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# CHARACTERIZATION OF NONPARAMETRIC CLASSES OF LIFE DISTRIBUTIONS[1]

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In this paper we obtain characterizations of large classes of nonparametric life distributions, such as the increasing (decreasing) failure rate, increasing (decreasing failure rate average, new better (worse) than used, etc., classes. The methods used differ from the usual functional equation methods used for the far more common characterizations of parametric families of life distributions.

**1. Introduction and summary.** Characterizations of particular *parametric families* of life distributions are quite common in the literature (see, for example, Kagan, Linnik and Rao, 1973, and Patil, Kotz and Ord, 1975). In this paper, by contrast, we present characterizations of large *classes* of *nonparametric* life distributions, such as the increasing (decreasing) failure rate, increasing (decreasing) failure rate average, new better (worse) than used, etc., classes. (See Section 2 for exact definitions.) Such characterizations are far less common and generally require quite different mathematical and statistical techniques.

Our characterizations are based on order statistics, weighted spacings between order statistics, and total time on test transforms; in most cases inequalities among limiting expected values determine the characterizations. Related results concerning total time on test transforms had been obtained earlier by Barlow and colleagues (exact references are given for each of these results as they appear in the text below), but not necessarily under the weakest assumptions on the distributions; they use plots of empirical total time on test transforms to help identify graphically the type of underlying distribution, i.e., IFR, DFR, etc. Since in characterization, emphasis is placed on obtaining results under the weakest assumptions on the distributions being characterized, we find it useful to prove stronger versions of a number of these known results—for example, a characterization which requires that a distribution be differentiable is not as appealing as one that requires that it only be continuous.

In Section 2, we present preliminaries consisting of definitions and notation. In Section 3, we present properties of the total time on test transform and a characterization of the IFR(SDFR) class of life distributions in terms of the concavity (convexity) of the total time on test transform. In Section 4, we present characterizations of the IFR(SDFR) classes based on the monotonicity of the expected values of the weighted spacings between successive order statistics; the number of sample sizes required is infinite. By using the fact that the shifted exponential distribution is both IFR and SDFR, we are able to obtain a strengthened characterization of the shifted exponential distribution, as compared with the earlier Saleh (1976) characterization. We also obtain additional characterizations of the IFR(SDFR) distribution requiring only a single sample size; of course, we must compensate by making the stronger assumption of stochastic monotonicity rather than expected value monotonicity. In Section 5, we present characterizations of distributions such as IFRA, NBU, NBUE, and their duals, which are similar in spirit to those in Section 4 for the IFR(SDFR) classes. In the same way that we obtain the (1976) Saleh characterization for the shifted exponential by using the fact the exponential is both IFR and SDFR, we may obtain characterizations of the exponential by using the fact that the exponential is both IFRA and DFRA, both NBU and NWU, and

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both NBUE and NWUE. The details are left to the reader.

One final remark should be made. Chandra and Singpurwalla (1978) have pointed out the close relationship between the total time on test transform and the Lorenz curve used by econometrists. Thus, some of our results of Section 3 concerning total time on test transforms can be used to obtain analogous results for the Lorenz curve, and may thus be of interest and value in fields other than reliability.

**2. Preliminaries.** Let $F$ be a life distribution, that is, $F(0-) = 0$. We use the following notation and conventions: $F^{-1}(t) \equiv \inf\{x: F(x) > t\}$, $t \in [0, 1)$; $F^{-1}(1) \equiv \sup\{x; F(x) < 1\}$; $\bar{F} \equiv 1 - F$; $R \equiv -\ln \bar{F}$. We use "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing".

Next we define the classes of life distributions to be considered in the sequel.

DEFINITION 2.1. (a) $F$ is *increasing failure rate* (IFR) if $\bar{F}(y + x)/\bar{F}(y)$ is decreasing in $y(-\infty < y < F^{-1}(1))$ for each $x > 0$.

(b) $F$ is (shifted) *decreasing failure rate* (SDFR), if $\bar{F}(y + x)/\bar{F}(y)$ is increasing in $y(F^{-1}(0) \leq y < \infty)$ for each $x > 0$ or if $F$ is degenerate.

(c) $F$ is *increasing failure rate average* (IFRA) if $(1/y)/R(y)$ is increasing in $y(0 < y < F^{-1}(1))$ or if $F$ is degenerate at 0.

(d) $F$ is *decreasing failure rate average* (DFRA) if $(1/y)R(y)$ is decreasing in $y > 0$ or if $F$ is degenerate at 0.

(e) $F$ is *new better than used* (NBU) if $\bar{F}(x + y) \leq \bar{F}(x)\bar{F}(y)$ for $x > 0, y > 0$.

(f) $F$ is *new worse than used* (NWU) if $\bar{F}(x + y) \geq \bar{F}(x)\bar{F}(y)$ for $x > 0, y > 0$.

(g) $F$ is *new better than used in expectation* (NBUE) if (i) $\int_0^\infty \bar{F}(x)\,dx) < \infty$; (ii) $\int_y^\infty \bar{F}(x)\,dx \leq (\int_0^\infty \bar{F}(x)\,dx)\bar{F}(y)$ for $y > 0$.

(h) $F$ is *new worse than used in expectation* (NWUE) if (i) $\int_y^\infty \bar{F}(x)\,dx < \infty$; (ii) $\int_y^\infty \bar{F}(x)\,dx \geq (\int_0^\infty \bar{F}(x)\,dx)\bar{F}(y)$ for $y > 0$.

The chain of implications IFR $\Rightarrow$ IFRA $\Rightarrow$ NBU $\Rightarrow$ NBUE is readily established assuming a finite mean for $F$ NBU (see Marshall and Proschan, 1972).

Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n$ from $F$. The $k$th *weighted spacing*, $W_{k:n}$, between order statistics $X_{k-1:n}$ and $X_{k:n}$ is defined by $W_{k:n} \equiv (n - k + 1)\cdot(X_{k:n} - X_{k-1:n})$ for $k = 1, 2, \cdots, n$, where $X_{0:n} \equiv 0$. The *total time on test* up to the $k$th order statistic, $T(X_{k:n})$, is defined by $T(X_{k:n}) \equiv \Sigma_{i=1}^k W_{i:n}$ for $k = 1, 2, \cdots, n$, and $T(X_{0:n}) \equiv 0$. If we assume that $n$ items are placed on test at time 0 and that successive failures are observed at times $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, then $W_{k:n}$ represents the total test time observed between $X_{k-1:n}$ and $X_{k:n}$, and $T(X_{k:n})$ represents the total test time observed between 0 and $X_{k:n}$ (see Barlow and Proschan, 1975, page 61).

**3. Properties of the total time on test and its transform.** Let $H_F^{-1}(t) \equiv \int_0^{F^{-1}(t)} \bar{F}(u)\,du$ for $0 \leq t \leq 1$. Barlow and Campo (1975) call $H_F^{-1}$ the *total time on test transform.* In this section we develop some of the properties of $H_F^{-1}$. These properites are used to prove some of the results in Section 4. However the results of this section are important in their own right. Barlow and Campo (1975) use plots of the total time on test transform of the empirical distribution as a method for tentative identification of IFR and DFR distributions. Incidentally, as mentioned in the introduction, Chandra and Singpurwalla (1978) have pointed out the close relationship between the total time on test transform and the Lorenz curve used by econometrists.

Before starting the first theorem we need a definition.

DEFINITION 3.1. A sequence $\{(k_r, n_r)\}_{r=1}^\infty$ of ordered pairs of natural numbers is a $t$-sequence $(0 < t < 1)$ if (i) $1 \leq k_r \leq n_r < n_{r+1}$ for all $r$, and (ii) $k_r/n_r \to t$ as $r \to \infty$.

THEOREM 3.2. *Let* $H_F^{-1}(\cdot)$ *(equivalently* $F^{-1}(\cdot)$*) be continuous at* $t \in (0, 1)$, *and let* $(k, n)$ *range over a t-sequence. Then as* $n \to \infty$,

$$\frac{1}{n} T(X_{k:n}) \to H_F^{-1}(t) \text{ a.s.}$$

PROOF. Let $F_n$ denote the empirical distribution function of $F$. Then

$$T(X_{k:n}) = nH_{F_n}^{-1}\left(\frac{k-1}{n}\right) = \int_0^{X_{k\,n}} n\bar{F}_n(u)\, du$$

(see Barlow and Campo, 1975). Also for $(k, n)$ ranging over a $t$-sequence, $X_{k:n} \to F^{-1}(t)$ a.s. as $n \to \infty$ since $F^{-1}(t)$ is the unique $x$ such that $F(x-) \le t \le F(x)$ if and only if $F^{-1}(\cdot)$ is continuous at $t$. The desired result follows by the Glivenko-Cantelli theorem. (Chung, 1974, page 123). □

Next we note that if $EX_1$ is finite, then $EX_{k:n}$, $EW_{k:n}$, and $ET(X_{k:n})$ are also finite, since $0 \le X_{k:n} \le T(X_{k:n}) \le \Sigma_{i=1}^n X_i \equiv n\bar{X}_n$. This observation can be used to show that whenever $EX_1$ is finite, $\{(1/n_r)T_{k_r:n_r}\}_{r=1}^{\infty}$ is uniformly integrable for every $t$-sequence $\{(k_r, n_r)\}_{r=1}^{\infty}$. Since a uniformly integrable sequence which converges almost surely converges in the mean (see Breiman, 1973, page 91), we can state the following result.

THEOREM 3.3. *Let $t$, $k$, and $n$ be as in Theorem* 3.2 *and let $EX_1$ be finite. Then* $E|(1/n)T(X_{k:n}) - H_F^{-1}(t)| \to 0$ *as $n \to \infty$. In particular,* $(1/n)/ET(X_{k:n}) \to H_F^{-1}$ *as $n \to \infty$.*

We remark that neither Theorem 3.2 nor Theorem 3.3 is true if $t$ is not *a point of continuity of* $F^{-1}(\cdot)$. In this case a counterexample to Theorem 3.2 and to Theorem 3.3 can easily be constructed using the facts that $\lim\inf_{n\to\infty} X_{[nt]:n} = F^{-1}(t\text{-})$ and $\lim\sup_{n\to\infty} X_{[nt]:n} = F^{-1}(t)$ and that $F^{-1}(t-) \neq F^{-1}(t)$. ($[\cdot]$ denotes the greatest integer function). To show that $\lim\inf_{n\to\infty} X_{[nt]:n} = F^{-1}(t-)$ and $\lim\sup_{n\to\infty} X_{[nt]:n} = F^{-1}(t)$, use the fact that $P[X_{[nt]:n} > x] = P[B(n, \bar{F}(x)) > n - [nt] + 1]$, where $B(n, \bar{F}(x))$ denotes a binomial random variable (see Mood, Graybill, and Boes, 1974, page 252), and the law of the iterated logarithm (see Breiman, 1968, page 291).

To state the next lemma we need a definition.

DEFINITION 3.4. A point $x$ is a *point of increase of $F$* if $F(x - h) < F(x) < F(x + h)$ for every $h > 0$.

Let ${}^+f(x_0)$ denote the right-hand derivative of $f$ at the point $x_0$.

LEMMA 3.5. *Let $(a, b)$ be an interval of points of increase and of continuity of $F$. Then* ${}^+H_F^{-1}(F(x))$ *exists and is nonzero for $x \in (a, b)$ if and only if* ${}^+R(x)$ *exists and is nonzero for $x \in (a, b)$. In either case,* ${}^+H_F^{-1}(F(x))\,{}^+R(x) = 1$.

PROOF. Note that in the interval $(a, b)$, $F^{-1}$ behaves like the usual inverse function of $F$. The result follows using standard differentiation results. □

The next proposition is easily verified.

PROPOSITION 3.6. *A nondegenerate life distribution $F$ is* (i) IFR *if and only if $R(x)$ is convex on* $(F^{-1}(0), F^{-1}(1))$ *and* $F(F^{-1}(0)) = 0$; (ii) SDFR *if and only if $R(x)$ is concave on* $(F^{-1}(0), \infty)$.

THEOREM 3.7. (Barlow and Campo, 1975). *The life distribution $F$ is* IFR(SDFR) *if and only if $H_F^{-1}$ is concave (convex) on* [0, 1].

PROOF. To show sufficiency let $H_F^{-1}$ be concave (convex). If $H_F^{-1}$ is constant on [0, 1], then $F$ is degenerate and thus IFR (SDFR). Now assume $H_F^{-1}$ is not constant on [0, 1]. Then (a) $F^{-1}$ is continuous on [0, 1]; (b) the interval $(F^{-1}(0), F^{-1}(1))$ consists of points of increase of $F$; (c) $H_F^{-1}(F(x))$ is strictly increasing on $(F^{-1}(0), F^{-1}(1))$; and (d) $F$ is continuous on $[0, F^{-1}(1))$ $((F^{-1}(0), \infty))$. Since the concavity (convexity) of $H_F^{-1}$ implies by (d) that $F(F^{-1}(0)) = 0$ $(F^{-1}(1) = \infty)$ we have by Lemma 3.5 and Proposition 3.6 that $F$ is IFR (SDFR).

To show necessity let $F$ be IFR (SDFR). If $F$ is degenerate, then $H_F^{-1}$ is constant on [0, 1] and thus concave (convex). Now assume $F$ is nondegenerate. Then by Proposition 3.6, $R$ is convex (concave) on $(F^{-1}(0), F^{-1}(1))$. Thus $R$ is continuous, strictly increasing, and has a positive right derivative on $(F^{-1}(0), F^{-1}(1))$. Consequently, necessity follows from Lemma 3.5. □

Theorem 3.7 is due to Barlow and Campo, (1975) (see also Barlow, 1977), but our proof is new. Our proof avoids some technical difficulties which arise in the limiting argument used in the Barlow and Campo proof of the "if" part of Theorem 3.7.

**4. Characterizations of the IFR(SDFR) class.** Barlow and Proschan (1966) have shown that if $F$ is IFR(SDFR), then for all $x > 0$, $P(W_{k:n} > x)$ is decreasing (increasing) in $k$ ($k = 2, 3, \cdots, n$) for all $n \geq 2$. If $F$ has a finite mean, then $EW_{k:n} < \infty$ for all choices of $k$ and $n$ and consequently $EW_{k:n}$ is decreasing (increasing) in $k (k = 2, 3, \cdots, n)$ for all $n \geq 2$. In this section we prove that a slightly weaker version of the last condition is sufficient for $F$ to be IFR (SDFR). Then we use this result to obtain a characterization of the shifted exponential obtained by Saleh (see Kotz, 1974), who required regularity conditions on $F$. Two other characterizations of the IFR (SDFR) are given.

The main result of this section follows.

THEOREM 4.1. *Let $F$ be a life distribution. Then $F$ is* IFR (SDFR *with finite mean*) *if and only if $F$ has a finite mean and $EW_{k:n}$ is decreasing (increasing) in $k$ ($k = 2, \cdots, n$) for infinitely many $n$.*

Since an IFR life distribution has a finite mean the "only if" part has already been shown. We prove the "if" part. We remark that this proof avoids all unnecessary assumptions on $F$.

PROOF OF SUFFICIENCY. Since $F^{-1}(\cdot)$ is an increasing function it has at most a countable number of discontinuities. Thus $C \equiv \{t \in (0, 1); F^{-1}(\cdot) \text{ is continuous at } t\}$ is all of (0, 1) except for at most a countable number of points. Now to show $F$ is IFR (SDFR) it suffices, by Theorem 3.7, to show that $H_F^{-1}$ is concave (convex). But since $H_F^{-1}$ is increasing and right continuous, to show that $H_F^{-1}$ is concave (convex) we need only show

$$H_F^{-1}(t_1 + h) - H_F^{-1}(t_1) \geq (\leq) H_F^{-1}(t_2 + h) - H_F^{-1}(t_2) \tag{4.1}$$

for all $t_1$, $t_2$, and $h$ such that $t_1, t_2, t_1 + h, t_2 + h \in C$ and $t_1 < t_2$.

Let $t_1$, $t_2$ and $h$ be as above. We show that inequality (4.1) holds. Since $T(X_{k\,n}) \equiv \sum_{i=1}^{k} W_{i:n}$, we obtain, for each one of the infinitely many $n$, that

$$\begin{aligned} &ET(X_{([nt_1]+[nh]:n)}) - ET(X_{[nt_1]:n}) \\ &\geq (\leq) ET(X_{([nt_2]+[nh]:n)}) - ET(X_{[nt_2]:n}). \end{aligned} \tag{4.2}$$

Applying Theorem 3.3 to both sides of inequality (4.2), we conclude that inequality (4.1) holds as was to be shown. □

It is clear from the proof of Theorem 4.1 that the following characterization of the IFR(SDFR) class is also true.

THEOREM 4.2. *Let $F$ be a life distribution with finite mean. Then $F$ is* IFR(SDFR) *if and only if for infinitely many $n \geq N$ and some $l (1 \leq l < N)$*

$$E \textstyle\sum_{i=k}^{k+l} W_{i:n}$$

*is decreasing (increasing) in $k (1 \leq k \leq n - l)$.* □

Note that $F$ is both IFR and SDFR if and only if $F$ is degenerate or if $F$ is shifted exponential, that is,

$$\begin{aligned} \bar{F}(x) &= \exp[-\lambda(x - F^{-1}(0))] & x \geq F^{-1}(0) \\ &= 1 & x < F^{-1}(0), \end{aligned}$$

for some positive $\lambda$. Hence as a corollary of Theorem 4.1 we obtain the following:

THEOREM 4.3. *Let $F$ be a life distribution with finite mean. Then $F$ is shifted exponential or degenerate with mean $\mu + F^{-1}(0)$ if and only if for infinitely many $n \geq 2$, $EW_{k:n} = \mu$ for $k = 2, 3, \cdots, n$.*

A similar characterization was obtained by Saleh (1976) but with the additional (unnecessary) condition that $\inf\{x: F(x) \geq t\}$ is differentiable on (0, 1).

Now we give another characterization of the IFR(SDFR) class which requires conditions for only one sample size. Since for $F$ continuous we have

$$P(X_{m+1:n} - X_{m:n} > u \mid X_{m:n} = x) = (\bar{F}(x+u)/\bar{F}(x))^{n-m} \tag{4.3}$$

(see David, 1970, page 18), the following result holds.

THEOREM 4.4. *The continuous life distribution $F$ is* IFR(SDFR) *if and only if for some fixed $n$ and $m$ $(2 \leq m + 1 \leq n)$, and all $u \geq 0$, $P(X_{m+1:n} - X_{m:n} > u \mid X_{m:n} = x)$ is decreasing (increasing) in $x(-\infty < x < F^{-1}(1))[(F^{-1}(0) \leq x < \infty)]$.*

Actually if $F$ is IFR(SDFR), then $P(X_{m+1:n} - X_{m:n} > u \mid X_{m:n} = x)$ is decreasing (increasing) in $x$ for *all* $n$ and $m$, where $2 \leq m + 1 \leq n$. However, since the emphasis of this paper is on characterizations, we omit this generalization from the statement of Theorem 4.4. A similar remark can be made about other theorems in this paper (see Theorem 5.1 for example).

Recall that a random variable $X$ is stochastically increasing (decreasing) in $Y$, another random variable, if for all $x$, $P(X > x \mid Y = y)$ is increasing (decreasing) in $y$. Hence Theorem 4.4 can be restated using this language. Similar remarks apply to other theorems in this paper (see, for example, Theorems 5.1 and 5.7).

**5. Characterization of classes of life distributions other than the IFR(SDFR).** In this section we characterize classes of life distributions other than the IFR(SDFR). The following characterization is similar in spirit to the characterization of the IFR(SDFR) class given in Theorem 4.4. By (4.3) we immediately obtain:

THEOREM 5.1. *The continuous life distribution $F$ is* NBU(NWU) *if and only if $P(X_{1:n-m} > u) \geq(\leq) P(X_{m+1:n} - X_{m:n} > u \mid X_{m:n} = x)$ for some fixed $n$ and $m$ $(1 \leq m < n)$, and all $u \geq 0$ and $x \geq 0$.*

To obtain another characterization of the NBU(NWU) class we need the following two lemmas:

LEMMA 5.2. *Let $F$ be a life distribution with finite mean and let $t \in (0, 1)$. Then $\{X_{[nt]:n}\}_{n=1}^{\infty}$ is uniformly integrable.*

PROOF. We have

$$P(X_{[nt]:n} > x) = P(B(n, \bar{F}(x)) \geq n - [nt] + 1),$$

where $B(n, \bar{F}(x))$ denotes a binomial random variable with parameters $n$ and $\bar{F}(x)$. Thus

$$P(X_{[nt]:n} > x) \leq \frac{n}{n - [nt] + 1} \bar{F}(x) \tag{5.1}$$

since $P(Z > t) \leq EZ/t$ for any nonnegative random variable $Z$. Hence

$$EX_{[nt]:n} I[X_{[nt]:n} \geq A]$$

$$= \int_A^\infty P[X_{[nt]:n} > x]\, dx + AP(X_{[nt]:n} \geq A] \text{ [by integration by parts]}$$

$$\le \frac{n}{n-[nt]+1}\left(\int_A^\infty \bar{F}(x)\,dx + A\bar{F}(A)\right) \qquad \text{[by (5.1)]}$$

$$= \frac{1}{1-\frac{[nt]}{n}+\frac{1}{n}}(EX_1 I[X_1 \ge A]).$$

Consequently the uniform integrability of the sequence $\{X_{[nt]:n}\}_{n=1}^{\infty}$ follows. □

LEMMA 5.3. *Let $F$ be a continuous life distribution. Let $M$ be a dense subset of the support of $F$. Assume $\bar{F}(x+y) \le (\ge)\ \bar{F}(x)\bar{F}(y)$ for all $x, y \in M$. Then $F$ is* NBU(NWU).

PROOF. Let $\bar{F}(x+y) \le \bar{F}(x)\bar{F}(y)$ for all $x, y \in M$. Then since $F$ is continuous we have that $\bar{F}(x+y) \le \bar{F}(x)\bar{F}(y)$ for all $x$ and $y$ in the support of $F$. Now for $z > F^{-1}(0)$ let $\tilde{z} \equiv F^{-1}(F(z)-)$. Let $x, y \in (F^{-1}(0), \infty)$. Since $\tilde{x}$ and $\tilde{y}$ are in the support of $F$, we have that $\bar{F}(x+y) \le \bar{F}(\tilde{x}+\tilde{y}) \le \bar{F}(\tilde{x})\bar{F}(\tilde{y}) = \bar{F}(x)\bar{F}(y)$. It follows that $F$ is NBU.

To show the NWU case let $\tilde{z} = F^{-1}(F(z))$, $z \ge 0$, and follow a similar argument. □

THEOREM 5.4. *Let $F$ be a continuous life distribution with finite mean. Then $F$ is* NBU(NWU) *if and only if for for every $t, s \in (0, 1)$, we have*

$$E(X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n} \mid X_{[nt]:n}) \le_{\text{a.s.}} (\ge_{\text{a.s.}}) EX_{[n(1-t)s]:n-[nt]}$$

*for infinitely many $n$.*

PROOF. We first prove sufficiency. There exists a subset $M$ of the support of $F$ with $F$ measure 1 such that for each $n$ in the infinite sequence of the hypothesis and each $y \in M$ we have:

$$E(X_{[nt]+[n(1-t)s]:n} - X_{[nt]} \mid X_{[nt]:n} = y) \le (\ge)\ EX_{[n(1-t)s]:n-[nt]}. \tag{5.2}$$

For $y$ such that $\bar{F}(y) > 0$, define a life distribution $G_y$ by $\bar{G}_y(\cdot) \equiv \bar{F}(\cdot + y)/\bar{F}(y)$. By Lemma 5.3, to show that $F$ is NBU(NWU) it is enough to prove

$$\bar{G}_y(x) \le (\ge)\ \bar{F}(x) \tag{5.3}$$

for all $x > 0$ and $y \in M/\{F^{-1}(1)\}$ (note that $y \in M/\{F^{-1}(1)\}$ implies $\bar{F}(y) > 0$.

Let $x > 0$ and $y \in M/\{F^{-1}(1)\}$. To show that inequality (5.3) holds let $Y_1, Y_2, \cdots, Y_{n-[nt]}$ be independent random variables with common distribution $G_y$. Then by the Markov property of order statistics (see David, 1970, page 18), the left-hand side of inequality (5.2) is equal to $EY_{[n(1-t)s]:n-[nt]}$. Thus we have

$$EY_{[n(1-t)s]:n-[nt]} \le (\ge)\ EX_{[n(1-t)s]:n-[nt]}. \tag{5.4}$$

By Lemma 5.2 we can let $n \to \infty$ (along the infinite sequence of the hypothesis) on both sides of inequality (5.4) to obtain that $G_y^{-1}(s) \le (\ge)\ F^{-1}(s)$ for all $s \in (0, 1)$ which are continuity points of both $F^{-1}(\cdot)$ and $G_y^{-1}(\cdot)$. Since $F^{-1}(\cdot)$ and $G_y^{-1}(\cdot)$ are right continuous on [0, 1) and left continuous at 1 and since the points of continuity of both $F^{-1}(\cdot)$ and $G_y^{-1}(\cdot)$ form a dense set in [0, 1], we see that $G_y^{-1}(s) \ge (\le)\ F^{-1}(s)$ for all $s \in [0, 1]$. This implies that $G_y(x) \ge (\le)\ F(x)$ which is equivalent to inequality (5.3).

To show necessity note that $F$ is NBU(NWU) if and only if for all $y$, $0 < y < F^{-1}(1)$, $Y_1$ is stochastically smaller (larger) than $X_1$, written $Y_1 \le_{\text{st}}(\ge_{\text{st}}) X_1$. Hence if $F$ is NBU(NWU), then for $y$, $t$, and $s$ such that $0 < y < F^{-1}(1)$, $0 < t < 1$, and $0 < s < 1$,

$$Y_{[n(1-t)s]:n-[nt]} \le_{\text{st}}(\ge_{\text{st}}) X_{[n(1-t)s]:n-[nt]}.$$

Since by the Markov property of order statistics the conditional random variable $X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n} \mid X_{[nt]:n} = y$ has the same distribution as the random variable $Y_{[n(1-t)s]:n-[nt]}$, necessity follows. □

For continuous $F$, we have by the Markov property of order statistics,

$$\frac{1}{n-[nt]} E\left(\sum_{k=1}^{n-[nt]} (X_{[nt]+k:n} - X_{[nt]:n}) \middle| X_{[nt]:n}\right) = \frac{\int_0^\infty \bar{F}(x + X_{[nt]:n})\,dx}{\bar{F}(X_{[nt]:n})} \text{ a.s.}$$

Hence we can use the methods of the proof of Theorem 5.4 to obtain the following characterization of the NBUE (NWUE) class.

THEOREM 5.5. *Let $F$ be a continuous life distribution with finite mean. Then $F$ is* NBUE (NWUE) *if and only if for every $t$ in* (0, 1) *we have*

$$\frac{1}{n-[nt]} \sum_{k=1}^{n-[nt]} E(X_{[nt]+k:n} - X_{[nt]:n} \mid X_{[nt]:n}) \leq_{\text{a.s.}} (\geq_{\text{a.s.}}) EX_i$$

*for infinitely many $n$.*

Since $E(X_{n:n} - X_{n-1:n} \mid X_{n-1:n} = x) = (\int_x^\infty \bar{F}(u)\,du)/\bar{F}(x)$ for continuous $F$ we have:

THEOREM 5.6. *Let $F$ be a continuous life distribution with finite mean. Then $F$ is* NBUE (NWUE) *if and only if $E(X_{n:n} - X_{n-1:n} = x) \leq (\geq) EX_1$ for some fixed $n \geq 2$, and all $0 \leq x < F^{-1}(1)$.*

Next we give a characterization of the IFRA(DFRA) class.

THEOREM 5.7. *Let $F$ be a life distribution. Then $F$ is* IFRA(DFRA) *if and only if for all $x > 0$, $P(W_{1:n} > x)$ is increasing in $n \geq N$ where $N$ is arbitrary.*

PROOF. We prove the theorem for the IFRA case. The proof of the DFRA case is similar.

We have for $0 \leq x < \infty$, $P(W_{1:n} > x) = \bar{F}^n(x/n)$. It follows that $P(W_{1:n} > x)$ is increasing in $n \geq N$ for all $x > 0$ if and only if

$$\bar{F}^n(x/n) \leq \bar{F}^m(x/m) \text{ for all } N \leq n < m \text{ and all } 0 < x < \infty. \tag{5.5}$$

Note that $F$ is IFRA if and only if

$$\bar{F}^{1/t_2}(t_2) \leq \bar{F}^{1/t_1}(t_1) \quad \text{for all} \quad 0 < t_1 < t_2 < \infty. \tag{5.6}$$

We show that (5.5) implies (5.6). Let $0 < t_1 < t_2$ both be rational, and let $N \leq n < m$ be natural numbers such that $t_2/t_1 = m/n$. Let $\alpha = (mt_1)^{-1}$. Then $n\alpha = 1/t_2$ and $m\alpha = 1/t_1$, so it is easily seen that (5.5) implies that $\bar{F}^{1/t_2}(x\alpha t_2) \leq \bar{F}^{1/t_1}(x\alpha t_1)$. Letting $x = \alpha^{-1}$, (5.6) follows since $\bar{F}$ is right continuous and the rationals are dense.

To show (5.6) implies (5.5), let $x$, $n$, and $m$ be such that $x > 0$ and $N \leq n < m$, then set $t_1 = x/m$ and $t_2 = x/n$ in (5.6) to obtain (5.5). □

To prove our next result we need a theorem of Barlow and Proschan (1966).

THEOREM 5.8. (Barlow and Proschan, 1966, Theorem 3.6). *Let $X(Y)$ have distribution $F(G)$. Assume that $F(0) = 0 = G(0)$, and that $F$ and $G$ are continuous. Assume also that the support of $F$ is an interval, possibly infinite, and that $G$ is strictly increasing on its support. Let $G^{-1}F(x)/x$ be increasing in $x$ in the support of $F$. Then $EX_{i:n}/EY_{i:n}$ is decreasing in $i$ ($i = 1, 2, \cdots, n$).*

THEOREM 5.9. *Let $F$ be a continuous life distribution with finite mean. Assume that the support of $F$ is an interval and that $F(0) = 0$. Then $F$ is* IFRA(DFRA) *if and only if $EX_{i:n}/\Sigma_{k=1}^{i} 1/(n - k + 1))$ is decreasing (increasing) in $i$ ($i = 1, 2, \cdots, n$) for infinitely many $n$.*

PROOF. Let $G$ in Theorem 5.8 be the exponential distribution with mean 1. Then $EY_{i:n} = \Sigma_{k=1}^{i} (n - k + 1)^{-1}$ (see Barlow and Proschan, 1975, page 60). Thus necessity follows from Theorem 5.8 (as does the dual result for the opposite direction of monotonicity) if we note that $G^{-1}(x) = -\ln(1 - x)$.

To prove sufficiency, first observe that every point in the interior of the support of $F(G)$ is a point of increase of $F(G)$. Hence $Y_{[nt]:n} \to G^{-1}(t) = -\ln(1 - t)$ and $X_{[nt]:n} \to F^{-1}(t)$ a.s. as $n \to \infty$ for $t \in (0, 1)$. By Lemma 5.2 it follows $EX_{[nt]:n} \to F^{-1}(t)$ and $EY_{[nt]:n} \to G^{-1}(t)$ as $n \to \infty$. Thus by hypothesis, $F^{-1}(t)/(-\ln(1 - t))$ is decreasing (increasing) in $t$ $(0 < t < 1)$. Equivalently

$$\frac{F^{-1}(F(x))}{-\ln(1 - \bar{F}(x))} = \frac{x}{-\ln\bar{F}(x)}$$

is decreasing (increasing in $x$ $(0 < x < F^{-1}(1))$. Sufficiency follows. □

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