The Annals of Probability
1980, Vol. 8, No. 4, 802–813

INEQUALITIES FOR THE PROBABILITY CONTENT OF A ROTATED SQUARE AND RELATED CONVOLUTIONS[1]

BY RICHARD L. HALL, MAREK KANTER, AND MICHAEL D. PERLMAN

Concordia University and University of Chicago

Let (X_1, X_2) be independent $N(0, 1)$ variables and let $P(v_1, v_2) = P[(X_1, X_2) \in C + (v_1, v_2)]$, where C is the square $\{|x_1| < a, |x_2| < a\}$. By demonstrating that $P[|X_i - v_i| \leqslant a]$ is log concave in v_i^2, the extrema of $P(v_1, v_2)$ on all circles $\{v_1^2 + v_2^2 = b^2\}$ are determined. The results are extended to determine the extrema of the probability of a cube in R^n. The proof is based on a log concavity-preserving property of the Laplace transforms.

1. Introduction. This paper originates with the following question. Let X_1, X_2 be independent $N(0, 1)$ random variables and let

$$C = \{(x_1, x_2) : |x_1| \leqslant a, |x_2| \leqslant a\}.$$

What are the extrema of the function

$$P(v_1, v_2) \equiv P[(X_1, X_2) \in C + (v_1, v_2)],$$

where (v_1, v_2) is restricted so that $v_1^2 + v_2^2 = b^2$? Since $P(\pm v_1, \pm v_2) = P(v_1, v_2)$, we only need consider $v_1 \geqslant 0$, $v_2 \geqslant 0$.

The investigation of the cases where either a or b is very small led one of us to conjecture and then prove [4] that for all fixed $a > 0$ and $b > 0$, $P(v_1, v_2)$ is maximized when $v_1 = v_2$ and minimized when $v_1 = 0$ or $v_2 = 0$. More specifically, in order to maximize its probability content the square $C + (v_1, v_2)$ should be located so that one of its diagonals (or the extension thereof) contains the origin $(0, 0)$. As the square is rotated from such a position, either by rotating its center (v_1, v_2) while keeping its sides parallel to the coordinate axes, or by fixing its center and rotating it about its center, the probability content decreases monotonically and achieves its minimum after rotation through an angle of $\pi/4$.

The argument used by Hall to prove this fact may be summarized as follows. Write

$$P(v_1, v_2) = q(v_1)q(v_2),$$

where

$$q(y) = P[|X_i - y| \leqslant a]. \tag{1.1}$$

If $r(y) \equiv q(y^{1/2})$ can be shown to be log concave in $y \geqslant 0$, then

$$P(v_1, v_2) \equiv r(v_1^2)r(v_2^2)$$

Received August 30, 1977; revised May 14, 1979.

[1]Research supported in part by NSERC Canada Grants No. A3438 and A8753, Quebec Action Concertée and U.S. National Science Foundation Grant No. MCS 76-81435.

AMS 1970 *subject classifications.* Primary 26A51; secondary 60D05, 60E05, 62H15.

Key words and phrases. Logarithmic concavity, logarithmic convexity, increasing failure rate, decreasing failure rate, Laplace transform, convolution, Gaussian density, noncentral chi-squared distribution, square, cube, Schur concavity.

is log concave in (v_1^2, v_2^2). Since P is invariant under exchange of coordinates, it follows that P is maximized when $v_1^2 = v_2^2 = \frac{1}{2}b^2$ and minimized when $(v_1^2, v_2^2) = (b^2, 0)$ or $(0, b^2)$. We note that the result immediately generalizes to cubes in $\mathbb{R}^n$ since

$$P[|X_i - v_i| \leqslant a, 1 \leqslant i \leqslant n] = \Pi_{i=1}^n r(v_i^2)$$

is a permutation-invariant log concave function of $(v_1^2, \cdots, v_n^2)$ (see Section 3). Other generalizations, such as the extension of the result to any scale mixture of spherically symmetric multivariate normal distributions on $\mathbb{R}^n$, are also presented in Section 3—cf. Theorem 3.1.

The crucial property that $r(y) \equiv q(y^{1/2})$ is log concave was established in [4] by a delicate discussion of inequalities between complicated expressions involving hyperbolic functions. In Section 2 of the present paper we give a simpler proof. Since

$$q(y) = (2\pi)^{-\frac{1}{2}} \int_{-a}^{a} e^{-(x \pm y)^2/2} dx = \hat{h}_6(y), \tag{1.2}$$

where $\hat{h}_6$ is given by (2.12) with $h = I_{[-a,a]}$, the desired result follows from Theorem 2.3. The latter result, together with the other theorems in Section 2, establishes more general log concavity and log convexity properties for Laplace transforms and related integral transforms.

2. Main results. Throughout this paper, g denotes a nonnegative measurable function on $[0, \infty)$ and G denotes the tail of its distribution function, i.e.,

$$G(x) = \int_x^\infty g(w)\,dw, \qquad x \in [0, \infty); \tag{2.1}$$

to avoid trivialities it is always assumed that $G(0) > 0$. Also, h shall denote a measurable function on $[0, \infty)$ or, later, on $(-\infty, \infty)$, such that h is nonnegative on $[0, \infty)$, and H is defined by

$$H(x) = \int_x^\infty w h(w)\,dw, \qquad x \in [0, \infty). \tag{2.2}$$

Notice that if

$$g(x) = h(x^{1/2}), \qquad x \in [0, \infty), \tag{2.3}$$

then

$$G(x) = 2H(x^{1/2}), \qquad x \in [0, \infty). \tag{2.4}$$

Next, define

$$K_1(x) = e^x$$
$$K_2(x) = \cosh(x)$$
$$K_3(x) = \sinh(x),$$

and for g, h as in the preceding paragraph define the transforms $\tilde{g}_i, \hat{h}_i$ by

$$\tilde{g}_i(y) = \int_0^\infty K_i((wy)^{1/2}) g(w) \frac{dw}{w^{1/2}}$$
$$\hat{h}_i(y) = \int_0^\infty K_i(xy) h(x)\,dx, \qquad i = 1, 2, 3.$$

Notice that $\tilde{g}_i$ and $\hat{h}_i$ are increasing on $[0, \infty)$, and that

$$g(x) = h(x^{1/2}) \text{ on } [0, \infty) \Rightarrow \tilde{g}(y) = 2\hat{h}_i(y^{1/2}) \text{ on } [0, \infty). \tag{2.5}$$

A function ψ is said to be log *concave* (log *convex*) on the interval $A \subseteq (-\infty, \infty)$ if $\psi : A \to [0, \infty)$ $(\psi : A \to (0, \infty])$ and

$$\psi(\theta x_1 + (1-\theta)x_2) \geqslant (\leqslant)[\psi(x_1)]^{\theta}[\psi(x_2)]^{1-\theta}$$

for all $x_1, x_2 \in A$ and $0 < \theta < 1$. Our main results are based on the following theorem.

THEOREM 2.1. (a) *If* $G(x)$ *is* log *concave on* $[0, \infty)$, *then* $\tilde{g}_i(y)$ *is* log *concave wherever finite on* $[0, \infty)$, $i = 1, 2, 3$.

(b) *If* $H(x^{1/2})$ *is* log *concave on* $[0, \infty)$, *then* $\hat{h}_i(y^{1/2})$ *is* log *concave wherever finite on* $[0, \infty)$, $i = 1, 2, 3$.

PROOF. By (2.5), we need only prove part (a). Let $b = \sup\{x \mid G(x) > 0\}$; then $0 < b \leqslant \infty$ and

$$\tilde{g}_i(y) = \int_0^b K_i((wy)^{1/2}) g(w) \frac{dw}{w^{1/2}}. \tag{2.6}$$

It can be verified (e.g., apply 3° on page 126 of [6]) that differentiation under the integral sign is permissible in (2.6) when y lies in the interior of the interval $\{\tilde{g}_i < \infty\}$, so in this case we obtain

$$2\tilde{g}_i'(y) = y^{-1/2} \int_0^b K_i'((wy)^{1/2}) g(w)\, dw.$$

Now integrate by parts to obtain

$$2\tilde{g}_i'(y) = y^{-1/2} G(0) K_i'(0) + \tfrac{1}{2} \int_0^b K_i''((wy)^{1/2}) G(w) \frac{dw}{w^{1/2}}.$$

Notice that G satisfies $G(b) = 0$ and $0 < G < \infty$ on $[0, b)$. Furthermore, $K_i'(0) \geqslant 0$ and $K_i'' = K_i$, $i = 1, 2, 3$. Thus

$$\frac{2\tilde{g}_i'(y)}{\tilde{g}_i(y)} = \frac{G(0)K_i'(0)}{y^{1/2}\tilde{g}_i(y)} - \frac{1}{2\int_0^b L_i(w,y) \dfrac{G'(w)}{G(w)} dw}, \tag{2.7}$$

where

$$L_i(w,y) = \frac{K_i((wy)^{1/2}) G(w)/w^{1/2}}{\int_0^b K_i((wy)^{1/2}) G(w)\, dw / w^{1/2}}.$$

In order to conclude that $\tilde{g}_i$ is log concave wherever finite, it is sufficient to show that (2.7) is decreasing in y. Clearly, the first term on the right of (2.7) is decreasing. To show that the second term is also decreasing, we argue that (i) $G'(w)/G(w)$ is decreasing on $(0, b)$, since G is log concave; (ii) $L_i(w,y)$ has monotone likelihood ratio (i.e., is totally positive of order $2 \equiv TP_2$) in (w,y), because $K_i(uv)$, being of the form $\Sigma_0^\infty a_k(uv)^k$ with $a_k \geqslant 0$, is TP_2 in (u, v) (cf. Karlin [5], page 101). In view of (i) and (ii) we may apply Theorem 3.4 of Karlin [5], page 285, to deduce that the second term on the right of (2.7) is decreasing in y. This completes the proof of Theorem 2.1.

REMARK 2.1. If g is such that G is log concave on $[0, \infty)$, then the probability distribution with density proportional to g is said to have *increasing failure* ($\equiv$ *hazard*) *rate* (cf. [2]). The support of g must be an interval $[a, b]$, $0 \leqslant a < b \leqslant \infty$. It is pointed out in [2] that if $G(0) < \infty$ then

$$g \text{ log concave on } [0, \infty) \Rightarrow G \text{ log concave on } [0, \infty), \tag{2.8}$$

but that the converse is not true. Since the indicator function $I_{[a,b]}$ of an interval is log concave and the product of log concave functions is log concave, Theorem 2.1 remains valid if $\int_0^\infty$ is replaced by $\int_a^b$ in the definitions of $\tilde{g}_i$ and $\hat{h}_i (0 \leqslant a < b \leqslant \infty)$, provided that the stronger assumptions that $g(x)$ and $h(x^{1/2})$ are log concave are imposed (as well as $G(0) < \infty$ and $H(0) < \infty$). (Concerning the terminology used in [2] and [5], it should be pointed out that a function ψ is log concave if and only if it is a Polya frequency function of order $2 (\equiv PF_2)$.)

REMARK 2.2. In contrast to Theorem 2.1(b), the log convexity of the Laplace transform, together with (2.9) and (2.10) below, implies that for any nonnegative measurable function h on $[0, \infty)$, $\hat{h}_i(y)$ is log *convex* on $(-\infty, \infty)$, $i = 1, 2, 3$. For example, if we take $g = h = I_{[0,1]}$, Theorem 2.1 and Remark 2.1 imply that each of

$$\tilde{g}_1(y) \equiv \frac{e^{y^{1/2}} - 1}{y^{1/2}} \equiv 2\hat{h}_1(y^{1/2})$$

$$\tilde{g}_2(y) \equiv \frac{\sinh(y^{1/2})}{y^{1/2}} \equiv 2\hat{h}_2(y^{1/2})$$

$$\tilde{g}_3(y) \equiv \frac{\cosh(y^{1/2}) - 1}{y^{1/2}} \equiv 2\hat{h}_3(y^{1/2})$$

are log concave, while $\hat{h}_1(y)$, $\hat{h}_2(y)$, and $\hat{h}_3(y)$ are log convex.

REMARK 2.3. In view of Remark 2.2, we point out the following relations between $\psi(y)$ and $\psi(y^{1/2})$: if ψ is an increasing function on $[0, \infty)$, then

$$\psi(y) \text{ log concave} \Rightarrow \psi(y^{1/2}) \text{ log concave}$$

$$\psi(y^{1/2}) \text{ log convex} \Rightarrow \psi(y) \text{ log convex},$$

while if ψ is decreasing on $[0, \infty)$, these implications are reversed.

REMARK 2.4. For any nonnegative measurable function h on $[0, \infty)$, consider the transform $\hat{h}_4$ defined by

$$\hat{h}_4(y) = \int_0^\infty e^{-xy} h(x)\, dx, \qquad y \in [0, \infty)$$

(compare $\hat{h}_4$ to $\hat{h}_1$). Since $\hat{h}_4(y)$ is a Laplace transform it is log convex in y, and furthermore it is decreasing, hence $\hat{h}_4(y^{1/2})$ is log convex (by Remark 2.3).

We now turn our attention to functions h defined on $(-\infty, \infty)$. If h is *even* on $(-\infty, \infty)$ (i.e., $h(x) = h(-x)$) then, provided the integrals exist,

$$\int_{-\infty}^{\infty} e^{xy} h(x)dx = 2\int_0^{\infty} \cosh(xy)h(x)dx = 2\hat{h}_2(y), \tag{2.9}$$

while if h is *odd* (i.e., $h(x) = -h(-x)$) then

$$\int_{-\infty}^{\infty} e^{xy} h(x)dx = 2\int_0^{\infty} \sinh(xy)h(x)dx = 2\hat{h}_3(y). \tag{2.10}$$

Thus if we define $\hat{h}_5$ by

$$\hat{h}_5(y) = \int_{-\infty}^{\infty} e^{xy} h(x)dx, \tag{2.11}$$

the following is an immediate corollary of Theorem 2.1.

THEOREM 2.2. *If h is an even or odd function on $(-\infty, \infty)$ such that $H(x^{1/2})$ is* log *concave on* $[0, \infty)$, *then $\hat{h}_5(y^{1/2})$ is* log *concave wherever finite on* $[0, \infty)$.

REMARK 2.5. As in Remark 2.1, a sufficient condition for the log concavity of $H(x^{1/2})$ on $[0, \infty)$ is the log concavity of $h(x^{1/2})$, provided $H(0) < \infty$. In this case, the conclusion of Theorem 2.2 remains valid if $\int_{-\infty}^{\infty}$ is replaced by $\int_{-a}^{a} (a > 0)$ in the definition of $\hat{h}_5$. Notice that $\hat{h}_5(y)$ is increasing and log *convex* in y.

Finally, for a function h on $(-\infty, \infty)$ such that the integral exists, define

$$\hat{h}_6(y) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2} h(x)dx \equiv (\phi * h)(y) \tag{2.12}$$

($\hat{h}_6(y)$ is the Weierstrass transform of h), where

$$\phi(x) = (2\pi)^{-\frac{1}{2}} e^{-x^2/2}. \tag{2.13}$$

For later reference notice that if h is even on $(-\infty, \infty)$ and $g(x) = h(x^{1/2})$ on $[0, \infty)$, then

$$\hat{h}_6(y) = Eh\left((\chi_1^2(y^2))^{1/2}\right) = Eg(\chi_1^2(y^2)), \tag{2.14}$$

where $\chi_1^2(y^2)$ denotes a noncentral chi-squared random variable with one degree of freedom and noncentrality parameter y^2.

THEOREM 2.3. *If h is an even or odd function on $(-\infty, \infty)$ such that $h(x^{1/2})$ is* log *concave on* $[0, \infty)$, *then $\hat{h}_6(y^{1/2})$ is* log *concave wherever finite on* $[0, \infty)$.

PROOF. From (2.12),

$$\hat{h}_6(y^{1/2}) = e^{-y/2} \widehat{(\phi h)}_5(y^{1/2}), \qquad y \in [0, \infty). \tag{2.15}$$

Since $\phi(y^{1/2})$ is log concave and the product of two log concave functions is again log concave, the result follows from Theorem 2.2 and Remark 2.5.

REMARK 2.6. From (2.15) we can write

$$\hat{h}_6(y) = e^{-y^2/2} \widehat{(\phi h)}_5(y),$$

a product of a log concave function and a log convex (cf. Remark 2.2) function. Thus, if no log concavity or log convexity assumptions are imposed on $h(x^{1/2})$ or $h(x)$, no general statement can be made about corresponding properties of $\hat{h}_6(y^{1/2})$ or $\hat{h}_6(y)$ (unlike $\hat{h}_i, i = 1, \cdots, 5$; cf. Remarks 2.2 and 2.4). However, since the convolution of two log concave ($\equiv PF_2$) functions on $(-\infty, \infty)$ is log concave (Karlin [5], Proposition 1.5, page 333) if $h(x)$ is log concave on $(-\infty, \infty)$ then $\hat{h}_6(y)$ is log concave on $(-\infty, \infty)$. On other hand, if $h(x)$ is log convex on $(-\infty, \infty)$ then $\hat{h}_6(y)$ is log convex on $(-\infty, \infty)$, because

$$\hat{h}_6(y) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-u^2/2}h(y+u)du$$

and the sum of log convex functions is log convex. Theorem 2.3 states that if h is even or odd and $h(x^{1/2})$ is log concave on $[0, \infty)$ then $\hat{h}_6(y^{1/2})$ is log concave, while Theorem 2.6 (to follow) states that if h is even and $h(x^{1/2})$ is log convex on $[0, \infty)$, then $\hat{h}_6(y^{1/2})$ is log convex. Furthermore, if h is even then h increasing (decreasing) on $[0, \infty) \Rightarrow \hat{h}_6$ increasing (decreasing) on $[0, \infty)$ (from (2.14)), so under such monotonicity conditions on h, Remark 2.3 provides information about the interrelationships among the various log concavity and log convexity properties of h and $\hat{h}_6$. Examples occur in the next two paragraphs. Finally, we record here that in view of (2.14), the preceding comments imply the following properties of the noncentral chi-squared distribution: if g is a nonnegative measurable function on $[0, \infty)$, then

$$(2.16)\quad g(x^2) \text{ log concave (log convex) in } x \text{ on } (-\infty, \infty)$$
$$\Rightarrow Eg\left(\chi_1^2(y^2)\right) \text{ log concave (log convex) in } y \text{ on } (-\infty, \infty),$$

while

$$(2.17)\quad g(x) \text{ log concave (log convex) in } x \text{ on } [0, \infty)$$
$$\Rightarrow Eg\left(\chi_1^2(y)\right) \text{ log concave (log convex) in } y \text{ on } [0, \infty),$$

whenever the expectations are finite. Nontrivial examples are obtained by setting $g(x) = e^{\pm cx}$, $e^{\pm cx^{1/2}}$, $x^{\pm c}$, $I_{[0,c]}(x)$, and $I_{[c,\infty)}(x)$, for $c > 0$.

As remarked at the end of Section 1, Theorem 2.3 implies that the function $q(y^{1/2})$ is log concave on $[0, \infty)$, where q is given in (1.1) and (1.2). Since $q = \hat{h}_6$, where $h(x) \equiv I_{[-a,a]}(x)$ is log concave on $(-\infty, \infty)$, Remark 2.6 shows that $q(y)$ is log convex on $(-\infty, \infty)$. However h, and hence q, is decreasing on $[0, \infty)$, so Remark 2.3 implies that log concavity of $q(y)$ is a weaker property than log concavity of $q(y^{1/2}), y \geqslant 0$.

By contrast, consider q^* given by

$$(2.18)\qquad q^*(y) = 1 - q(y) = P[|X_i - y| \geqslant a]$$

(see (1.1)). Since $q^*(y) = \hat{h}_6(y)$ where now $h = I_{(-\infty,-a)\cup(a,\infty)}$, Theorem 2.3 implies that $q^*(y^{1/2}) \equiv 1 - q(y^{1/2})$ is log concave for $y \geqslant 0$. Here, q^* is increasing on $[0, \infty)$, and for any such function (log) concavity of $q^*(y)$ would be a stronger property than (log) concavity of $q^*(y^{1/2})$, $y \geqslant 0$. However, while $q^*(y)$ is log concave for large values of y, it must be log convex for values of y near 0, since log $q^*(y^{1/2})$ is increasing on $[0, \infty)$ yet its first derivative vanishes at $y = 0$ (by symmetry).

Finally consider the single-tail probability

$$q^{**}(y) = P[X_i \geqslant y] = (2\pi)^{-\frac{1}{2}}\int_0^\infty e^{-(x+y)^2/2}dx. \tag{2.19}$$

Since

$$q^{**}(y^{1/2}) = e^{-y/2}\hat{\phi}_4(y^{1/2}),$$

Remark 2.4 shows that $q^{**}(y^{1/2})$ is log convex on $[0, \infty)$. On the other hand, by (2.8), $q^{**}(y)$ is log *concave* on $(-\infty, \infty)$.

REMARK 2.7. Suppose h is an even function on $(-\infty, \infty)$ such that $h(x^{1/2})$ is log concave on $[0, \infty)$, as in Theorem 2.3. Since $\hat{h}_6 = \phi * h$ where ϕ is also even and $\phi(x^{1/2})$ is log concave on $[0, \infty)$, Theorem 2.3 would be a rather weak result if it were true that $(h_1 * h_2)(y^{1/2})$ is log concave on $[0, \infty)$ whenever h_i is even and $h_i(x^{1/2})$ is log concave on $[0, \infty)$, $i = 1, 2$. To see that this is not the case, take $h_1 = h_2 = I_{[-1,1]}$. Then for $y \geqslant 0, (h_1 * h_2)(y^{1/2}) = (1 - y^{1/2})I_{[0,1]}(y)$, which is not log concave on $[0, 1]$.

We conclude this section by reexamining Theorems 2.1–2.3 if the assumption of log concavity is replaced by log convexity.

THEOREM 2.4. (a) *If* G *is* log *convex on* $[0, \infty)$, *then* $\tilde{g}_2(y)$ *is* log *convex on* $[0, \infty)$.
(b) *If* $H(x^{1/2})$ *is* log *convex on* $[0, \infty)$, *then* $\hat{h}_2(y^{1/2})$ *is* log *convex on* $[0, \infty)$.

PROOF. Proceeding as in the proof of Theorem 2.1, we find that (2.7) is again valid when y lies in the interior of $\{\tilde{g}_i < \infty\}$ (now $b = \infty$, by Remark 2.8). When $i = 2$, however, $K_2'(0) = 0$. Furthermore, $G'(w)/G(w)$ is now increasing on $[0, \infty)$, so that (2.7) is increasing in y in $[0, \infty)$, hence $\tilde{g}_2(y)$ is log convex wherever finite. As $\tilde{g}_2(y)$ is increasing, it is therefore log convex on $[0, \infty)$.

REMARK 2.8. If g is such that G is log convex on $[0, \infty)$, then the probability distribution with density proportional to g is said to have *decreasing failure* ($\equiv$ *hazard*) *rate* [2]. In this case, G must be strictly decreasing on $[0, \infty)$, hence $G > 0$ on $[0, \infty)$ and $g > 0$ a.e. on $[0, \infty)$. Furthermore, since

$$G(x) = \int_0^\infty g(x + u)du$$

and the sum of log convex functions is convex,

$$g \text{ log convex on } [0, \infty) \Rightarrow G \text{ log convex on } [0, \infty). \tag{2.20}$$

To illustrate (2.8) and (2.20), note that if g is the gamma density proportional to $x^{\alpha-1}e^{-x}$, then the distribution has increasing (decreasing) failure rate if $\alpha > 1$ $(0 < \alpha < 1)$.

REMARK 2.9. In Theorem 2.4(b), log convexity of $\hat{h}_2(y^{1/2})$ is a stronger property (cf. Remark 2.3) than log convexity of $\hat{h}_2(y)$, which holds for any nonnegative h (cf. Remark 2.2). (The same remark applies to $\hat{h}_5$ in Theorem 2.5.)

The final two theorems of this section follow from Theorem 2.4 in the same way that Theorem 2.2 and 2.3 followed from Theorem 2.1.

THEOREM 2.5. *If h is an even function on $(-\infty, \infty)$ such that $H(x^{1/2})$ is* log *convex on* $[0, \infty)$, *then $\hat{h}_5(y^{1/2})$ is* log *convex on* $[0, \infty)$.

THEOREM 2.6. *If h is an even function on $(-\infty, \infty)$ such that $h(x^{1/2})$ is* log *convex on* $[0, \infty)$, *then $\hat{h}_6(y^{1/2})$ is* log *convex on* $[0, \infty)$.

3. Applications. We begin by considering applications of our results to the probability content of n-dimensional cubes under scale mixtures of spherically symmetric multivariate normal distributions. Let $X = (X_1, \cdots, X_n)$ where the X_i are independent $N(0, 1)$ random variables and let C be the cube given by

$$C = \{(x_1, \cdots, x_n) : |x_i| \leq a, 1 \leq i \leq n\}. \tag{3.1}$$

For any $v \equiv (v_1, \cdots, v_n) \in \mathbb{R}^n$ define

$$P(v) = P[X \in C + v] = \Pi_{i=1}^n r(v_i^2), \tag{3.2}$$

where $r(y) = q(y^{1/2})$ as in Section 1. We now insert a scale parameter $t \geq 0$ and define

$$P_t(v) = P[tX \in C + v] \equiv \Pi_{i=1}^n r_t(v_i^2), \tag{3.3}$$

where for $y \geq 0$ and $t > 0$,

$$r_t(y) \equiv q_t(y^{1/2}) = (2\pi)^{-\frac{1}{2}} t^{-1} \int_{-a}^{a} e^{-(x-y^{1/2})^2/2t^2} dx. \tag{3.4}$$

Since Theorem 2.3 remains true in the presence of a scale parameter $t \neq 1$, $q_t(y^{1/2})$, like $q(y^{1/2})$, is log concave on $[0, \infty)$. It follows that $P_t(v)$ is log concave in $(v_1^2, \cdots, v_n^2)$ on the nonnegative orthant $\mathbb{R}_+^n$. Furthermore, $P_t(v)$ is invariant under permutations of $v_1^2, \cdots, v_n^2$.

Because the set of log concave functions is not convex, we shall interest ourselves in the property of Schur concavity (cf. [7]) which is closed under combinations. First, for $u, w \in \mathbb{R}^n$, we say that u *majorizes* w if $w = Du$ for some doubly stochastic matrix D. A function $S(u) \equiv S(u_1, \cdots, u_n)$ is *Schur concave* on $\mathbb{R}^n(\mathbb{R}_+^n)$ if $S(u) \leq S(w)$ whenever u majorizes w with $u, w \in \mathbb{R}^n(\mathbb{R}_+^n)$. Equivalently, $S(u)$ is Schur concave on $\mathbb{R}^n(\mathbb{R}_+^n)$ if it is invariant under permutations of $u_1, \cdots, u_n$ and if for all c (all $c > 0$), $S(c + \lambda, c - \lambda, u_3, \cdots, u_n)$ is decreasing in λ for $\lambda \geq 0$ (for $c \geq \lambda \geq 0$). We say S is *Schur convex* if $-S$ is Schur concave. It is easy to see (e.g. [8]) that log concavity (log convexity) together with permutation invariance implies Schur concavity (Schur convexity). Therefore, for each $t \geq 0, P_t(v)$ is a Schur concave function of $(v_1^2, \cdots, v_n^2) \in \mathbb{R}_+^n$.

Suppose now that the distribution of the random vector $Z \equiv (Z_1, \cdots, Z_n)$ is a scale mixture of spherically symmetric multivariate normal distributions, i.e., there exists a probability measure μ on $[0, \infty)$ such that for all Borel sets B of $\mathbb{R}^n$ we have

$$P[Z \in B] = \int_0^\infty P[tX \in B]\mu(dt).$$

Therefore, for the cube C,

$$P[Z \in C + v] = \int_0^\infty P_t(v)\mu(dt),$$

so we have the following result.

THEOREM 3.1. *If the distribution of Z is a scale mixture of spherically symmetric multivariate normal distributions on $\mathbb{R}^n$, then for the cube C given in* (3.1), *$P[Z \in C + v]$ is a Schur concave function of $(v_1^2, \cdots, v_n^2)$.*

In particular, on the set $\{v | v_1^2 + \cdots + v_n^2 = b^2\}$ the function $P[Z \in C + v]$ is maximized when $v_1^2 = \cdots = v_n^2 = b^2/n$ and minimized when $v_1^2 = b^2, v_2^2 = \cdots = v_n^2 = 0$. Many intermediate comparisons are also available; for example, $P[Z \in C + v]$ decreases as $(v_1^2, \cdots, v_n^2)$ successively assumes the values

$$\left(\frac{b^2}{n}, \cdots, \frac{b^2}{n}\right), \left(\frac{b^2}{n-1}, \cdots, \frac{b^2}{n-1}, 0\right), \cdots, \left(\frac{b^2}{2}, \frac{b^2}{2}, 0, \cdots, 0\right), (b^2, 0, \cdots, 0).$$

The family of all scale mixtures of spherically symmetric multivariate normal distributions has been discussed, for example, by Strawderman [10] and includes, for instance, the multivariate t distribution.

One application of Theorem 3.1 is to hypothesis testing. Suppose that we observe $Z' = Z + v$, where Z is as in Theorem 3.1 and v is an unknown centering vector. We wish to test the null hypothesis that $v = 0$ against the composite alternative hypothesis that $\Sigma v_i^2 = b^2$. (This is certainly a sensible alternative in the case of spherically symmetric random vectors.) Consider now the test which accepts the null hypothesis if $Z' \in C$, C given by (3.1). (Such acceptance regions are often associated with simultaneous confidence intervals.) We have shown that the power function of this test is minimized when $v_1^2 = \cdots = v_n^2$ and maximized when $v_1^2 = b^2$, $v_2^2 = \cdots = v_n^2 = 0$.

Moving away from cubes, let $Z = (Z_1, \cdots, Z_n)$ be as above and suppose that h is an even or odd function on $(-\infty, \infty)$ such that $h(x^{1/2})$ is log concave (log convex) on $[0, \infty)$. By Theorem 2.3 (Theorem 2.6) and the argument leading to Theorem 3.1, $E[\Pi_{i=1}^n h(Z_i + v_i)]$ is a Schur concave (Schur convex) function of $(v_1^2, \cdots, v_n^2)$.

It is natural to ask whether Theorem 3.1 is true for any spherically symmetric distribution on $\mathbb{R}^n$. That this is false, even for unimodal distribution, is demonstrated by consideration of the uniform distribution on a disk in $\mathbb{R}^2$ having a suitably chosen radius, as illustrated in Figure 1.

It is interesting to compare the result proved here with results in previous papers. For simplicity let $n = 2$ and let $Z = (Z_1, Z_2)$ be as in Theorem 3.1. For $b > 0$ consider the four points $v^{(1)} = (b, b)$, $v^{(2)} = (2^{1/2}b, 0)$, $v^{(3)} = (2^{1/2}b, 2^{1/2}b)$, and $v^{(4)} = (2b, 0)$ in the $x_1 - x_2$ plane. Note that $v^{(1)}$ and $v^{(2)}$ lie on the circle of radius $2^{1/2}b$, while $v^{(3)}$ and $v^{(4)}$ lie on the circle of radius $2b$. Let K be a convex body in the plane and for any point v define

$$P(v) = P[Z \in K + v].$$

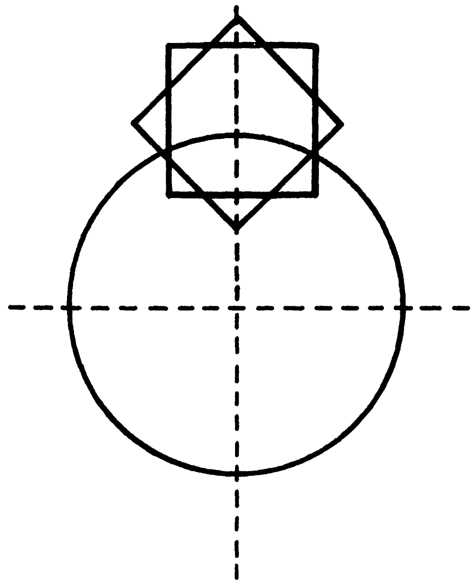

FIG. 1

Anderson's theorem [1] implies that (i) if K is symmetric about the origin ($K = -K$) then $P(v)$ decreases as v moves along the line segments from $v^{(1)}$ to $v^{(3)}$ and from $v^{(2)}$ to $v^{(4)}$—in particular $P(v^{(1)}) > P(v^{(3)})$ and $P(v^{(2)}) > P(v^{(4)})$. Each of the results of Mudholkar [8], Davidovic, et al [3], Prekopa [9], Marshall and Olkin [7] (and others) implies that (ii) if K is symmetric about the x_1-axis then $P(v)$ decreases as v traverses the line segment from $v^{(2)}$ to $v^{(3)}$—hence $P(v^{(2)}) > P(v^{(3)})$; and (iii) if K is symmetric about the line $\{x_1, x_2 \mid x_1 = x_2\}$ then $P(v)$ decreases as v traverses the line segment from $v^{(1)}$ to $v^{(4)}$—so $P(v^{(1)}) > P(v^{(4)})$. Finally, our result shows that (iv) if K is in fact the square $C = \{|x_1| \leqslant a, |x_2| \leqslant a\}$ with sides parallel to the coordinate axes, then $P(v)$ decreases as v traverses the circular arcs from $v^{(1)}$ to $v^{(2)}$ and from $v^{(3)}$ to $v^{(4)}$—hence in this case (using (ii))

$$P(v^{(1)}) > P(v^{(2)}) > P(v^{(3)}) > P(v^{(4)}). \tag{3.5}$$

(The reader is urged to draw a diagram showing the squares $C + v^{(i)}$, $1 \leqslant i \leqslant 4$.)

Lastly, we compare the behavior of the probability content of cubes with that of several other types of sets. For simplicity we consider $X = (X_1, \cdots, X_n)$ as given in the first paragraph of this section, although each of the conclusions in the remainder of this section holds if X is replaced by Z as given in Theorem 3.1. First, define

$$C^* = \{(x_1, \cdots, x_n) : |x_i| \geqslant a, 1 \leqslant i \leqslant n\}$$

and

$$P^*(v) = P[X \in C^* + v] = \Pi_{i=1}^n q^*(v_i),$$

where q^* is defined in (2.18). Since $q^*(y^{1/2})$ is log concave on $[0, \infty)$, P^* is a Schur concave function of $(v_1^2, \cdots, v_n^2)$ on $\mathbb{R}_+^n$; in particular, when $n = 2$, $P^*(v^{(1)}) > P^*(v^{(2)})$ and $P^*(v^{(3)}) > P^*(v^{(4)})$. Since q^* is increasing on $[0, \infty)$, $P^*(v^{(3)}) > P^*(v^{(1)})$ and $P^*(v^{(4)}) > P^*(v^{(2)})$. Also, since $q^*(y)$ is log convex in a neighborhood of $0, P^*(v)$ is Schur *convex* in $(v_1, \cdots, v_n)$ on a neighborhood of the origin in $\mathbb{R}^n$; in particular, when $n = 2$ and b is small, $P^*(v^{(4)}) > P^*(v^{(1)})$. Summarizing, when $n = 2$ we have

$$(3.6) \qquad \begin{gathered} P^*(v^{(3)}) > P^*(v^{(1)}) > P^*(v^{(2)}) \\ P^*(v^{(3)}) > P^*(v^{(4)}) > P^*(v^{(2)}) \end{gathered}$$

for all $b > 0$ and

$$(3.7) \qquad P^*(v^{(3)}) > P^*(v^{(4)}) > P^*(v^{(1)}) > P^*(v^{(2)})$$

when b is small, while it can be shown that

$$(3.8) \qquad P^*(v^{(3)}) > P^*(v^{(1)}) > P^*(v^{(4)}) > P^*(v^{(2)})$$

when b is large.

It is curious to notice that since all inequalities satisfied by the function

$$\bar{P}^*(v) \equiv 1 - P^*(v) = P\left[X \in (C^*)^c + v\right]$$

are opposite to those satisfied by $\bar{P}^*, \bar{P}^*$ satisfies (3.6), (3.7), and (3.8) with the inequalities reversed; in particular

$$(3.9) \qquad \bar{P}^*(v^{(2)}) > \bar{P}^*(v^{(1)}) > \bar{P}^*(v^{(4)}) > \bar{P}^*(v^{(3)})$$

when b is small, while

$$(3.10) \qquad \bar{P}^*(v^{(2)}) > \bar{P}^*(v^{(4)}) > \bar{P}^*(v^{(1)}) > \bar{P}^*(v^{(3)})$$

when b is large. Since $(C^*)^c \supset C$, it might be expected that $\bar{P}^*$ and P have similar monotonicity properties, but (3.10) does not closely resemble (3.5). A diagram shows that $(C^*)^c$ and C are sufficiently dissimilar that this difference in the orderings of $\bar{P}^*$ and P is not surprising.

Finally, define

$$C^{**} = \{(x_1, \cdots, x_n) | x_i \geqslant 0, 1 \leqslant i \leqslant n\}$$

and

$$P^{**}(v) = P[X \in C^{**} + v] = P[X_i \geqslant v_i, 1 \leqslant i \leqslant n] = \Pi_{i=1}^n q_i^{**}(v_i),$$

where q^{**} is defined in (2.19). Since $q^{**}(y^{1/2})$ is log convex on $[0, \infty)$, $P^{**}(v)$ is a Schur convex function of $(v_1^2, \cdots, v_n^2)$ on $\mathbb{R}_+^n$. On the other hand, $q^{**}(y)$ is log concave on $(-\infty, \infty)$, so $P^{**}(v)$ is a Schur concave function of $(v_1, \cdots, v_n)$ on $\mathbb{R}^n$. In particular, when $n = 2$,

$$(3.11) \qquad P^{**}(v^{(2)}) > P^{**}(v^{(1)}) > P^{**}(v^{(4)}) > P^{**}(v^{(3)}).$$

In view of the dissimilarity of C^{**} and $(C^*)^c$, the agreement between (3.9) and (3.11) is somewhat unexpected.

REFERENCES

[1] ANDERSON, T. W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* **6** 170–176.

[2] BARLOW, R. E., MARSHALL A. W. and PROSCHAN F. (1963). Properties of probability distributions with monotone hazard rate. *Ann. Math. Statist.* **34** 375–389.

[3] DAVIDOVIC, J. S., KORENBLJUM B. I. and HACET B. I. (1969). A property of logarithmically concave functions. *Soviet Math. Dokl.* **10** 477–480.

[4] HALL, R. (1980). Logarithmic concavity and the probability content of a rotated square. Unpublished manuscript.

[5] KARLIN, S. (1968). *Total Positivity*. Stanford Univ. Press.

[6] LOÈVE, M. (1963). *Probability Theory, 3rd ed.* Van Nostrand Co., Princeton.

[7] MARSHALL, A. W. and I. OLKIN (1974). Majorization in multivariate distributions. *Ann. Statist.* **2** 1189–1200.

[8] MUDHOLKAR, G. (1966). The integral of an invariant unimodal function over an invariant convex set—an inequality and applications. *Proc. Amer. Math. Soc.* **17** 1327–1333.

[9] PREKOPA, A. (1973). On logarithmically concave measures and functions. *Acta. Sci. Math.* **34** 335–343.

[10] STRAWDERMAN, W. E. (1974). Minimax estimation of location parameters for certain spherically symmetric distributions. *J. Multivariate Anal.* **4** 255–264.

RICHARD L. HALL AND MAREK KANTER
DEPARTMENT OF MATHEMATICS
CONCORDIA UNIVERSITY
1455 DE MAISONNEUVE BLVD. WEST
MONTREAL, QUEBEC H3G 1M8
CANADA

MICHAEL D. PERLMAN
DEPARTMENT OF STATISTICS
UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS 60637