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SOME TRANSFORMATIONS OF DIFFUSIONS BY TIME REVERSAL[1]

BY M. J. SHARPE

University of California, San Diego

The method of time reversal of a Markov process from a cooptional time, introduced by Nagasawa, is used to show that certain occupation time and last exit time problems for one linear diffusion are equivalent to first passage time problems for certain other diffusions. Another proof of Nagasawa's theorem is given, based on the measures of Revuz.

1. Introduction. In some investigations [2] concerning Bessel diffusions on the line it was found by explicit calculations that the distributions of the last exit time from a state and occupation time of an interval for one Bessel diffusion are identical with first passage time distributions for certain other Bessel diffusions. The object of this paper is to show that these phenomena persist for a very general class of linear diffusions. The proofs are free of explicit calculations and depend only on some theorems on transformations of Markov processes by means of time change and reversal. At the end of the paper a new proof, based on the measures of Revuz, is given for the principal tool—Nagasawa's theorem on time reversal.

The basic process will be a regular diffusion X with lifetime ζ on the interval $(0, \infty)$, satisfying the hypotheses

(1.1) The end point 0 is an entrance point but not an exit point.

$$X_t \to \infty \quad \text{as} \quad t \nearrow \zeta. \tag{1.2}$$

Let $T_a = \inf\{t > 0: X_t = a\}$ denote the hitting time for the state $a \in (0, \infty)$. Let s be a scale function for X. Then if $0 < a \leq x \leq b < \infty$,

$$P^x\{T_a < T_b\} = \frac{s(b) - s(x)}{s(b) - s(a)}$$

In view of (1.1) and (1.2), as $a \to 0$, $P^x\{T_a < T_b\} \to 0$ and consequently $s(0+) = -\infty$. For fixed $a > 0$, $\lim_{b\to\infty} P^x\{T_a < T_b\} = P^x\{T_a < \zeta\} < 1$ otherwise X would be recurrent. It follows that $s(\infty) < \infty$. We may assume therefore that s satisfies.

$$s(0+) = -\infty, \qquad s(\infty) = 0. \tag{1.3}$$

Let Γ denote the infinitesimal generator of X, with domain Δ, and let m be a speed measure for X normalized so that $\Gamma = \frac{1}{2}(d/dm)(d/ds)$. Standard results in diffusion theory ([3], page 130) show that, under (1.1),

$$\int_0^\epsilon (s(\epsilon) - s(t))m(dt) < \infty \quad \text{for every} \quad \epsilon > 0. \tag{1.4}$$

In fact, (1.3) and (1.4) are together equivalent to (1.1) and (1.2).

We now define two other diffusions in terms of the scale function s and speed measure m for X. Let Y denote the diffusion on $(0, \infty)$ having scale function $\bar{s} = -1/s$ and speed measure $\bar{m}(dx) = s^2(x)m(dx)$ with death occuring only as the process reaches 0. Let Y^b denote, for a fixed $b > 0$, the diffusion on $(0, b]$ obtained from Y by placing a reflecting barrier at b. More precisely, Y^b has speed measure $1_{(0,b)}d\bar{m}$, scale function $\bar{s}|_{(0,b]}$, and every

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function f in the domain of the generator of Y^b satisfies $f(b) - f(x) = o(s(b) - s(x))$ as $x \nearrow b$. It will be apparent from the proof of Theorem (1.5) that the endpoint 0 is an exit point for Y and Y^b. This could be shown directly from the forms of $\bar{s}$ and $\bar{m}$.

We now let $A_t = \int_0^t 1_{(0,b)}(X_s)\, ds$, so that A_∞ denotes the total time spent in $(0, b)$ by the process X, and we let $L = L_b = \sup\{t\colon X_t = b\}$ denote the last exit time of X from the interval $[0, b)$. We set also $T = \inf\{t\colon Y_t = 0\}$ and $T^b = \inf\{t\colon Y_t^b = 0\}$.

(1.5) THEOREM. (i) *The distribution of L_b for the process X starting at 0 is identical to the distribution of T for the process Y starting at b.*

(ii) *The distribution of A_∞ for the process X starting at 0 is identical to the distribution of T^b for the process Y^b starting at b.*

2. Proofs. The principal result needed for the proof of Theorem (1.5) is the theorem of Nagasawa on time reversal for dual processes. The theorem is stated in [5] and a proof can be found in [4], page 41. Another proof that seems simpler is given in Section 3.

The form needed here can be stated as follows.

(2.1) *Let X and $\hat{X}$ be standard Markov processes in duality on their common state space E relative to a σ-finite reference measure ξ. (See [1], VI). Let $u(x, y)$ denote the potential kernel density relative to ξ so that $Uf(x) = E^x \int_0^\infty f(X_t)\, dt = \int u(x, y) f(y) \xi(dy)$. Let L be a cooptional time for X; that is, L is a positive random variable satisfying $L \le \zeta$ and $L \circ \theta_t = (L - t)^+$. Define $\tilde{X}$ by*

$$\begin{aligned}\tilde{X}_t &= X_{(L-t)-} && \text{on } \{0 < L < \infty\}, \text{ for } 0 < t < L\\ &= \Delta && \text{otherwise.}\end{aligned}$$

Fix an initial law λ and let $v(y) = \int \lambda(dx) u(x, y)$. Then, under P^λ, the process $(\tilde{X}_t)_{t>0}$ is homogeneous Markov on E with transition semigroup $(\tilde{P}_t)$ given by

$$\begin{aligned}\tilde{P}_t f(y) &= \hat{P}_t(fv)(y)/v(y) && \text{if } 0 < v(y) < \infty\\ &0 && \text{if } v(y) = 0 \text{ or } \infty.\end{aligned}$$

In case $\lambda = \epsilon_x$, we have $v(y) = u_x(y) = u(x, y)$.

For a diffusion X satisfying (1.1) and (1.2), we take $\xi = m$, the speed measure. Let U denote the potential operator for X. The potential kernel density $u(x, y)$ for X is easy to compute in this case. Firstly, for $0 < a < b < \infty$, let $R_{ab} = \inf\{t\colon X_t \notin (a, b)\}$. By elementary results in diffusion theory, if $a < x < b$

$$E^X \int_0^{R_{ab}} f(X_t)\, dt = \int_{(a,b)} u_{ab}(x, y) f(y) m(dy)$$

where $u_{ab}(x, y)$ is symmetric in (x, y) and for $a < x \le y < b$,

(2.2) $$u_{ab}(x, y) = (s(x) - s(a))(s(b) - s(y))/(s(b) - s(a)).$$

Because of (1.3), we obtain as $a \searrow 0$ and $b \nearrow \infty$ that for $x > 0$

(2.3) $$Uf(x) = E^x \int_0^\infty f(X_t)\, dt = \int_{(0,\infty)} u(x, y) f(y)\, du(dy)$$

where

(2.4) $$u(x, y) = |s(y)| \wedge |s(x)|.$$

This proves that X is self dual relative to the reference measure m, and in particular, when $x = 0$, we obtain $u_0(y) = |s(y)|$. The random variable $L = \sup\{t\colon X_t = b\}$ (= 0 if $\{t\colon X_t = b\}$

is empty) is obviously a cooptional time. It follows then from (2.1) that the process X reversed at L is, under P^0, a diffusion with initial point b with transition function $Q_t f(x) = P_t(fs)(x)/s(x)$ for $x > 0$. The reversed process $\tilde{X}$ dies only at 0, and its potential operator V is given by

$$Vf(x) = U(fs)(x)/s(x) \quad \text{for} \quad f \in \mathscr{E}_+ \quad \text{and} \quad x > 0.$$

Using (2.4), one sees that for $f \in \mathscr{E}_+$ and $x > 0$,

$$Vf(x) = \int |s(x)| \wedge |s(y)| f(y) \frac{s(y)}{s(x)} m(dy)$$
$$= \int \bar{s}(x) \wedge \bar{s}(y) f(y) s^2(y) m(dy)$$

where $\bar{s} = -1/s$. It is, therefore, a simple matter to see that $\bar{s}$ is a scale function for the reversed process and $\bar{m}(dy) = s^2(y)m(dy)$ is the corresponding speed measure. In effect, $V1_{(a,c)} < \infty$ if $0 < a < c < \infty$, and the identity $V1_{(a,c)}(x) = V1_{(a,c)}(a)\tilde{P}^x\{\tilde{T}_a < \zeta\}$ for $0 < x < a$ leads to $\tilde{P}^x\{\tilde{T}_a < \zeta\} = \bar{s}(x)/\bar{s}(a)$ so $\bar{s}$ is a scale function for $\tilde{X}$. It is almost a matter of definition then that $\bar{m}$ is the speed measure for $\tilde{X}$. We have proved, therefore, that $\tilde{X}$ is equivalent to the process Y described in the introduction, and so (1.5) (i) is proven.

The proof of (1.5) (ii) is quite similar. We shall prove that if the process X is time changed via the additive functional A to give a process X^A whose lifetime is A_∞, then the process $\tilde{X}^A$ obtained from X^A by reversal at its lifetime has, under P^0, the same distribution as the process Y^b starting at b. From this will follow obviously (1.5) (ii).

The potential operator for the time-changed process is

$$U_A f(x) = E^x \int_0^\infty f(X_t)\, dA_t = E^x \int_0^\infty [f1_{(0,b)}](X_t)\, dt$$
$$= U(f1_{(0,b)})(x)$$
$$= \int u(x, y) f(y) 1_{(0,b)}(y) m(dy)$$

where $u(x, y)$ is given by (2.4). From this we deduce that the process X^A, which lives on $[0, b]$, is self dual relative to the reference measure $1_{(0,b)}(y)m(dy)$ and the restriction of $u(x, y)$ to $[0, b] \times [0, b]$ is its potential kernel density. Applying (2.1) again, we see that under P^0, $\tilde{X}^A$ is continuous and strong Markov with potential operator $\tilde{U}_A$ given by

$$\tilde{U}_A f(x) = U_A(fu_0)(x)/u_0(x) = U(fs1_{(0,b)})(x)/s(x), \qquad 0 < x \le b.$$

Thus $\tilde{U}_A f(x) = V(f1_{(0,b)})(x)$ for $0 < x \le b$, where V is the potential operator for the process Y. Since Y dies only at 0, the process Y^b can be obtained from Y by time change based on the additive functional $\int_0^t 1_{(0,b)}(Y_s)\, ds$—see [2], (4.8), for example. It follows just as above that the potential operator for Y^b is given by $V(f1_{(0,b)})$ and consequently Y^b and $\tilde{X}^A$ have the same potential operator. As in the proof of (i), this shows that Y^b and $\tilde{X}^A$ are equivalent processes. Since, under P^0, the initial value of $\tilde{X}^A$ is b, the assertion (1.5) (ii) is now evident.

3. Nagasawa's theorem. In this section we give a proof of (2.1). Recall that the hypothesis of duality of X and $\hat{X}$ relative to ξ gives us the existence, for each $\alpha \ge 0$, of functions $u^\alpha(x, y)$ satisfying, for $f \in \mathscr{E}_+$,

$$U^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dt = \int u^\alpha(x, y) f(y) \xi(dy); \tag{3.1}$$

$$\hat{U}^{\alpha}f(y) = \hat{E}^{y}\int_0^{\infty} e^{-\alpha t}f(\hat{X}_t)\,dt = \int \xi(dx)f(x)u^{\alpha}(x, y); \tag{3.2}$$

(3.3) for all y, $x \to u^{\alpha}(x, y)$ is α-excessive;

(3.4) for all x, $y \to u^{\alpha}(x, y)$ is α-coexcessive (i.e., α-excessive for $\hat{X}$).

Under these hypotheses, one has the following important theorem of Revuz [6] connecting additive functionals and measures on $(E, \mathscr{E})$: let A be a natural additive functional of X with, say, finite potential $u_A(x) = E^x A_{\infty}$. Then there exists a σ-finite measure ν_A on $(E, \mathscr{E})$ satisfying

$$E^x \int_0^{\infty} e^{-\alpha t}f(X_t)\,dA_t = \int u^{\alpha}(x, y)f(y)\nu_A(dy); \; f \in \mathscr{E}_+, \; \alpha \geq 0. \tag{3.5}$$

We shall show that $(\tilde{X}_t)_{t>0}$ as defined in (2.1) is Markov relative to a filtration $(\tilde{\mathscr{F}}_t)$ defined below. To begin with, let $\mathscr{H}$ denote the class of all real measurable processes $(Z_t)_{t>0}$ on Ω having the property of homogeneity on $(0, \infty)$:

$$Z_t \circ \theta_s = Z_{t+s} \quad \text{for all} \quad t > 0, \, s \geq 0. \tag{3.6}$$

For example, if $f \in \mathscr{E}$, the process $Z_t = f(X_{t-})1_{(0,\zeta)}(t)$ is in $\mathscr{H}$. In addition, if $Z \in \mathscr{H}$ and if $r \geq 0$, the process $t \to Z_{t+r}$ is in $\mathscr{H}$. A necessary and sufficient condition on a random variable $L \leq \zeta$ in order that it be a cooptional time is that $1_{(0,L)} \in \mathscr{H}$ or $1_{(0,L]} \in \mathscr{H}$.

One defines then, for any cooptional time L, the σ-algebra

$$\mathscr{G}_L = \{ Z_L 1_{\{0<L<\infty\}} + F1_{\{L=0\}} + c1_{\{L=\infty\}} : Z \in \mathscr{H}, F \in \mathscr{F}, c \text{ scalar}\}. \tag{3.7}$$

From the examples quoted above, one sees that $X_{L-}1_{\{0<L<\zeta\}}$ is $\mathscr{G}_L$ measurable and that if $t > 0$, $(L-t)^+$ is also a cooptional time and $\mathscr{G}_{(L-t)^+} \supset \mathscr{G}_L$. We now define

$$\tilde{\mathscr{F}}_t = \mathscr{G}_{(L-t)^+}. \tag{3.8}$$

Replacing L by $(L - s)^+$ in the inclusion noted above, we see that $(\tilde{\mathscr{F}}_t)_{t\geq 0}$ is indeed a filtration, and $(\tilde{X}_t)_{t>0}$ is adapted to $(\tilde{\mathscr{F}}_t)_{t>0}$.

In order to prove the theorem, it is enough to prove that if $g \in b\mathscr{C}_+(E)$ and if $0 < s < t + s$, then with $\tilde{P}_t$ as defined in (2.1)

$$E^{\mu}\{g(\tilde{X}_{t+s}) \mid \tilde{\mathscr{F}}_s\} = \tilde{P}_t g(\tilde{X}_s). \tag{3.9}$$

One notes that one may assume $s = 0$ upon replacing L by $(L - s)^+$. Having made this change, (3.9) is equivalent to the equality of the Laplace transforms in t,

$$E^{\mu}\left\{\int_0^{\infty} e^{-\alpha t}g(\tilde{X}_t)\,dt \mid \mathscr{G}_L\right\} = \tilde{U}^{\alpha}g(\tilde{X}_0). \tag{3.10}$$

Because of (3.7), (3.10) is equivalent to

$$E^{\mu}\left\{Z_L 1_{\{0<L<\infty\}} \int_0^{L} e^{-\alpha t}g(\tilde{X}_t)\,dt\right\} = E^{\mu}\{Z_L \tilde{U}^{\alpha}g(X_{L-})1_{\{0<L<\infty\}} \tag{3.11}$$

for all bounded positive $Z \in \mathscr{H}$.

Note that since L was replaced by $(L - s)^+$, we have $L < \zeta$ a.s. on $\{L < \infty\}$.

Let B_t denote the increasing process given by

$$B_t = Z_L 1_{\{0<L<\infty\}} 1_{[L,\infty)}(t). \tag{3.12}$$

It is trivial to verify from properties of $\mathscr{H}$ that B_t is an additive functional, not adapted to $(\mathscr{F}_t)$ in general. The potential of B_t is $u_B(x) = E^x(Z_L 1_{\{0<L<\infty\}})$, an excessive function of X which is a natural potential since $L < \zeta$. There exists, therefore, a natural additive functional A of X having this potential. One has then

$$E^x \int_{(0,\zeta)} e^{-\alpha t} f(X_{t-})\, dB_t = E^x \int_{(0,\zeta)} e^{-\alpha t} f(X_{t-})\, dA_t\,;\, f \in \mathscr{E}_+,\ \alpha \geq 0, \tag{3.13}$$

because the integrand is predictable, and both B and A have the same dual predictable projection. We may then compute the left side of (3.11). Substituting $L - t$ for t in the integrand and then using the fact that $Z_L = Z_L \circ \theta_t$ for $0 < t < L$, one has for $\alpha > 0$

$$\begin{aligned} E^\mu\left\{Z_L 1_{\{0<L<\infty\}} \int_0^L e^{-\alpha t} g(\tilde{X}_t)\, dt\right\} &= E^\mu\left\{Z_L 1_{\{0<L<\infty\}} \int_0^L e^{-\alpha L \circ \theta_t} g(X_t)\, dt\right\} \\ &= E^\mu \int_0^\infty 1_{\{L\circ\theta_t > 0\}} Z_L \circ \theta_t e^{-\alpha L \circ \theta_t} g(X_t)\, dt \\ &= E^\mu \int_0^\infty h(X_t) g(X_t)\, dt \end{aligned}$$

where $h(y) = E^y\{1_{\{L>0\}} Z_L e^{-\alpha L}\}$.
However,

$$\begin{aligned} E^\mu \int_0^\infty h(X_t) g(X_t)\, dt &= \int \mu(dx) U(hg)(x) \\ &= \int \mu(dx) \int u(x, y) h(y) g(y) \xi(dy). \end{aligned}$$

We obtain another expression for h making use of (3.13) and (3.5). Evidently, if ν is the Revuz measure for A,

$$\begin{aligned} h(y) &= E^y \int_0^\infty e^{-\alpha t}\, dB_t \\ &= E^y \int_0^\infty e^{-\alpha t}\, dA_t \\ &= \int u^\alpha(y, z) \nu(dz). \end{aligned}$$

Substituting above, we find after inverting the order of integration

$$\begin{aligned} E^\mu\{Z_L 1_{\{0<L<\infty\}} \int_0^L e^{-\alpha t} g(\tilde{X}_t)\, dt\} &= \int \nu(dz) \int \xi(dy) u^\alpha(y, z) \int \mu(dx) u(x, y) g(y) \\ &= \int \nu(dz) \hat{U}^\alpha(\nu g)(z). \end{aligned}$$

On the other hand, the right side of (3.11) can be written as

$$E^\mu \int_0^\infty \tilde{U}^\alpha g(X_{t-})\, dB_t = E^\mu \int_0^\infty \tilde{U}^\alpha g(X_{t-})\, dA_t$$

$$= \int \mu(dx) \int u(x, z)\nu(dz)\tilde{U}^{\alpha}g(z)$$

using (3.12) and (3.13) again. The last expression is clearly equal to

$$\int \nu(dz)v(z)\tilde{U}^{\alpha}g(z) = \int \nu(dz)\hat{U}^{\alpha}(vg)(z)1_{\{v(z)<\infty\}}.$$

However, because

$$\int \nu(dz)v(z) = \int \mu(dx) \int u(x, z)\nu(dz)$$

$$= \int \mu(dx)E^{x}B_{\infty} = E^{\mu}\{Z_L 1_{\{0<L<\infty\}}\} < \infty,$$

v is finite ν a.e., and the equality (3.11) follows.

REFERENCES

1. BLUMENTHAL, R. K. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic, New York.
2. GETOOR, R. K. and SHARPE, M. J. (1979). Excursions of Brownian motion and Bessel processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **47** 83–106.
3. ITO, K. and MCKEAN, H. P. (1965). *Diffusion Processes and Their Sample Paths.* Academic, New York.
4. MEYER, P. A. (1968). Processus de Markov: la frontière de Martin. *Lecture Notes in Mathematics* **77** Springer, Berlin.
5. NAGASAWA, M. (1964). Time reversions of Markov processes. *Nagoya Math J.* **24** 177–204.
6. REVUZ, D. (1970). Mesures associées aux fonctionnelles additives de Markov, I. *Trans. Amer. Math Soc.* **148** 501–531.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CALIFORNIA 92093