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ZERO-RANGE INTERACTION AT BOSE-EINSTEIN SPEEDS UNDER A POSITIVE RECURRENT SINGLE PARTICLE LAW

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The equilibrium states and the time asymptotic behavior of Spitzer's zero-range interaction scheme are studied in the case of a positive-recurrent, irreducible single particle law and an attractive speed function. It is shown that within the natural phase space of finitely many particles per cell, clusters occur which lead time-asymptotically to infinite occupancy by certain individual cells. These cells are then identified in terms of the parameters of the model.

1. Introduction. The zero-range interaction is one among several of the Markov processes introduced by Spitzer (1970) to describe the evolution of a system of infinitely many indistinguishable particles which move along a countable set Λ according to random dynamics. The evolution depends on an irreducible transition probability matrix $P = (p(x, y) : x, y \in \Lambda)$ and a positive real-valued function, m, defined on the set, $\mathfrak{N}$, of natural numbers. The elements of Λ are called *cells*, P is called the *single particle law*, and m is called the *speed function*. Think of there being some initial distribution of finitely many particles per cell on Λ which evolve according to the following scheme: given that there are k particles in the cell x, after an exponentially distributed amount of time with parameter $m(k)$, a particle jumps from cell x to a cell y with probability $p(x, y)$.

As noted by Spitzer (1970), there are some interesting special cases to consider in the zero-range model. If m is constant then the particles evolve independently according to a common jump transition matrix. This is the case of no-interaction. When Λ is finite, Spitzer (1970) has shown that the Maxwell-Boltzmann distribution is the equilibrium distribution for the system. This case had been studied earlier, with Λ countably infinite, in a doctoral thesis by Derman (1955) and, as is now to be expected, it was shown that equilibrium distributions could be constructed by distributing a "Poisson" number of particles independently in each cell. Another interesting case arises when m is decreasing inversely with increasing occupancy number. In this case we can think of there being attractive forces between the particles which increase the mean holding times in cells occupied by more than one particle. When Λ is finite, Spitzer (1970) obtained the Bose-Einstein distribution as the equilibrium distribution for the system.

In this paper we shall investigate the infinite particle zero-range model for the Bose-Einstein speed function $m(k) = 1/k$. Aside from existence, there are two

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typical problems which are central to the theory of infinite particle systems. Specifically, the problems are the identification of the totality of equilibria for the system and, secondly, the identification of the time-asymptotic behavior of the system.

2. Mathematical preliminaries and statements of results. Let $\overline{\mathfrak{N}} = \mathfrak{N} \cup \{\infty\}$ be the one-point compactification of the discrete topology on $\mathfrak{N} = \{0, 1, 2, \cdots\}$. For topological reasons, define the phase space for the system by $\bar{S}_\Lambda = \overline{\mathfrak{N}}^\Lambda$ and give $\bar{S}_\Lambda$ the corresponding product topology. The Borel sigma-field of subsets of $\bar{S}_\Lambda$ is denoted by $\overline{\mathfrak{B}}_\Lambda$. The space of observables, $E(\bar{S}_\Lambda)$, is the Banach space consisting of all continuous real-valued functions of $\bar{S}_\Lambda$ equipped with the uniform norm. The sub-class of $E(\bar{S}_\Lambda)$ which consists of those observables which depend on the occupation numbers at finitely many cells is denoted by $E_f(\bar{S}_\Lambda)$. It follows from the Stone-Weierstrass theorem that $E_f(\bar{S}_\Lambda)$ is dense in $E(\bar{S}_\Lambda)$.

According to Liggett's existence theorem (Liggett, 1972), if we assume

$$\sup_y \Sigma_{x \in \Lambda} p(x, y) < \infty, \tag{2.1}$$

then there is a unique strongly continuous, Feller transition function, $(P(t, \sigma; B) : t > 0, \sigma \in \bar{S}_\Lambda, B \in \overline{\mathfrak{B}}_\Lambda)$ with infinitesimal generator $(\bar{A}, \mathfrak{D}_{\bar{A}})$ such that

$$\bar{A} g(\sigma) = \Sigma_{x \in \Lambda} \Sigma_{y \in \Lambda} I_{B_x}(\sigma) p(x, y) [\, g(\sigma_{x,y}) - g(\sigma)], \; g \in E_f(\bar{S}_\Lambda), \tag{2.2}$$

where

$$\begin{aligned} o_{x,y}(z) &= \sigma(x) - 1 \quad && \text{if } z = x \quad && \text{and if } \sigma(x) > 1 \\ &= \sigma(y) + 1 && \text{if } z = y && \text{and if } \sigma(x) > 1 \\ &= \sigma(u), && \text{otherwise.} \end{aligned} \tag{2.3 }$$

$$B_x = \{\sigma \in \bar{S}_\Lambda : \sigma(x) > 1\}. \tag{2.4}$$

The conditions on the speed function given in Liggett's existence theorem are easily checked in the present case where

$$\begin{aligned} m(k) &= 1/k, \quad && k = 1, 2, \cdots \\ &= 0, && k = 0, k = \infty. \end{aligned} \tag{2.5 }$$

In passing, it should be mentioned that there are important classes of speed functions for which Liggett's existence theorem fails. However, Liggett has shown that by "diluting" the class of allowable initial distributions, it is possible to prove the existence of a system with the desired dynamics under relaxed conditions on the speed function (cf. Liggett, 1973).

A measure $\bar{a} = (\bar{a}(x) \colon x \in \Lambda)$ on the sigma-field of all subsets of the set Λ such that

$$\bar{a}(x) > 0 \qquad \text{for each } x \in \Lambda \tag{2.6}$$

$$\bar{a}(\Lambda) > 0 \tag{2.7}$$

$$\Sigma_{x \in \Lambda} \bar{a}(x) p(x, y) = \bar{a}(y) \qquad \text{for each } y \in \Lambda \tag{2.8}$$

is called a nontrivial, nonnegative, invariant measure for P. In view of the irreducibility of the single particle law, P, the condition (2.8) makes the conditions

(2.6) and (2.7) equivalent to the single condition $\bar{a}(x) > 0$ for each $x \in \Lambda$. We let $\mathcal{I}(P)$ denote the set of all nontrivial nonnegative invariant measures of P. If P is recurrent then $\mathcal{I}(P) \neq \phi$ and is given by constant multiples of a single nontrivial invariant measure. In the case that P is positive-recurrent, this single measure is finite and, consequently, has a bounded density. In the case that P is null-recurrent, it is not a finite measure and may or may not have a bounded density. In the case that P is transient, $\mathcal{I}(P)$ may be empty or P may have infinitely many nontrivial invariant measures.

In Section 3 we shall examine states of equilibria for the system which correspond to those invariant measures $\bar{a} \in \mathcal{I}(P)$ for which

$$\sup_{x\in\Lambda}\bar{a}(x) < \infty. \tag{2.9}$$

Apart from condition (2.1) for existence, we assume that there is an $\bar{a} \in \mathcal{I}(P)$ for which (2.9) is valid. For each real number ρ, $0 \leqslant \rho < [\sup_x \bar{a}(x)]^{-1}$, let $\pi_{\rho\bar{a}}$ be the product measure on $(\bar{S}_\Lambda, \bar{\mathfrak{B}}_\Lambda)$ with marginal factors $\nu^x_{\rho\bar{a}}$, $x \in \Lambda$, given by

$$\begin{aligned} \nu^x_{\rho\bar{a}}(k) &= [1 - \rho\bar{a}(x)]\cdot[\rho\bar{a}(x)]^k, \quad && k = 0, 1, 2, \cdots \\ &= 0 && k = \infty. \end{aligned} \tag{2.10}$$

For $\rho = \rho_M = [\sup_x \bar{a}(x)]^{-1}$, let $\pi_{\rho_M\bar{a}}$ be the product measure on $(S_\Lambda, \mathfrak{B}_\Lambda)$ with marginal factors $\nu^x_{\rho_M\bar{a}}$, $x \in \Lambda$, given by

$$\begin{aligned} \nu^x_{\rho_M\bar{a}}(k) &= [1 - \rho_M\bar{a}(x)][\rho_M\bar{a}(x)]^k, \quad && k = 0, 1, 2, \cdots \\ &= 0, && k = \infty \end{aligned} \tag{2.11}$$

if $\bar{a}(x)\rho_M < 1$, and

$$\begin{aligned} \nu^x_{\rho_M\bar{a}}(k) &= 0, \quad && k = 0, 1, 2, \cdots \\ &= 1, && k = \infty \end{aligned} \tag{2.12}$$

if $\bar{a}(x)\rho_M = 1$.

Since $\bar{S}_\Lambda$ is compact the family $\{\pi_{\rho\bar{a}} : 0 \leqslant \rho \leqslant \rho_M\}$ is tight. Moreover, it is not hard to show that

$$\pi_{\rho_M\bar{a}} = \lim_{\rho\uparrow\rho_M}\pi_{\rho\bar{a}}. \tag{2.13}$$

THEOREM (2.14). *Assume that there is an $\bar{a} \in \mathcal{I}(P)$ such that* (2.9) *holds. Then each $\pi_{\rho\bar{a}}$, $0 \leqslant \rho \leqslant \rho_M$ and $\rho = \infty$, is an equilibrium state for $(P(t, \sigma, B)$: $t > 0$, $\sigma \in \bar{S}_\Lambda$, $B \in \bar{\mathfrak{B}}_\Lambda)$ where $\pi_{\infty\bar{a}}$ is the point mass representing infinite occupancy in each cell.*

The remainder of Section 3 is devoted to properties of the equilibria given in (2.14) under the condition that P is positive recurrent.

The investigation of the long-time behavior of the system is carried out in Section 5 under the assumption that P is positive-recurrent. However, this investigation is based on certain monotonicity results which are given in Section 4. The techniques employed in Section 4 are popular in the study of particle systems and they are essentially due to Liggett and Spitzer.

THEOREM (2.15). *Assume that the single particle law, P, is positive-recurrent and satisfies* (2.1). *Let* $\bar{a} \in \mathcal{I}(P)$ *and let* y_M *denote a solution to* $\bar{a}(y) = \max_x \bar{a}(x)$. *Define* $C\binom{k}{y_M} = \{\sigma \in S_\Lambda: \sigma(y_M) \geq k\}$. *If* $\sigma \in S_\Lambda$ *and if* $\Sigma_x \sigma(x) = \infty$, *then*

$$\lim_{t\to\infty} \bar{P}\left(t, \sigma; C\binom{k}{y_M}\right) = 1$$

for each $k = 0, 1, 2, \cdots$.

The natural state space for the system is S_Λ rather that $\bar{S}_\Lambda$. However, according to Theorem (2.15), clustering will occur. Consequently, as a corollary to Theorem (2.15) we can identify the class of all equilibria for the zero-range system restricted to S_Λ.

COROLLARY (2.16). *Fix* ρ *such that* $0 \leq \rho < \rho_M$ *and define*

$$\mu^{(N)} = \pi_{\rho\bar{a}}(\cdot \mid \Sigma_{x\in\Lambda}\sigma(x) = N), \qquad N = 0, 1, 2, \cdots.$$

Then $\{\mu^{(N)}: N = 1, 2, \cdots\}$ *is precisely the collection of all extremal equilibria for* $\bar{P}$ *restricted to* $(S_\Lambda, \bar{\mathfrak{B}}_\Lambda \cap S_\Lambda)$.

3. **Equilibria.** In this section we shall examine states of equilibria for the zero-range system which correspond to those invariant measures $\bar{a} \in \mathcal{I}(P)$ for which (2.9) is valid. The assumption that P is positive-recurrent is not necessary in Theorem (2.14), however, condition (2.9) is certainly valid under this assumption.

PROOF OF (2.14). The two cases $\rho = 0$ and $\rho = \infty$ are clear. The case in which $\rho = \rho_M$ follows from the case $0 < \rho < \rho_M$ because the class of equilibrium states is closed in the weak* topology. So, we fix ρ between 0 and ρ_M. Let $g \in E_f(\bar{S}_\Lambda)$ and let $C(g)$ denote the coordinate set of g. Then $g(\sigma) = g(\eta)$ if $\sigma = \eta$ (pointwise) on $C(g) \subset \Lambda$, and

$$\begin{aligned}\bar{A}g(\sigma) = &\Sigma_{x\in C(g)}\Sigma_{y\in\Lambda} I_{B_x}(\sigma)p(x,y)[g(\sigma_{x,y}) - g(\sigma)] \\ &+ \Sigma_{x\in\Lambda-C(g)}\Sigma_{y\in C(g)} I_{B_x}(\sigma)p(x,y)[g(\sigma_{x,y}) - g(\sigma)].\end{aligned}$$

Convergence of each of the above series is automatic by (2.1) and the fact that $\Sigma_{y\in\Lambda}p(x,y) = 1$ for each $x \in \Lambda$. Since by construction (cf. Liggett, 1972), $(\bar{A}, \mathfrak{D}_{\bar{A}})$ is the closure of the operator $(\bar{A}, E_f(S_\Lambda))$, it suffices to show $\int_{S_\Lambda} \bar{A}g(\sigma)\pi_{\rho\bar{a}}(d\sigma) = 0$ for all $g \in E_f(\bar{S}_\Lambda)$.

Consider first the integral

$$I_1 = \int_{\bar{S}_\Lambda}\{\Sigma_{x\in C(g)}\Sigma_{y\in\Lambda} p(x,y)\cdot I_{B_x}(\sigma)[g(\sigma_{x,y}) - g(\sigma)]\}\pi_{\rho\bar{a}}(d\sigma).$$

Since $\pi_{\rho\bar{a}}(S_\Lambda) = 1$ for $0 \leq \rho < \rho_M$, the integral I_1 is given by the corresponding integral over S_Λ. We have, under the change of variables $\sigma_{x,y} \to \sigma$,

$$\begin{aligned}I_1 = &\Sigma_{x\in C(g)}\Sigma_{y\in\Lambda\setminus C(g)}\rho\cdot p(x,y)\bar{a}(x)\int_{S_\Lambda} g(\sigma)\pi_{\rho\bar{a}}(d\sigma) \\ &+ \Sigma_{x\in C(g)}\Sigma_{y\in C(g)} p(x,y)\frac{\bar{a}(x)}{\bar{a}(y)}\int_{S_\Lambda\cap B_y} g(\sigma)\pi_{\rho\bar{a}}(d\sigma) \\ &- \Sigma_{x\in C(g)}\int_{S_\Lambda\cap B_x} g(\sigma)\pi_{\rho\bar{a}}(d\sigma),\end{aligned}$$

where this simplification is made by using the independence of $g(\sigma)$ and $\sigma(y)$ when $y \in \Lambda \setminus C(g)$ and the invariance of $\bar{a}$ under P. Now consider,

$$I_2 = \int_{S_\Lambda}\{\Sigma_{x\in\Lambda\setminus C(g)}\Sigma_{y\in C(g)}p(x,y)I_{B_x}(\sigma)[g(\sigma_{x,y}) - g(\sigma)]\}\pi_{\rho\bar{a}}(d\sigma)$$
$$= \Sigma_{x\in\Lambda\setminus C(g)}\Sigma_{y\in C(g)}p(x,y)\int_{S_\Lambda\cap B_x}g(\sigma_{x,y})\pi_{\rho\bar{a}}(d\sigma)$$
$$-\Sigma_{x\in\Lambda\setminus C(g)}\Sigma_{y\in C(g)}p(x,y)\int_{S_\Lambda\cap B_x}g(\sigma)\pi_{\rho\bar{a}}(d\sigma).$$

If again we use independence and the invariance of $\bar{a}$, then we get,

$$I_2 = \Sigma_{y\in C(g)}\int_{S_\Lambda\cap B_y}g(\sigma)\pi_{\rho\bar{a}}(d\sigma)$$
$$-\Sigma_{x\in C(g)}\Sigma_{y\in C(g)}\frac{\bar{a}(x)}{\bar{a}(y)}p(x,y)\int_{S_\Lambda\cap B_y}g(\sigma)\pi_{\rho\bar{a}}(d\sigma)$$
$$-\Sigma_{x\in\Lambda\setminus C(g)}\Sigma_{y\in C(g)}\rho\bar{a}(x,y)\int_{S_\Lambda}g(\sigma)\pi_{\rho\bar{a}}(d\sigma).$$

Combining these expressions and observing that

$$\Sigma_{x\in C(g)}\Sigma_{y\in\Lambda\setminus C(g)}\rho\bar{a}(x)p(x,y) = \Sigma_{x\in C(g)}\Sigma_{y\in\Lambda\setminus C(g)}\rho\bar{a}(y)p(y,x)$$

we obtain the result. □

PROPOSITION (3.1). *Assume that P is positive-recurrent and let $\bar{a} \in \mathcal{G}(P)$. For $0 \leqslant \rho < \rho_M$ we have $\pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) < \infty) = 1$.*

PROOF. The event $\{\sigma \in \bar{S}_\Lambda: \Sigma_x\sigma(x) < \infty\}$ is a $\pi_{\rho\bar{a}}$ zero-one event. It therefore suffices for us to observe that

$$\pi_{\rho\bar{a}}(\Sigma_x\sigma(x) < \infty) \geqslant \pi_{\rho\bar{a}}(\Sigma_x\sigma(x) = 0)$$
$$= \Pi_{x\in\Lambda}(1 - \rho\bar{a}(x)) > 0$$

since $0 < \bar{a}(x) < 1$ and $\Sigma_{x\in\Lambda}\bar{a}(x) < \infty$. □

Using Liggett's construction (Liggett, 1972), it is not hard to show that the particle number of the system is conserved. Also, it is to be noted that, since $\bar{a}$ is summable, we have for $0 < \rho < \rho_M$, $\pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) = N) > 0$ for each $N = 0, 1, 2, \cdots$.

4. Monotonicity of states. The evolution of the state of the system in the space, $\bar{\mathfrak{S}}_\Lambda$, consisting of all probabilities on $(\bar{S}_\Lambda, \bar{\mathfrak{B}}_\Lambda)$, takes place according to the adjoint to the semigroup of linear operators furnished by $\{\bar{P}(t,\cdot,\cdot)\}$. In this section we shall investigate the monotonicity of the evolution of states under this adjoint law. The techniques adopted in this section are quite familiar in the study of particle systems (cf. Spitzer, 1974; Liggett, 1974).

Let $M = \{f \in E(\bar{S}_\Lambda): f(\sigma) \leqslant f(\eta) \text{ if } \sigma \leqslant \eta \text{ (pointwise)}\}$. Let $\mu, \nu \in \bar{\mathfrak{S}}_\Lambda$. Then $\mu \leqslant \nu$ if and only if

$$(4.1) \qquad \int_{\bar{S}_\Lambda}f(\sigma)\nu(d\sigma) \leqslant \int_{\bar{S}_\Lambda}f(\sigma)\nu(d\sigma) \quad \text{for all } f \in M.$$

Note that (4.1) defines a partial ordering on $\bar{\mathfrak{S}}_\Lambda$.

THEOREM (4.2). *Let $\mu, \nu \in \bar{\mathfrak{S}}_\Lambda$ such that $\mu \leqslant \nu$. Define $\nu_t = \int_{S_\Lambda} \bar{P}(t\ \sigma, \cdot)\mu(d\sigma)$, $t \geqslant 0$ and $\nu_t = \int_{S_\Lambda} \bar{P}(t, \sigma, \cdot)\nu(d\sigma)$, $t \geqslant 0$. Then*

$$\mu_t \leqslant \nu_t \qquad \text{for all} \quad t \geqslant 0.$$

PROOF. Let (σ_t, η_t) denote the coupled process on $(\bar{S}_\Lambda \times \bar{S}_\Lambda \mathfrak{B}_\Lambda \times \mathfrak{B}_\Lambda)$ with infinitesimal generator given by

$$\begin{aligned}\Omega g(\sigma, \eta) = {} & \Sigma_{(x,y)\,:\,\sigma(x)\geqslant 1,\,\eta(x)\geqslant 1}\, p(x, y)\big[\, g(\sigma_{x,y}, \eta_{x,y}) - g(\sigma, \eta)\big] \\ & + \Sigma_{(x,y)\,:\,\sigma(x)\geqslant 1,\,\eta(x)=0}\, p(x, y)\big[\, g(\sigma_{x,y}, \eta) - g(\sigma, \eta)\big] \\ & + \Sigma_{(x,y)\,:\,\sigma(x)=0,\,\eta(x)\geqslant 1}\, p(x, y)\big[\, g(\sigma, \eta_{x,y}) - g(\sigma, \eta)\big]\end{aligned}$$

for $g \in E_f(\bar{S}_\Lambda \times \bar{S}_\Lambda)$, $(\sigma, \eta) \in \bar{S}_\Lambda \times \bar{S}_\Lambda$. Then the infinitesimal generators of the marginal processes are given by $(\bar{A}, \mathfrak{D}_{\bar{A}})$. The existence of the coupled process comes from Liggett's existence theorem. Now, let $\tilde{S} = \{(\sigma, \eta) \in \bar{S}_\Lambda \times \bar{S}_\Lambda : \sigma \leqslant \eta$ (pointwise)$\}$. Then, $\Omega I_{\tilde{S}} = 0$. So, in particular, $\tilde{S}$ is closed under the evolution of the coupled process. It now follows that for $f \in M$,

$$\begin{aligned}\int_{\bar{S}_\Lambda} f(\sigma)\mu_t(d\sigma) - \int_{\bar{S}_\Lambda} f(\sigma)\nu_t(d\sigma) &= \int_{\bar{S}_\Lambda} E_\sigma f(\sigma_t)\mu(d\sigma) - \int_{\bar{S}_\Lambda} E_\sigma f(\sigma_t)\nu(d\sigma) \\ &= \int_{\bar{S}_\Lambda \times \bar{S}_\Lambda} E_{(\sigma,\eta)}\big[f(\sigma_t) - f(\eta_t)\big]d(\mu \times \nu)(\sigma, \eta) \\ &\leqslant 0. \qquad \square\end{aligned}$$

REMARKS. The key to the proof of Theorem (4.2) is in showing that $I_{\tilde{S}} \in \mathfrak{D}_\Omega$. That this is true follows from Liggett's construction of the process with the closure of $(\Omega, E_f(S_\Lambda))$ as its generator, (cf. Liggett, 1972). Liggett (1974) has shown that the evolution of states for simple exclusion is monotonic and it is from there that the idea behind Theorem (4.2) was adopted.

5. The long-time behavior of the system. We recall that for each $N = 0, 1, 2, \cdots, S_\Lambda(N)$ is closed under the evolution of the system. The restriction of the transition law to $S_\Lambda(N)$ shall be referred to as the N-particle system. Throughout this section the single particle law is positive-recurrent. It is easy to apply induction to see that the N-particle system is an irreducible Markov chain. Fix ρ such that $0 < \rho < \rho_M$ and define,

$$\mu^{(N)} = \pi_{\rho\bar{a}}(\cdot|\Sigma_{x\in\Lambda}\sigma(x) = N), \qquad N = 0, 1, 2, \cdots. \tag{5.1}$$

It is well known that an irreducible Markov chain is positive-recurrent if and only if the transition law possesses a positive summable invariant measure. Since $\mu^{(N)}$ is such a measure for the N-particle system, it follows that the N-particle system is positive recurrent. So, as a consequence, if $\pi \in \bar{\mathfrak{S}}_\Lambda$ and if $\pi(S_\Lambda^{(N)}) = 1$ for some N, then π lies in the domain of attraction of $\mu^{(N)}$. That is

$$\lim_{t\to\infty} \int_{\bar{S}_\Lambda} \bar{P}(t, \sigma, \cdot)\pi(d\sigma) = \mu^{(N)}, \qquad \text{(weak* sense)}, \tag{5.2}$$

(cf. Breiman, 1969).

It is now necessary that we consider the case when the infinite volume system supports an infinite number of particles. For $x_1, \cdots, x_r \in \Lambda$, $k_1, \cdots, k_r \in \mathfrak{N}$, define

$$C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix} = \{\sigma \in \bar{S}_\Lambda \sigma(x_i) \geq k_i,\ 1 \leq i \leq r\}.$$

Note that the indicator

$$I_{C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}},$$

belongs to the set, M, defined in Section 4.

PROPOSITION 5.3. *For each $N = 0, 1, 2, \cdots$, we have*

$$\mu^{(N)}\left(C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right) \leq \mu^{(N+1)}\left(C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right),$$

for all $x_1 \cdots x_r \in \Lambda$ and $k_1 \cdots k_r \in \mathfrak{N}$.

PROOF. Since the N-particle system is positive-recurrent with equilibrium state $\mu^{(N)}$, we have

$$\lim_{t\to\infty}\bar{P}\left(t, \sigma; C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right) = \mu^{(N)}\left(C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right) \text{ for } \sigma \in S_\Lambda(N).$$

Define $\sigma_N \in S_\Lambda(N)$ by selecting $x \in \Lambda$ and setting

$$\begin{aligned} \sigma_N(y) &= N \qquad \text{if } y = x \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Clearly $\sigma_N \leq \sigma_{N+1}$. Suppose that $\Sigma_{i=1}^r k_1 \leq N$. Then

$$\begin{aligned} \mu^{(N)}\left(C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right) &= \lim_{t\to\infty}\bar{P}\left(t, \sigma_N; C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right) \\ &\leq \lim_{t\to\infty}\bar{P}\left(t, \sigma_{N+1}; C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right) \\ &= \mu^{(N+1)}\left(C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right), \end{aligned}$$

since

$$I_{C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}} \in M.$$

In the case that $\Sigma_{i=1}^r k_i > N$, we have

$$\mu^{(N)}\left(C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right) = 0. \qquad \square$$

LEMMA 5.4. *Let*

$$\lambda_N = \mu^{(N)}\left(C\begin{pmatrix} k_1 \cdots k_r \\ x_1 \cdots x_r \end{pmatrix}\right)$$

for $x_1 \cdots x_r \in \Lambda$, $k_1 \cdots k_r \in \mathfrak{N}$. Then

$$\lim_{N\to\infty}\lambda_N = \Pi_{j=1}^r[\rho\bar{a}(x_j)]^{k_j} + \Sigma_{n=0}^{\infty}[\lambda_{n+1} - \lambda_n]\pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) \leq n),$$

for $0 < \rho < \rho_M$.

PROOF. In view of Proposition (3.1), we have

$$\lim_{m\to\infty}\pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) \leqslant m) = 1, \qquad \text{for } 0 \leqslant \rho < \rho_M.$$

So for $0 < \rho < \rho_M$,

$$\begin{aligned}\lim_{m\to\infty}\lambda_m &= \lim_{m\to\infty}\lambda_m \cdot \pi_{\rho\bar{a}}(\Sigma_x\sigma(x) \leqslant m)\\ &= \lim_{m\to\infty}\big[\lambda_m \cdot \pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) \leqslant m)\\ &\quad + \Sigma_{n=0}^{m-1}[\lambda_n - \lambda_{n+1}]\pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) \leqslant n)\big]\\ &\quad - \Sigma_{n=0}^{\infty}[\lambda_n - \lambda_{n+1}]\pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) \leqslant n)\\ &= \Sigma_{n=0}^{\infty}\lambda_n\pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) = n)\\ &\quad - \Sigma_{n=0}^{\infty}[\lambda_n - \lambda_{n+1}]\pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) \leqslant n)\end{aligned}$$

by the summation by parts formula. Now, since $0 < \rho < \rho_M$,

$$\begin{aligned}\Sigma_{n=0}^{\infty}\lambda_n\pi_{\rho\bar{a}}(\Sigma_{x\in\Lambda}\sigma(x) = n) &= \Sigma_{n=0}\pi_{\rho\bar{a}}\left(C\begin{pmatrix}k_1 & \cdots & k_r\\ x_1 & \cdots & x_r\end{pmatrix}\Big|S_\Lambda^{(n)}\right)\cdot\pi_{\rho\bar{a}}(S_\Lambda^{(n)})\\ &= \pi_{\rho\bar{a}}\left(C\begin{pmatrix}k_1 & \cdots & k_r\\ x_1 & \cdots & x_r\end{pmatrix}\right)\\ &= \Pi_{j=1}^{r}[\rho\bar{a}(x_j)]^{k_j}.\end{aligned}$$

□

Consider the system of equations

$$\text{(M)} \qquad \bar{a}(y) = \sup_x\bar{a}(x).$$

Since $\lim_{y\to\infty}\bar{a}(y) = 0$, the system (M) has at least one solution $y = y_M$. We shall use y_M generically to denote a solution to M.

PROOF OF (2.15). Let $\sigma \in \bar{S}_\Lambda$ assume $\Sigma_{x\in\Lambda}\sigma(x) = \infty$. Then we will demonstrate that $\lim_{t\to\infty}\bar{P}(t, \sigma; C(\begin{smallmatrix}k\\ y_M\end{smallmatrix})) = 1$, for each $k = 0, 1, 2, \cdots$.

PROOF. Since $\Sigma_{x\in\Lambda}\sigma(x) = \infty$, for each $N = 0, 1, 2, \cdots$, we can select $\sigma_N \in S_\Lambda^{(N)}$ such that $\sigma_N \leqslant \sigma$. Now, for any $k \in \mathfrak{N}$, we have by Theorem 4.2,

$$\begin{aligned}\liminf_{t\to\infty}\bar{P}\left(t, \sigma; C\begin{pmatrix}k\\ y_M\end{pmatrix}\right) &\geqslant \lim_{t\to\infty}\bar{P}\left(t, \sigma_N; C\begin{pmatrix}k\\ y_M\end{pmatrix}\right)\\ &= \mu^{(N)}\left(C\begin{pmatrix}k\\ y_M\end{pmatrix}\right).\end{aligned}$$

Now, in view of Lemma 5.4, we have

$$\lim_{N\to\infty}\mu^{(N)}\left(C\begin{pmatrix}k\\ y_M\end{pmatrix}\right) \geqslant [\rho\bar{a}(y_M)]^k \quad \text{for all } k = 0, 1, 2, \cdots.$$

So, letting N tend to infinity, we obtain

$$\liminf_{t\to\infty}\bar{P}\left(t, \sigma; C\begin{pmatrix}k\\ y_M\end{pmatrix}\right) \geqslant [\rho\bar{a}(y_M)]^k \quad \text{for all } 0 < \rho < \rho_M.$$

Now let ρ increase to ρ_M. Then

$$\lim_{\rho\uparrow\rho_M}[\rho\bar{a}(y_M)]^k = 1 \qquad \text{for all } k = 0, 1, 2, \cdots,$$

so that

$$\lim_{t\to\infty}\bar{P}\left(t, \sigma; C\binom{k}{y_M}\right) = 1 \quad \text{for all } k = 0, 1, 2, \cdots,$$

COROLLARY. *Let $\mu^{(N)}$ be the state given in* (5.1) *for each $N = 0, 1, 2 \cdots$. Then $\{\mu^{(N)} : N = 0, 1, 2, \cdots\}$ is the collection of all extremal equilibria for the zero-range model at Bose-Einstein speeds with positive-recurrent single particle law when restricted to $(S_\Lambda, \mathcal{B}_\Lambda)$.*

PROOF. Suppose that π is an extremal equilibrium state for the model and that $\pi \neq \mu^{(N)}$ for $N = 0, 1, 2 \cdots$. Then $\alpha = \pi(\sigma \in S_\Lambda : \Sigma_x\sigma(x) = \infty) > 0$. Now for each $k = 0, 1, 2, \cdots,$ we have

$$\pi\left(C\binom{k}{y_M}\right) = \int_{S_\Lambda} P\left(t, \sigma; C\binom{k}{y_M}\right)\pi(d\sigma)$$

$$\geqslant \int_{\{\sigma\in S_\Lambda:\ \Sigma_x\sigma(x)=\infty\}} P\left(t, \sigma; C\binom{k}{y_M}\right)\pi(d\sigma).$$

Letting t tend to infinity, it now follows that

$$\pi\left(C\binom{k}{y_M}\right) \geqslant \alpha > 0 \qquad \text{for } k = 0, 1, 2, \cdots.$$

But this is impossible since $C\binom{k}{y_M}$ decreases to $C\binom{\infty}{y_M}$ as k increases to ∞; i.e.,

$$\pi\left(C\binom{\infty}{y_M}\right) = \lim_{k\to\infty}\pi\left(C\binom{k}{y_M}\right) \geqslant \alpha > 0,$$

and therefore $\pi(S_\Lambda) < 1$. □

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