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A GAUSSIAN MEASURE ON l^∞

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We give an example of a Gaussian measure on the σ-algebra of l^∞ generated by *all* the continuous linear functionals on l^∞, which is not a Radon measure.

Let E be a Banach space. The *cylindrical σ-algebra* of E is the smallest σ-algebra which measures all the continuous linear functionals on E. There exists very neat examples of probabilities μ on the cylindrical σ-algebra of a Banach space E which are such that $\mu(K) = 0$ for all norm compact sets K of E (and hence $\mu(L) = 0$ for every weak compact L, since the Radon measures coincide for the weak and norm topologies of a Banach space) and this situation is clearly understood, especially since the work of G. A. Edgar [2]. The purpose of this paper is to show that this situation can arise even for a centered Gaussian probability μ, (that is, a μ such that the law of each f in the topological dual E' of E is Gaussian with mean zero).

THEOREM. *There exists a Gaussian probability μ on the cylindrical σ-algebra of l^∞ such that the measure of every ball of radius* 1 *is zero* (*and hence μ is not Radon*).

PROOF. Let us first recall the well-known elementary fact that if $\alpha \geq 1$, $\theta(\alpha) = (2\pi)^{-1/2} \int_\alpha^\infty e^{-t^2/2}\,dt$ lies between $1/2\alpha(2\pi)^{1/2}e^{\alpha^2/2}$ and $1/\alpha(2\pi)^{1/2}e^{\alpha^2/2}$.

For $n \geq 0$ let λ_n be the Gaussian probability on $\mathbb{R}$ of mean zero and variance γ_n^2, where $\gamma_n = (\log(n+2))^{-1/2}$. Let μ_o be the product probability of the λ_n on $\mathbb{R}^N$. We have

$$\mu_o\{x;\, \|x\|_\infty \leq \alpha\} = \prod_n \left(1 - 2\theta\left(\frac{\alpha}{\gamma_n}\right)\right).$$

Hence

$$\mu_o\{x;\, \|x\|_\infty \leq 1\} = \prod_n \left(1 - 2\theta\left(\frac{1}{\gamma_n}\right)\right) = 0$$

since

$$\sum_n \gamma_n e^{-1/2\gamma_n^2} = \sum_n (\log(n+2))^{-1/2}(n+2)^{-1/2} = \infty.$$

However, for $\alpha \geq 1$ we have

$$\begin{aligned}\mu_o\{x;\, \|x\|_\infty \leq \alpha\} &\geq 1 - 2\sum_n \theta\left(\frac{\alpha}{\gamma_n}\right)\\ &\geq 1 - \frac{2}{\alpha(2\pi)^{1/2}} \sum_n (\log(n+2))^{-1/2} \frac{1}{(n+2)^{\alpha^2/2}}.\end{aligned}$$

Thus $\mu_o(l^\infty) = 1$, but every ball in l^∞ of radius 1 has measure 0, since

$$\begin{aligned}\mu_o\{x;\, \|x-y\|_\infty \leq 1\} &= \prod_n \mu_n[y(n)-1,\, y(n)+1]\\ &\leq \prod_n \mu_n[-1, 1] = 0.\end{aligned}$$

Let μ be the restriction of μ_o to subsets of l^∞. To show that μ is defined on the cylindrical σ-algebra of l^∞, it is enough to show that every $f \in l^{\infty\prime}$ is μ-measurable. Such an f is the sum of

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an element of l^1 and an element null on c_o [1]. So it is enough to consider the case $f_{|c_o} = 0$, we shall show that for such an f we have $f = 0$ μ a.e. It is enough to consider the case $f \geq 0$, $\|f\| = 1$, and to show that for all ϵ, $\alpha > 0$:

$$\mu^*\{x \in l^\infty;\ \|x\|_\infty \leq \alpha, f(|x|) \geq 2\epsilon\} = 0$$

since $|f(x)| \leq f(|x|)$. Write $\nu(a) = f(\chi_a)$ for $a \subseteq \mathbb{N}$, where χ_a is the characteristic function of a, and $a(x) = \{i \in \mathbb{N};\ |x(i)| \geq \epsilon\}$ for $x \in l^\infty$. Then

$$f(|x|) \leq \epsilon + \|x\|_\infty \nu(a(x)) \qquad x \in l^\infty.$$

Now suppose that $\mu^*\{x;\ \|x\|_\infty \leq \alpha, f(|x|) \geq 2\epsilon\} > 0$. Then $\mu^*A > 0$ where $A = \{x;\ \nu(a(x)) \geq \epsilon/\alpha\}$. Now for any $n \in \mathbb{N}$ let $\alpha_n = \mu\{x;\ n \in a(x)\} = 2\theta(\epsilon/\gamma_n)$. Let k be so large that $k\epsilon^2/2 > 1$. Then

$$\textstyle\sum_n \left((\log(n+2))^{-1/2}\frac{1}{(n+2)^{\epsilon^2/2}}\right)^k < \infty;$$

that is $\sum_n \alpha_n^k < +\infty$. This means exactly that

$$\mu^k\{(x_1, \cdots, x_k);\ a(x_1) \cap \cdots \cap a(x_k) \text{ is infinite}\} = 0$$

where μ^k is the product measure in $(l^\infty)^k$.

Now let m be so large that $m > k\alpha/\epsilon$. Then for μ^m-almost all $(x_i)_{i\leq m}$, $\cap_{i\in I}\, a(x_i)$ is finite for all subset I of $\{1, \cdots, m\}$ of cardinality k. However $(\mu^m)^*(A^m) = (\mu^*A)^m > 0$. So there exists a family $(x_i)_{i\leq m}$ in A^m such that for all subsets I of $\{1, \cdots, m\}$ of cardinality k we have $\cap_{i\in I}\, a(x_i)$ is finite, and hence $\nu(\cap_{i\in I}\, a(x_i)) = 0$. Since

$$\{\textstyle\sum_{i\leq m} \chi_{a(x_i)} \geq k\} \subset \cup_{\text{card.} I=k} \cap_{i\in I}\, a(x_i)$$

we have $\nu(\{\sum_{i\leq m} \chi_{a(x_i)} \leq k-1\}) = 1$, and hence $\sum_{i\leq m} \nu(a(x_i)) \leq k-1$. But since $\nu(a(x_i)) \geq \epsilon/\alpha$ for all i and $m(\epsilon/\alpha) \geq k$, this is a contradiction. □

REMARKS. 1. Since all the closed balls of l^∞ are $\sigma(l^\infty, l^1)$ compact and μ is supported by large balls, μ is Radon for $\sigma(l^\infty, l^1)$.

2. If Σ is the σ-algebra of all μ measurable sets on l^∞, the identity map $(l^\infty, \Sigma, \mu) \to l^\infty$ is scalarly measurable. Its indefinite Pettis integral exists, i.e., for $B \in \Sigma$ there is $b \in l^\infty$ such that $f(b) = \int_B f(t)\, d\mu\,(t)$ for $f \in l^{\infty\prime}$. In fact if $b \in l^\infty$ is given by $b(n) = \int_B \delta_n$ where $\delta_n \in l^{\infty\prime}$ is the nth coordinate functional, then $b \in c_0$ since $\delta_n \to 0$ in $L^1(\mu)$, and $f(b) = \int_B f(t)\, d\mu\,(t)$ since it holds for $f \in l^1$ and $f \in c_0^0$.

3. Of course (l^∞, Σ, μ) is perfect, since it is a Radon measure on a K_σ.

4. Since μ is not Radon, it is not τ-regular for the weak topology of l^∞ [2].

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