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BOREL SETS VIA GAMES

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A family of games $G = G(\sigma, u)$ is defined such that (a) for each σ the set of all u for which Player I can force a win in $G(\sigma, u)$ is a Borel set $B(u)$ and (b) every Borel set is a $B(u)$ for some u.

1. Introduction. The subsets of the line now called Borel sets were defined by Borel (1898) as the smallest class $\mathscr{S}$ of sets that (a) includes all intervals and (b) is closed under countable disjoint union and under proper difference; i.e., if $S_1, S_2, \cdots$ are disjoint sets in $\mathscr{S}$, then $\cup_n S_n$ is also in $\mathscr{S}$, and if S and S^* are in $\mathscr{S}$ and $S^* \subset S$, then S/S^* is also in $\mathscr{S}$. Borel called these sets *ensembles mesurables*, and his definition was motivated by measure: we know the measure of an interval—its length—and if the measures of S_n, S, S^* in (b) are a_n, a, a^*, then we want the measures of $\cup_n S_n$ and S/S^* to be $\sum a_n$ and $a - a^*$ respectively.

If we replace (b) by (b′) closed under countable union and countable intersection or by (b″) closed under countable union and under complementation, we get the same class $\mathscr{S}$. These definitions of Borel sets, especially via (a) and (b″), are now standard. The definition is not only simply stated; it is technically convenient. To prove that all Borel sets have a certain property, e.g., the Baire property, we have only to check that intervals have the property and that the class of sets with the property is closed under countable union and under complementation.

Nevertheless, the definition seems to me somehow unsatisfactory. For instance it is not immediately clear from the definition that there are only c Borel sets; just as, if we defined the rationals as the smallest field containing the integers, it would not be immediately clear that there are only countably many rationals. We present here a direct description of Borel sets, based on games. Our description was suggested by one given by Hausdorff (1937).

2. The main result. Denote by X the set of all finite sequences $x = (x(1), \cdots, x(k))$ of positive integers. We include in X the empty sequence e of length O. A subset Y of X is a *stop rule* if every infinite sequence $w = (w(1), w(2), \cdots)$ of positive integers has exactly one segment on Y. (The set whose only element is e is a stop rule.) We shall associate with each stop rule Y, function f defined on Y whose values are intervals, and real number u a game $G(Y, f, u)$ played as follows. Two players, I and II, alternately choose positive integers $n_1, n_2, \cdots$. I chooses first, and all choices are made knowing the results of all previous choices. As soon as a position $y = (n_1, \cdots, n_k)$ in Y is reached, play stops. I wins if the number u is in the interval $f(n_1, \cdots, n_k)$, and II wins otherwise. Denote by $B(Y, f)$ the set of all real numbers u for which I can force a win in the game $G(Y, f, u)$.

THEOREM. *The sets $B(Y, f)$ are just the Borel sets.*

SKETCH OF PROOF. The class $\mathscr{B}$ of sets $B(Y, f)$ includes all intervals I, since for any Y if we take $f \equiv$ I we get $B(Y, f) =$ I. To show $\mathscr{B} \supset \mathscr{S}$, it suffices to show that, if $B_1, B_2, \cdots$ are in $\mathscr{B}$, so are $U = \cup_n B_n$ and $V = \cap_n B_n$. Say $B_n = B(Y_n, f_n)$. For U, associate with each real number u the game played as follows: I chooses a positive integer r, then II chooses a

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positive integer s, then they play $G(Y_r, f_r, u)$. This is just the game $G(Y^*, f^*, u)$, where Y^* is the set of all $y^* = rsy$ with $y \in Y_r$ and $f^*(rsy) = f_r(y)$. And I can force a win in $G(Y^*, f^*, u)$ iff there is an r for which he can force a win in $G(Y_r, f_r, u)$; i.e., iff $u \in \cup_r B(Y_r, f_r) = U$. So $B(Y^*, f^*) = U$. Similarly for V, except that after r and s are chosen, they play $G(Y_s, f_s, u)$.

To show that $\mathscr{B} \subset \mathscr{S}$, note that we can associate with any stop rule Y, set-valued f (not just interval-valued), and real number u a game $G(Y, f, u)$ and corresponding set $B(Y, f)$. We show that, *if all values of f are in $\mathscr{S}$, then $B(Y, f)$ is also in $\mathscr{S}$.* So, in particular, any $B(Y, f)$ with f interval-valued is in $\mathscr{S}$. Suppose, then, that we have a Y and set-valued f with $B(Y, f) \notin \mathscr{S}$. We shall show that there is a y with $f(y) \notin \mathscr{S}$. For all x that are segments of some $y \in Y$, denote by $B(Y, f, x)$ the set of all u for which I can force a win in $G(Y, f, u)$, starting from x. Then

(1) $$B(Y, f, x) = f(x) \qquad \text{for} \quad x \in Y;$$

(2) $$B(Y, f, x) = \cup_n B(Y, f, xn), \qquad \text{for} \quad x \notin Y, \quad \text{length of } x \text{ even};$$

(3) $$B(Y, f, x) = \cap_n B(Y, f, xn), \qquad \text{for} \quad x \notin Y, \quad \text{length of } x \text{ odd}.$$

It follows from (2) and (3) that if $B(Y, f, x) \notin \mathscr{S}$ and $x \notin Y$, there is an n for which $B(Y, f, xn) \notin \mathscr{S}$. So, starting with $B(Y, f) = B(Y, f, e)$ not in $\mathscr{S}$, we get a sequence $n_1, n_2, \cdots$ such that $B(Y, f, (n_1, \cdots, n_k)) \notin \mathscr{S}$. The sequence $n_1, n_2, \cdots$ continues until Y is hit, say at y; yielding $B(Y, f, y) = f(y)$ not in $\mathscr{S}$.

A referee has given the following short proof that $\mathscr{B} \subset \mathscr{S}$. The set $B(Y, f)$ is clearly analytic, being the projection of the set of all pairs (u, r) such that r is a winning strategy for Player I in $G(Y, f, u)$. Since, from a theorem of Gale and Stewart, the clopen game $G(Y, f, u)$ is determined, the complement of $B(Y, f)$ is also analytic. So, from a theorem of Lusin, $B(Y, f)$ is Borel.

COMMENT. The classical constructive definition of Borel sets is based on ordinals. We have in effect replaced ordinals by stop rules.

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