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OPTIMAL STOPPING WITH SAMPLING COST: THE SECRETARY PROBLEM

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A secretary problem is an optimal stopping problem based on relative ranks. To the usual formulation of the secretary problem we add a cumulative interview cost function $h(\cdot)$, no longer obtaining "cutoff point" rules. For an appealing form of $h(\cdot)$ we examine the limiting results using the infinite secretary problem. It is shown that the other appealing form of $h(\cdot)$ leads to trivial limiting results. A large class of problems is considered and recursive equations leading to the limiting solution are given. In particular we solve the problem of minimizing expected rank with a linear interview cost function. An approximation to the rank problem with fixed cost c per interview is obtained (for all values of c) through the solution of a single differential equation.

1. Introduction. A finite secretary problem is usually defined in the following way: n candidates apply for a secretarial position. Some criterion is established to rank the candidates from 1 (best) to n (worst) with no ties. The candidates arrive in a random fashion and only their relative ranks are observed. On the basis of their observed relative ranks and some loss function, each candidate is either hired (and the process stops) or passed by. Passed candidates can never be recalled. If the $(n-1)$st candidate is passed, the last candidate must be selected.

To consider the limiting case $(n \to \infty)$, Gianini and Samuels (1976), following a suggestion of Rubin (1966), defined an infinite secretary problem. Here an infinite number of rankable candidates arrive at times that are i.i.d. (identically independently distributed) uniform on $[0, 1]$. Again only relative ranks can be observed and there is no recall. Gianini (1977) and more generally Lorenzen (1979) showed that the infinite problem is the limit of the corresponding finite problems.

The formulation considered in this paper has a risk consisting of two parts, a loss function $q(\cdot)$ based on absolute ranks and a cumulative interview cost function $h_n(\cdot)$. Our goal is to find the minimal risk and a stopping rule that attains this risk. A large class of problems will be explicitely solved.

2. The finite problem. Let $X(i)$ and $Y(i)$ be the absolute and relative ranks of the ith candidate to appear. Let τ be a stopping rule on $\{1, 2, \cdots, n\}$ based on $Y(i)$ and let $V_n = \inf_\tau E[q(X(\tau)) + h_n(\tau)]$ be the minimal expected loss. Since at each time r we must either accept the present candidate or take another observation, the backward induction argument of Lindley (1961) applies and gives

$$C(n-1) = n^{-1} \textstyle\sum_{k=1}^{n} [q(k) + h_n(n)] \tag{2.1}$$

$$C(r-1) = r^{-1} \textstyle\sum_{k=1}^{r} \min[Q_n(r, k) + h_n(r), C(r)]$$

where $C(r)$ is the minimal going on cost at time r and

$$Q_n(r, k) = E[q(X(r)) \mid Y(r) = k] = \sum q(l) \binom{l-1}{k-1}\binom{n-l}{r-k} \Big/ \binom{n}{r}$$

is the expected cost for stopping on a candidate of relative rank k at time r. Then the

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minimal risk is $V_n = C(0)$ and the optimal stopping rule is to stop on the rth candidate having relative rank k if it is cheaper to do so than go on (i.e., if $Q_n(r, k) + h_n(r) \le C(r)$). Equivalently, and this is the form used throughout this paper, let $I_k = \{r \mid Q_n(r, k) + h(r) \le C(r)\}$. The optimal rule is to stop the first time a candidate of relative rank k arrives in I_k for some k.

The difficulty comes when we let $n \to \infty$. We let $q(\cdot)$ be an infinite nondecreasing sequence and consider two natural forms for the loss function. The first is to define an infinite nondecreasing sequence $h(\cdot)$ and let $h_n(i) = h(i)$ for $i = 1, \cdots, n$. The second is to define an increasing function $h(\cdot)$ on [0, 1] and let $h_n(i) = h(i/n)$. Strange as it may seem, the limiting solution to the first is always trivial. Only the second form is interesting. (In fact, we later use the second form to approximate the solution to the fixed cost per interview problem $h_n(i) = ci$ for an arbitrary constant c.)

Consider the first form.

PROPOSITION 2.1. *If $h(1) \le h(2) \le \cdots$ is given and $h_n(i) = h(i)$, $h(\infty) = \lim h(i)$, $q(\infty) = \lim q(i)$ and V^* is the minimal risk for the no-cost problem, then*

$$\lim_{n\to\infty} V_n = \min[V^* + h(\infty), q(\infty) + h(1)].$$

That is, in the limit, either take the first candidate and obtain a risk of $q(\infty) + h(1)$ or use the optimal no-cost policy paying the maximal interview cost and obtain a risk of $V^* + h(\infty)$.

PROOF. The inequality $\limsup V_n \le \min[V^* + h(\infty), q(\infty) + h(1)]$ follows from the remark following the proposition. For the other inequality, first note that

$$\lim_{n\to\infty} Q_n(r, k) = q(\infty) \quad \text{for any fixed } k \text{ and } r. \tag{2.2}$$

Now let V_n, τ_n, V_n^* and τ_n^* be the optimal risks and stopping rules for the secretary problem with and without interview cost. Fix N and use a property of the no-cost policy to choose n so large that $\tau_n^* \ge \mathrm{N}$ with probability one. Since $Q_n(r, k)$ decreases in r and $h_n(r)$ increases in r, we have

$$\begin{aligned} V_n &= E[q(X(\tau_n)) + h(\tau_n)] \\ &\ge [Q_n(N, 1) + h(1)]P(\tau_n \le N) + E[q(X(\tau_n))]I_{(\tau_n \ge N)} \\ &\quad + h(N)P(\tau_n > N) \\ &\ge [Q_n(N, 1) + h(1)]P(\tau_n \le N) + [V_n^* + h(N)]P(\tau_n > N) \\ &\ge \min[Q_n(N, 1) + h(1), V_n^* + h(N)] \\ &\to \min[q(\infty) + h(1), V^* + h(N)] \end{aligned}$$

This holds for all N. □

The rest of the paper is concerned with the interesting case where $h(\cdot)$ is an increasing function defined on [0, 1] and $h_n(i) = h(i/n)$. To obtain an heuristic limit we let $r/n \to x$ and let

$$C(r) \to f(x),$$

$$Q_n(r, k) \to R_k(x) = \textstyle\sum_{l=k}^{\infty} q(l) \binom{l-1}{k-1} x^k (1-x)^{l-k} \quad \text{and}$$

$$\mathrm{h}_n(\mathrm{r}) \to \mathrm{h}(\mathrm{x}).$$

If we rewrite (2.1) as a difference equation, divide each side by $1/n$ and let $n \to \infty$ we get a differential equation of the form

$$f'(x) = x^{-1} \textstyle\sum_{k=1}^{\infty} [f(x) - R_k(x) - h(x)]^+, \qquad f(1) = q(\infty) + h(1). \tag{2.3}$$

(Note for the no-cost problem $h(x) = 0$ and we get the differential equation obtained by Mucci (1973) as the limit of the finite problems.)

3. The infinite problem. To verify that the differential equation (2.3) gives the limiting solution of the finite problems we turn to the infinite secretary problem of Gianini and Samuels (1976). Here we let $f(x)$ be the minimal going on cost at time x, $h(x)$ the interview cost for observing the process up to time x and $R_k(x)$ the computed expected cost for stopping on a candidate of relative rank k at time x.

THEOREM 3.1. *If $q(\infty) < \infty$, $h(1) < \infty$ and $h(\cdot)$ is continuous then the differential equation* (2.3) *holds and $V_n \to f(0)$ the minimal risk for the infinite problem.*

PROOF. This is a simple application of Theorem 2.1 and Corollary 2 to Theorem 4.1 in Lorenzen (1979). □

This is easily extendable to the case $q(\infty) = \infty$. Undoubtedly even this can be strengthened if $h(\cdot)$ tends to infinity slowly enough but no proof is offered.

THEOREM 3.2. *If $h(1) < \infty$ then* (2.3) *again holds and furthermore $V_n \to f(0)$.*

PROOF. That (2.3) holds again follows from Theorem 2.1 in Lorenzen (1979). That $V_n \to f(0)$ can be seen by considering an intermediary process equivalent to the infinite secretary problem where we are told whether a candidate is one of the n best or not. (This is not a valid secretary problem.) We show the difference in minimal risk between this process and the finite problem tends to zero and the lim sup of the minimal risk for this process is bounded by $f(0)$. The other inequality is established in Corollary 1 of Theorem 4.1 in Lorenzen (1979). □

The limit of the finite problem stopping rules (after normalizing to the unit interval) is given by the island rule (a term first coined by Presman and Sonin (1972) for a similar problem)

$$I_k = \{x \mid R_k(x) + h(x) \le f(x)\}. \tag{3.1}$$

This is the optimal policy for the infinite problem; stop the first time a candidate of relative rank k arrives in I_k for some k.

Unlike the no-cost problem, more than one island can exist. As an example consider the rank problem $q(i) = i$ with 10 applicants. The algorithm in Chow, et al. (1964) gives $I_3 = \{8, 9, 10\}$ for the no-cost problem. If we let $h_{10}(i) = 0$ for $i = 1, 2, 3$ and $h_{10}(i) = 6$ for $i = 4, \cdots, 10$ equations (2.1) give $I_3 = \{3, 8, 9, 10\}$ which has two islands. For the infinite problem similar results hold.

Two simplifying properties of the optimal procedure exist. The first is intuitive and is suggested by the previous example; you will stop sooner with interview cost than without interview cost. The second gives conditions under which the optimal procedure is a single island rule.

PROPOSITION 3.1. *The optimal stopping rule will stop sooner with an interview cost than without an interview cost.*

PROOF. Let I_k^* be the optimal policy for the no-cost problem and recall x in I_k^* implies $\inf_{\tau > x} E[R_{Y(\tau)}(\tau)] \ge R_k(x)$. Then,

$$\begin{aligned} f(x) &= \inf_{\tau > x} E[R_{Y(\tau)}(\tau) + h(\tau)] \\ &\ge \inf_{\tau > x} E[R_{Y(\tau)}(\tau)] + h(x) \\ &\ge R_k(x) + h(x). \end{aligned}$$

Therefore x is in I_k. A similar result holds for the finite case. □

PROPOSITION 3.2. *Suppose $f(0) < R_1(0) + h(0)$ (so the game is worth starting), $q(k) < q(\infty)$ for all k and $h(\cdot)$ is differentiable. Let*

$$W_k(x) = x^{-1} \textstyle\sum_{j=1}^{k} [R_{k+1}(x) - R_j(x)] - h'(x). \tag{3.2}$$

If, for each k, $W_k(x)$ has at most one sign change, from + to −, then the optimal rule is a single island rule. That is, $I_k = [\beta_k, 1]$ for some β_k.

PROOF. If $g_k(x) = f(x) - R_k(x) - h(x)$ we can show $g'_k(z) \geq W_k(z)$ for all z in I_k and $g_k(x) = 0$ implies $g'_k(x) = W_k(x)$. Since $g_k(x) < 0$ for x not in I_k and $g_k(x) > 0$ for x in I_k, more than one island would imply $g'_k(z) < 0$ for some z in I_k. This would then contradict our hypothesis. □

4. A class of q(•)'s and h(•)'s. Let us consider the following class of secretary problems.

$$q_\xi(i) = \Gamma(\xi + i - 1)/[\Gamma(\xi)\cdot\Gamma(i)], \qquad \xi \geq 2,$$

$$h(x) = \textstyle\sum_{i=1}^{s} a_i x^i, \qquad s \text{ arbitrary}, a_i \geq 0.$$

Mucci (1973) has shown $R_M(x) = q_\xi(M)x^{1-\xi}$. Since $W_M(x)$ has the same sign as $\hat{W}_M(x) = \sum_{j=1}^{M} q_\xi(M+1) - q_\xi(j) - \sum_{i=1}^{s} i a_i x^{\xi+i-2}$ and the derivative of $\hat{W}_M(x)$ is negative for $x > 0$, W_M has at most one sign change. We conclude I_M is a single island rule and let $I_M = [\beta_M, 1]$. Following the procedure outlined in Mucci (1973), we obtain the recursive equations

$$\begin{aligned}(\beta_{M+1})^{-M} &\textstyle\sum_{j=1}^{M-1} [R_{M+1}(\beta_{M+1}) - R_j(\beta_{M+1})] \\ &= (\beta_M)^{-M} \textstyle\sum_{j=1}^{M-1} [R_M(\beta_M) - R_j(\beta_M)] - (M-1)\displaystyle\int_{\beta_M}^{\beta_{M+1}} y^{-M}h'(y)\,dy.\end{aligned} \tag{4.1}$$

Using the equalities $\sum_{j=1}^{M-1} [R_{M+1}(x) - R_j(x)] = [(M+\xi)/M]\sum_{j=1}^{M-1} [R_M(x) - R_j(x)] = [(M+\xi)/M][(\xi-1)/\xi](M-1)q_\xi(M)x^{1-\xi}$ and substituting for $R_M(\cdot)$ gives

$$\beta_M^{\xi+M-1} = \left[\frac{[(\xi-1)/\xi]q_\xi(M) + \beta_M^{\xi+M-1}\sum i a_i u_{i-M-1}(\beta_M)}{[(M+\xi)/M][(\xi-1)/\xi]q_\xi(M) + \beta_{M+1}^{\xi+M-1}\sum i a_i u_{i-M-1}(\beta_{M+1})}\right]\beta_{M+1}^{\xi+M-1} \tag{4.2}$$

where $u_l(x) = x^{l+1}/(l+1)$ for $l \neq -1$ and $u_{-1}(x) = \ln(x)$.

The cutoff point for the no-cost problem (call it α_M) is an upper bound for β_M (Proposition 3.1) and is derived in Mucci (1973). For $M > s$ (so that $u_l(x) < 0$) a lower bound on β_M is obtained by setting $\beta_{M+1} = 0$ and $\beta_M = 1$ in the complicated expression within brackets. Using the fact that $\beta_i \to 1$ gives

$$\beta_M \geq \prod_{l=M}^{\infty}\left(\frac{l}{l+\xi} - \sum_{i=1}^{m}\frac{i a_i l\xi}{(l-i)(l+\xi)(\xi-1)q_\xi(l)}\right)^{1/(l+\xi-1)} \quad \text{and}$$

$$\beta_M \leq \alpha_M = \prod_{l=M}^{\infty}\left(\frac{l}{l+\xi}\right)^{1/(l+\xi-1)}$$

These bounds squeeze together as $M \to \infty$. To obtain the solution we choose M, compute the upper and lower bounds and use an iterative numerical technique to successively solve backwards for $\beta_{M-1}, \cdots, \beta_1$, also obtaining upper and lower bounds. If the bounds on β_1 are not sufficiently close, go back and start with a bigger M. When the bounds on β_1 are sufficiently close, we compute $V = f(0) = f(\beta_1) = R_1(\beta_1) + h(\beta_1)$, the minimal risk. The β_i themselves give the optimal policy.

As a particular case let us consider the rank problem ($q(i) = i$ or $\xi = 2$) with linear cost $h(x) = Kx$. We solve the finite problem using (2.1) and the infinite problem using (4.1).

Since $\prod_{l=1}^{n} [l/(l+2)]^{1/(l+1)} - \prod_{l=1}^{\infty} [l/(l+2)]^{1/(l+1)}$ is of the order $1/n$, a direct evaluation is not feasible. Instead, the Euler-Maclaurin sum formula can be used to approximate the log of the infinite product and decrease computational time. Minimal risks for the finite and infinite problems are given in the next table.

Suppose we are interested in the limiting value for the rank problem where the cost per interview is some fixed constant c. Since $q(i) = i \to \infty$ and $h_n(i) = ci \to \infty$, Proposition 2.1 proves $V_n \to \infty$. Unfortunately it tells us nothing about the rate. To obtain this information we approximate the finite problem by the infinite problem with $K = cn$. (This approximation is adequate as long as it is not optimal to select the first candidate. It will not be optimal whenever $c < (0.088)(n + 1)$.) Letting $V_{(K)}$ be the risk for that infinite problem, we are interested in the rate at which $V_{(K)} \to \infty$. The last column of Table 4.1 indicates $V_{(K)} \simeq 2.477K^{1/2}$ for K large.

PROPOSITION 4.1. *For the rank problem with linear cost* $h(x) = Kx$, $\lim V_{(K)}/K^{1/2} = g(0) \simeq 2.477$ *where* $g(\cdot)$ *is defined on* $\mathbb{R}^+$ *and satisfies*

$$g'(x) = x^{-1} \textstyle\sum_{i=1}^{\infty} [g(x) - i/x - x]^+, \qquad g(x) < \infty, g(\infty) = \infty. \tag{4.3}$$

PROOF. Let $f_K(x)$ satisfy $f_K'(x) = x^{-1} \sum [f_K(x) - i/x - Kx]^+$, $f_K(\cdot)$ finite on $[0, 1)$ and $f(1) = \infty$. Define $g_K(x)$ on $[0, K^{1/2}]$ by $g_K(x) = f_K(x/K^{1/2})/K^{1/2}$. Then $g_K(\cdot)$ satisfies $g_K'(x) = x^{-1} \sum [g_K(x) - i/x - x]^+$, $g_K(\cdot)$ finite on $[0, K^{1/2})$ and $g_K(K^{1/2}) = \infty$. Since the $g_K(\cdot)$ satisfy the same differential equation, differing only at the right endpoint, it is easy to see $g_K(z)$ is decreasing in K for all $K \geq \mathbf{K} > z^2$. Thus, on a fixed interval $[0, z]$ the functions $g_K(\cdot)$, $K > \mathbf{K}$, are bounded which, by the form of the differential equation, shows they are equi-Lipschitzian and therefore equicontinuous. This shows $g_K(x)$ tends to some function $g(x)$, $g_K'(x) \to g'(x)$ and $g(\cdot)$ satisfies (4.3). Since z was arbitrary this proves the proposition. Call $g(\cdot)$ the infinite infinite problem and solve this the same way we solved the infinite problem. To seven decimal places we obtain $g(0) = 2.4768709$. □

To get an estimate of the optimal procedure for the finite problem we use the same trick. Compute $\gamma_1, \gamma_2, \cdots$ the "cutoff points" for the infinite infinite problem and approximate the β_i^K ("cutoff points" for the infinite problem) by $\beta_i^K \simeq \gamma_i/K^{1/2}$. Then approximate the finite procedure by $I_i^n \simeq \{n\beta_i^K, \cdots, n\}$. Thus for n large V_n behaves like $2.477\ (nc)^{1/2}$ and the cutoff points are approximately equal to $n\gamma_i/(nc)^{1/2} = n^{1/2}\gamma_i/c^{1/2}$. To further aid in the approximation, the first 10 γ_i are given in the next table.

As an example, suppose there are 100 applicants and the cost per interview is $c = 0.1$. (Note c is small enough to make the approximation valid.) The exact result from Table 4.1 is $V_n = 7.855$. The approximation gives $V_n \simeq 7.833$. The approximation to the optimal strategy is equally as good.

TABLE 4.1
Minimal costs for the linear cost rank problem.

K\n	10	100	1000	10000	∞
0	2.5579	3.6032	3.8324	3.8649	3.8695
1	3.1415	4.1189	4.3320	4.3619	4.3661
5	4.9110	5.9592	6.1505	6.1641	6.1674
10	6.5000*	7.8553	8.0489	8.0716	8.0745
100	15.5000*	23.0564	24.6013	24.7520	24.7688
500	55.5000*	46.0512	54.5161	55.2973	55.3845
1000	105.5000*	60.5000*	76.5554	78.1496	78.3255
5000	505.5000*	100.5000*	166.0788	174.2509	175.1412
10000	1005.5000*	150.5000*	229.4733	245.9021	247.6871

* Select first candidate

TABLE 4.2
Cutoff points for the infinite infinite problem.

γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}
0.5079	1.1621	1.8439	2.5361	3.2332	3.9332	4.6349	5.3378	6.0415	6.7459

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