \mu_j \). (Here \( \Lambda = \{ \mu_1, \mu_2, \cdots \} \).) On the parameter sequences \( R = \{ \Lambda \} \) let \( M \) be the measure so that \( \{ \mu_j \}_{j=1}^\infty \) is the required strictly stationary, strongly mixing sequence—each \( \mu_j \) distributed with distribution \( G \). Now for \( A \subset \) parameter sequences \( R \) and \( B \subset \) set of component lifetimes \( T \), each measurable with respect to the \( \sigma \)-fields generated by the cylinder sets, let
\[
P(A \times B) = \int_A Q_\Lambda(B)(d\Lambda).
\]
As in [4] where a similar model is considered, it is routine to show that \( P \) is well defined and extends to a probability measure on \( (R \times T, F) \). And the very definition implies

**Lemma 3.** Let \( B \) be measurable \( \subset \) set of component lifetimes \( T \). Then \( Q_\Lambda(B) = 1 \) for a.e. environment \( \Lambda \) if and only if \( P(R \times B) = 1 \).

Returning to the proof of (c), let \( T^*_i \) be the random variable on \( R \times T \) defined by \( T^*_i(\Lambda, \omega) = T_i(\omega) = \omega_j \) (= \( j \)th component of \( \omega \). Hence
\[
P(T^*_i < t) = \int_R Q_\Lambda(T^*_i < t)(d\Lambda)
\]
\[= \int R^\infty (1 - \exp(-tv^{-1}))G(\phi_t).
\]
So \( E(T^*_i) = E(\mu_1) \). A straightforward verification shows that strict stationarity and
the strong mixing of $\mu_1$, $\mu_2$, $\cdots$ imply these properties hold for $T_1^*$, $T_2^*$, $\cdots$. Thus the ergodic theorem implies as $n \to \infty$

$$n^{-1}S_n^* = n^{-1}(T_1^* + \cdots + T_n^*) \to E(\mu_1) \quad \text{a.e.}$$

But $(\Lambda, \omega) : n^{-1}S_n^*(\Lambda, \omega) \to E(\mu_1)$ as $n \to \infty) = \{(\Lambda, \omega) : n^{-1}S_n^*(\omega) \to E(\mu_1)\}$ as $n \to \infty) = R \times (\omega : n^{-1}S_n^*(\omega) \to E(\mu_1))$ as $n \to \infty)$. Therefore Lemma 3 implies as $n \to \infty n^{-1}S_n^* \to E(\mu_1)$ a.e. for a.e. fixed parameter sequence. Since $N_t$ increases with $t$, $\{N_t \to \infty \text{ as } t \to \infty\} \subset \bigcap_{n=1}^\infty \bigcup_{j=1}^\infty (N_j \geq n) = \bigcap_{k=1}^\infty \bigcup_{j=1}^\infty \{S_j < j\}$ which is a set of measure 1 since each $S_n$ has a proper probability distribution for each parameter sequence. Thus $N_t \to \infty$ a.e. Now $S_{N_t} \leq t < S_{N_t+1}$. So

$$(N_t)^{-1}S_{N_t} \leq (N_t)^{-1}t < (N_t)^{-1}S_{N_t+1} = (N_t)^{-1}(S_{N_t} + T_{N_t+1}).$$

Hence it remains to show that $n^{-1}T_{n+1} \to 0$ a.e. in the case where $E(\mu_1) < \infty$. But

$$P(\{n^{-1}T_{n+1} > \varepsilon\}) = P(T_{n+1} > n\varepsilon) = \exp(-n\varepsilon\lambda_{n+1})$$

(recalling that $T_n$ has exponential distribution with mean $\mu_n$ when the parameter sequence is fixed). Thus the first Borel-Cantelli lemma [1], page 69, implies $n^{-1}T_{n+1} \to 0$ a.e. if $\sum_{k=0}^\infty \exp(-n\varepsilon\lambda_{n+1}) < \infty$. But this is true for a.e. parameter sequence since this series has a finite expectation: By monotone convergence

$$E(\sum_{k=0}^\infty \exp(-n\varepsilon\lambda_{n+1})) = \sum_{k=0}^\infty E(\exp(-n\varepsilon\lambda_{n+1})) = \sum_{k=0}^\infty \int_0^\infty \exp(-n\varepsilon y^{-1})G(dy).$$

This converges by the integral test since

$$\int_0^\infty \int_0^\infty \exp(-n\varepsilon y^{-1})G(dy) \, dt = e^{-1}E(\mu_1).$$

Thus $t(N_t)^{-1} \to (E(\mu_1))^{-1}$ a.e. which completes the proof of Theorem 1.

The proof of Theorem 2 depends on

**Lemma 4.** Let $X_1$, $X_2$, $\cdots$ be independent, identically distributed each with finite expectation $m$ and finite variance $\sigma^2$. Set

$$Y(a) = \Sigma_{j=0}^\infty a(1-a)^jX_{j+1}.$$ 

Then $\lim_{a \to 0} Y(a) = m$ a.e.

**Proof.** Setting $X_j = X_j^+ + X_j^-$ where $X_j^\pm = \max(\pm X_j, 0)$, and

$$Y(a)^\pm = a\Sigma_{j=0}^\infty (1-a)^jX_{j+1}^\pm$$

shows that a proof for nonnegative random variables $X_j$ suffices. Hence assume throughout that each $X_j$ is nonnegative.

By monotone convergence $E(Y(a)) = m$; hence for each fixed $0 < a < 1$, $Y(a)$ converges a.e. Also by direct calculation $E(Y(a)^2) = m^2 + \sigma^2a/(2-a)$. So

$$\sigma^2(Y(a)) = \sigma^2a/(2-a).$$

Hence $P(1\{Y(a) - m| > \varepsilon\}) \leq \sigma^2a/(\varepsilon^2(2-a))$ by Chebyshev's inequality. Now the first Borel-Cantelli lemma implies that the sequence $(Y(n)^{1/2})_{n=1}^\infty$ converges to $m$ everywhere on a set $\Omega$ of probability 1. The
claim is that the full set \( \{ Y(a) \} \) converges to \( m \) a.e. as \( a \downarrow 0 \). To see this suppose that \((n + 1)^{-2} < a < n^{-2}\). Now \( h(x) = x(1 - x)^j \) is increasing on \([0, 1/(j + 1)]\) and decreasing on \([1/(j + 1), 1]\). Thus

\[
a(1 - a)^j < n^{-2}(1 - n^{-2})^j \quad \text{for } n^{-2} < 1/(j + 1) \quad \text{or } j < n^2 - 1
\]

\[
< (n + 1)^{-2}(1 - (n + 1)^{-2})^j \quad \text{for } (n + 1)^{-2} > 1/(j + 1) \quad \text{or } j > n^2 + 2n.
\]

When \( n^2 < j < n^2 + 2n \) a bound for \( h(a) = a(1 - a)^j \) is obtained in this way: \( h \) is concave on \((n + 1)^{-2} < a < n^{-2}\) (for \( n > 3 \)); so

\[
h(a) < h((n + 1)^{-2}) + h'((n + 1)^{-2})(a - (n + 1)^{-2})
\]

\[
< h((n + 1)^{-2}) + h'((n + 1)^{-2})(n^{-2} - (n + 1)^{-2}).
\]

But a routine calculation shows the second term in the last right-hand side is less than \( h(n^{-2}) \) for \( n \) large. Hence for \( n^2 < j < n^2 + 2n, n \) large

\[
a(1 - a)^j < n^{-2}(1 - n^{-2}) + (n + 1)^{-2}(1 - (n + 1)^{-2}).
\]

So, for \( a \) close enough to 0

\[
Y(a) < \sum_{j=0}^{n^2+2n} n^{-2}(1 - n^{-2})^j X_{j+1} + \sum_{j=n^2}^{\infty} (n + 1)^{-2}(1 - (n + 1)^{-2})^j X_{j+1}
\]

\[
= Y((n + 1)^{-2}) + Z_1 + Z_2
\]

where

\[
Z_1 = \sum_{j=0}^{n^2-1} n^{-2}(1 - n^{-2})^j - (n + 1)^{-2}(1 - (n + 1)^{-2})^j X_{j+1}
\]

\[
Z_2 = \sum_{j=n^2}^{\infty} n^{-2}(1 - n^{-2})^j X_{j+1}.
\]

Now \( \max_{0 < x < 1} |h'(x)| = 1 \). Thus the mean value theorem implies that the term multiplying \( X_{j+1} \) in the series defining \( Z_1 \) is \( < \) in absolute value

\[
|n^{-2} - (n + 1)^{-2}| = (2n + 1)/(n^2(n + 1)^2)
\]

\[
< 2/n^3.
\]

Hence

\[
|Z_1| < \sum_{j=0}^{n^2-1} (2/n^3) X_{j+1}
\]

\[
= (2/n)n^{-2} \cdot \sum_{j=0}^{n^2-1} X_{j+1}
\]

\[
\rightarrow 0 \text{ a.e.}
\]

by the law of large numbers. Also,

\[
|Z_2| < n^{-2} \sum_{j=n^2}^{\infty} n^{-2} X_{j+1}
\]

\[
= (2n + 1/n^2) \cdot (2n + 1)^{-1} \sum_{j=n^2}^{\infty} n^{-2} X_{j+1}
\]

\[
\rightarrow 0 \text{ a.e.}
\]

using the first Borel-Cantelli lemma and Chebyshev's inequality. (Note: for

\[
Z' = (2n + 1)^{-1} \sum_{j=n^2}^{\infty} n^{-2} X_{j+1},
\]
\[ E(Z') = m, \sigma^2(Z') = \sigma^2/(2n + 1) \]. Combining this with the analogous reverse inequality yields \( Y(n^{-2}) + W_n < Y(a) < Y((n + 1)^{-2}) + W_n' \) where \( W_n \) and \( W_n' \) → 0 as \( n \uparrow \infty \) a.e., say on set \( \Omega' \) of measure 1. Therefore on \( \Omega \cap \Omega' \), \( Y(a) \rightarrow m \) as \( a \downarrow 0 \) a.e.

**Proof of Theorem 2.** (d)

\[
P(\mu(t) < x) = \sum_{n=1}^{\infty} P(\mu(t) < x | N_t = n - 1) \cdot P(N_t = n - 1)
\]

\[
= \sum_{n=1}^{\infty} \epsilon_n \mu_n f_1 \ast \cdots \ast f_n(t)
\]

where \( \epsilon_n = 1, 0 \) respectively if \( \mu_n < , > x \). So

\[
\Theta(s) = \int_0^\infty e^{-st} P(\mu(t) < x) \, dt
\]

\[
= \sum_{n=1}^{\infty} \epsilon_n \mu_n \phi_1(s) \cdots \phi_n(s)
\]

\[
= \sum_{n=1}^{\infty} \epsilon_n \mu_n (1 + s\mu_1)^{-1} \cdots (1 + s\mu_n)^{-1}.
\]

In the same way as in (b),

\[
\lim_{s \downarrow 0} s\Theta(s) = \lim_{s \downarrow 0} s\sum_{n=1}^{\infty} \epsilon_n \mu_n e^{-sE(\mu_n)}
\]

\[
= \lim_{s \downarrow 0} s/(1 - e^{-sE(\mu_1)}) \cdot (1 - e^{-sE(\mu_1)}) \sum_{n=1}^{\infty} \epsilon_n \mu_n e^{-sE(\mu_n)}.
\]

Lemma 4 now applies with the result

\[
\lim_{s \downarrow 0} s\Theta(s) = (E(\mu_1))^{-1} E(\epsilon_1 \mu_1) = (E(\mu_1))^{-1} \int_0^\infty yG(dy).
\]

Application of the same Tauberian theorem yields result (d).

(c) Details are similar in all three cases and much the same as in (d); so only the outline for the case \( H_t = \) residual waiting time \( S_{N_t+1} - t \) is here presented. Now

\[
P(H_t > \xi) = \sum_{n=0}^{\infty} P(H_t > \xi | N_t = n - 1) \cdot P(N_t = n - 1)
\]

\[
= \sum_{n=0}^{\infty} P(T_n > \xi) \cdot \mu_n f_1 \ast \cdots \ast f_n(t)
\]

by the "memoryless" property of exponential random variables. Let \( \rho(s) \) be

\[
\int_0^\infty e^{-st} P(H_t > \xi) \, dt = \sum_{n=0}^{\infty} \mu_n e^{-\lambda t} \phi_1(s) \cdots \phi_n(s).
\]

As in (d)

\[
\lim_{s \downarrow 0} s\rho(s) = \lim_{s \downarrow 0} \sum_{n=0}^{\infty} \mu_n e^{-\lambda t} \phi_1(s) \cdots \phi_n(s)
\]

\[
= (E(\mu_1))^{-1} E(\mu_1 e^{-\lambda t}).
\]

Application of the same Tauberian theorem completes the proof. (Note that the proofs apply with the usual modifications when \( E(\mu_1) = \infty \).)

**3. Randomizing the parameter sequence.** The above process may be compared with the process in which the \( \mu_j \)'s are random independent, identically distributed rather than preselected and fixed. The probabilistic setting for this new process has been defined at the beginning of the proof of Theorem 1: \( T_1, T_2, \cdots \) are independent, identically distributed each with density

\[
f(t) = \int_0^\infty ye^{-yG(dy)}
\]
for $t > 0$. So the model reduces to the standard renewal model of [3], chapter 11. Still it may be of interest to calculate the distribution of $\mu(t) = \mu_{N_i+1}$ = parameter of the component in operation at time $t$.

**Theorem 5.** In the renewal model in which $\{\mu_i\}_{i=1}^{\infty}$ is a sequence of independent, identically distributed random variables with distribution $G$ concentrated on $(0, \infty)$, (a) $\{\mu(t)\}_{t>0}$ is a Markov process; (b) $\mu(t)$ approaches in distribution $(E(\mu_i))^{-1} \cdot yG(dy)$ when $E(\mu_i) < \infty$.

**Proof.** It is clear that $\{\mu_i\}$ is Markovian since each $T_j$ is exponentially distributed. Now $\{\mu(t)\}_{t=0}^{\infty}$ constitutes a jump process. Given $\mu(t) = x$, the waiting time till the next jump is exponential with mean $1/x$ at which time the process jumps to another state according to distribution $G$ independent of $x$. Hence with $Q_t(x, \Omega) = P(\mu(t) \in \Omega | \mu(0) = x)$ Kolmogorov's backward equations are

$$\frac{\partial Q_t(x, \Omega)}{\partial t} = x^{-1}Q_t(x, \Omega) + x^{-1}\int_0^\infty Q_t(y, \Omega)G(dy)$$

[3], page 317. The infinitesimal generator associated with $Q_t$ is thus

$$Uu(x) = -x^{-1}[u(x) - \int_0^\infty u(y)G(dy)].$$

So the resolvent operator is

$$R_xw(x) = (1 + \tau x)^{-1}[xw(x) + C]$$

where

$$C = \left(\int_0^\infty \tau y(1 + \tau y)^{-1}G(dy)\right)^{-1} \cdot \int_0^\infty yw(y)(1 + \tau y)^{-1}G(dy)$$

since $R_x$ is the inverse of $\tau - U$. Or

$$R_xw(x) = (1 + \tau x)^{-1}xw(x) + L(h_1 * h_2 * U)(\tau)$$

where $L$ indicates the Laplace transform of the function $h_1 * h_2 * U$ and

$$h_1(s) = x^{-1}e^{-s/x}, \quad s > 0$$

$$h_2(s) = \int_0^\infty w(y)e^{-s/y}G(dy), \quad s > 0$$

$$U(t) = \sum_{k=0}^\infty F*e^{n}(t), \quad F(t) = \int_0^\infty (1 - e^{-t/y})G(dy) \quad \text{for } t > 0.$$ 

Now $P(\mu(t) < x | \mu(0) = \mu_0) = \int_0^\infty w(y)Q_t(\mu_0, dy)$ where $w(y) = 1$ if $0 < y < x$ and 0 otherwise. Since the Laplace transform of this function (as a function of $t$) is $R_xw(x)$, taking inverse transforms implies

$$P(\mu(t) < x | \mu(0) = \mu_0) = e^{-t/x}w(x) + h_1 * h_2 * U(t).$$

Since $h_1 * h_2$ is directly Riemann integrable, the renewal theorem of [3], page 349, implies as $t \to \infty$

$$P(\mu(t) < x | \mu(0) = \mu_0) = (E(\mu_1))^{-1} \int_0^\infty h_1 * h_2(t) dt$$

$$= (E(\mu_1))^{-1} \int_0^\infty yG(dy).$$
REFERENCES


DEPARTMENT OF MATHEMATICS
STATE UNIVERSITY OF NEW YORK
COLLEGE AT PURCHASE
PURCHASE, NEW YORK 10577