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NECESSARY AND SUFFICIENT LIFETIME CONDITIONS FOR NORMED CONVERGENCE OF CRITICAL AGE-DEPENDENT PROCESSES WITH INFINITE VARIANCE[1]

BY MARTIN I. GOLDSTEIN AND FRED M. HOPPE

Université de Montréal, and Cornell University and University of Michigan

The critical age-dependent branching process with offspring p.g.f. of the form $f(s) = s + (1 - s)^{1+\alpha}L(1 - s)$, $0 < \alpha \le 1$, L slowly varying at 0, is investigated. We generalize Kesten's unpublished necessary condition to establish N.A.S.C. on the tail of the lifetime distribution for existence of a nondegenerate normalized conditioned limit law and pose several related questions.

1. Introduction. Let $Z(t)$ be a critical age-dependent branching process with life-time distribution G satisfying $G(0^+) = 0$,

$$\int_0^\infty [1 - G(t)]\, dt = \mu < \infty$$

and offspring p.g.f. of the form

$$f(s) = s + (1 - s)^{1+\alpha}L(1 - s)$$

where $0 < \alpha \le 1$ and L is slowly varying at 0. In [4] we generalized to such processes a result of Slack [6], namely if $\exists\, \gamma > 0$ such that

$$t^{1+(1/\alpha)+\gamma}[1 - G(t)] \to 0 \quad \text{as} \quad t \to \infty \tag{0}$$

then there is a nondegenerate normed conditioned limit law. This condition is slightly stronger than necessary, since, for example, if the offspring distribution has a finite variance σ^2 (in which case necessarily $\alpha = 1$ and $L(0^+) = \sigma^2/2$) then

$$t^2[1 - G(t)] \to 0$$

is both necessary [Kesten, unpublished] and sufficient [3]. The purpose of this paper is to derive the necessary and sufficient condition in the general case.

In what follows $F(s, t)$ denotes the p.g.f. of $Z(t)$ given $Z(0) = 1$, $f_n(s)$ the nth iterate of $f(s)$, and $\{X_n\}$ the underlying Galton-Watson process of generations. Also, if t is not an integer then occasionally t will mean $[t]$.

2. Results. The following are equivalent:

$$\lim_{t\to\infty} P[(1 - F(0, t))Z(t) \le x \mid Z(t) \ne 0] = H(x), \qquad x > 0, \tag{A}$$

where H is a proper distribution satisfying $H(0^+) = 0$;

$$\lim_{t\to\infty} \frac{1 - F(0, t)}{1 - f_{t/\mu}(0)} = 1; \tag{B}$$

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(C) $$\lim_{t\to\infty} \frac{P[Z(t) > 0, X_{t/(\mu+\gamma)} = 0]}{1 - f_{t/\mu}(0)} = 0$$

for some (and hence any) $\gamma > 0$;

(D) $$\lim_{t\to\infty} \frac{t[1 - G(t)]}{1 - f_t(0)} = 0.$$

COROLLARY. (1) *The limit in* (A) *has* L.S. *transform given by* $1 - u(1 + u^\alpha)^{-1/\alpha}$; (2) $(\alpha t/\mu)[1 - F(0, t)]^\alpha L(1 - F(0, t)) \to 1$ *as* $t \to \infty$.

The equivalence of (A) and (D) is the main assertion. We have delineated four equivalent conditions for two reasons. First, (B) and (C) are intermediate steps in a rather technical proof, and break the discussion into convenient sections. Secondly they contain additional probabilistic insight. For example, (B) makes precise the intuitive statement that at time t the dominant generation will be t/μ and hence the probability of extinction of $Z(t)$ will be essentially the same as that of $X_{t/\mu}$. We shall proceed by the following sequence: (B) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (B) $\Rightarrow$ (A) $\Rightarrow$ (B).

After completing this work we were informed by V.A. Vatutin that he had shown (D) $\Rightarrow$ (A) by methods similar to ours. Since this part is similar to what we also did in [4] it will only be sketched. The implications (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D) are, however, the main novelty of the paper.

We shall require some results concerning the asymptotic behaviour of the iterates f_k which may be found in [6];

(1) $$\lim_{n\to\infty} \alpha n(1 - f_n(0))^\alpha L(1 - f_n(0)) = 1,$$
$$\lim_{n\to\infty} \frac{1 - f_n(\exp[-u(1 - f_n(0))])}{1 - f_n(0)} = \frac{u}{(1 + u^\alpha)^{1/\alpha}}, \qquad u > 0.$$

It was shown in [2] that (1) implies

(2) $$1 - f_n(0) \sim n^{-1/\alpha} L_1(n), \qquad n \to \infty$$

where L_1 is some function slowly varying at ∞. Therefore, rewriting (D) as

$$\lim_{t\to\infty} t^{1+1/\alpha} L_1^{-1}(t)(1 - G(t)) \to 0$$

and invoking the well-known property of slowly varying functions,

$$L_1^{-1}(t) < t^\gamma \text{ asymptotically as } n \to \infty \text{ for any } \gamma > 0,$$

the connection between (D) and our previous sufficient condition (0) is apparent.

3. Proofs. In the sequel, $0 < \gamma, \gamma' < \mu$, $n = [t/(\mu + \gamma)]$, $n' = [t/(\mu - \gamma')]$, and G^{*k} is the k-fold convolution of G.

(B) $\Rightarrow$ (C).

$$P[Z(t) > 0, X_n = 0] = P[Z(t) > 0] - P[X_n > 0] + P[Z(t) = 0, X_n > 0]$$
$$= P[Z(t) > 0] - P[X_n > 0] + P[Z(t) = 0, X_n > 0, X_{n'} > 0] + P[Z(t) = 0, X_n > 0, X_{n'} = 0]$$
$$\leq P[Z(t) > 0] - P[X_n > 0] + P[Z(t) = 0, X_{n'} > 0] + P[X_n > 0, X_{n'} = 0]$$
$$= P[X(t) > 0] - P[X_{n'} > 0] + P[Z(t) = 0, X_{n'} > 0]$$

yielding

$$P[Z(t) > 0, X_n = 0] \leq 1 - F(0, t) - (1 - f_{n'}(0)) + P[Z(t) = 0, X_{n'} > 0].$$

On the other hand

$$P[Z(t)=0, X_{n'}>0] \le P[X_{n'}>0 \quad \text{and each particle in generation } n' \text{ dies by time } t]$$
$$\le P[X_{n'}>0 \quad \text{and a fixed particle in generation } n' \text{ dies by time } t]$$
$$= (1-f_{n'}(0))G^{*n'}(t).$$

We obtain

$$\frac{P[Z(t)>0, X_n=0]}{1-f_{t/\mu}(0)} \le \frac{1-F(0,t)}{1-f_{t/\mu}(0)} - \frac{1-f_{n'}(0)}{1-f_{t/\mu}(0)}(1-G^{*n'}(t)).$$

$G^{*n'}(t) \to 0$ by virtue of the weak law of large numbers,

$[1-f_n(0)]/[1-f_{t/\mu}(0)] \to [1-(\gamma'/\mu)]^{1/\alpha}$ by (2) and

$[1-F(0,t)]/[1-f_{t/\mu}(0)] \to 1$ by B).

Taking the limit on t and then on $\gamma' \downarrow 0$, keeping γ fixed, (C) ensues.

(C) $\Rightarrow$ (D). We shall use a subscript on P to indicate the initial population size. c, d, D and $M>1$ are constants to be specified and a_n denotes $[1-f_n(0)]^{-1}$.

$P[Z(t)>0, X_n=0]$

$\ge P[Z(t)>0, X_n=0, da_n \le X_k \le Da_n$ for all $t/4\mu \le k \le t/2\mu$, $a_n \le X_{t/4\mu} \le Ma_n]$

$\ge P[a_n \le X_{t/4\mu} \le Ma_n, da_n \le X_k \le Da_n$ for all $t/4\mu \le k \le t/2\mu$, $X_n =$ 0 and at least one of the particles in generations between $t/4\mu$ and $t/2\mu$ has a lifetime exceeding $t]$

$\ge P[\cup_{1\le i\le da_n t/4\mu}\{Y_i>t\}]P[X_n=0 \mid X_{t/2\mu}=Da_n]P[da_n \le X_k \le Da_n$ for all $t/4\mu \le k \le t/2\mu \mid a_n \le X_{t/4\mu} \le Ma_n]P[a_n \le X_{t/4\mu} \le Ma_n]$

where $\{Y_i\}$ are independent random variables with distribution G. Therefore,

$$a_nP[Z(t)>0, X_n=0] \ge a_nA_nB_nC_nD_n \tag{3}$$

where A_n, B_n, C_n, and D_n are the respective probabilities in the product above.

By (C) the left side of (3) tends to 0 as $n\to\infty$ $(t\to\infty)$. We will show that each of a_nD_n, C_n, and B_n stays bounded away from 0 as $n\to\infty$ forcing $A_n \to 0$.

First consider

$$a_nD_n = \left(\frac{a_n}{a_{t/4\mu}}\right)P\left[\frac{a_n}{a_{t/4\mu}} \le \frac{X_{t/4\mu}}{a_{t/4\mu}} \le \frac{Ma_n}{a_{t/4\mu}} \,\middle|\, X_{t/4\mu} \ne 0\right].$$

By (2),

$$\frac{a_n}{a_{t/4\mu}} \to \left(\frac{4\mu}{\mu+\gamma}\right)^{1/\alpha} = b > 0$$

and it is shown in [6] that the conditional probability converges to $G(Mb)-G(b)$ where G is a distribution function with L.S. transform $1-u(1+u^\alpha)^{-1/\alpha}$. For $\alpha=1$, G is exponential and for $0<\alpha<1$, evaluating the second derivative of the L.S. transform at 0 shows that G has infinite variance. In either case G is not concentrated on a compact set so we may fix M sufficiently large that $G(Mb)-G(b)>0$. This shows that a_nD_n is bounded away from 0. Next

$$C_n \le P[X_k \ge da_n \quad \text{for all} \quad t/4\mu \le k \le t/2\mu \mid X_{t/4\mu} = a_n]$$

$$- P[X_k > Da_n \quad \text{for some} \quad t/4\mu \le k \le t/2\mu \,|\, X_{t/4\mu} = Ma_n]$$

$$= P_{a_n}[T_n > t/4\mu] - P_{Ma_n}[S_n \le t/4\mu]$$

where T_n is the first time that X_k drops below da_n and S_n is the first time the X_k exceeds Da_n(equal to ∞ in case either level is not achieved). Notice that both S_n and T_n are stopping times adapted to the martingale $\{X_k\}$. Now

$$\lim P[X_{t/4\mu} \le xa_n \,|\, X_0 = a_n], \qquad x > 0$$

exists as a proper, nondegenerate distribution. This can be seen by noting that $H_n(x) = P[X_{t/4\mu} \le xa_n \,|\, X_0 = a_n]$ has L.S. transform $\Psi_n(u) = [1 - (1 - \varphi_n(u))a_{t/4\mu}^{-1}]^{a_n}$ when φ_n is the L.S. transform of the conditional distribution $P[X_{t/4\mu} \le xa_n \,|\, X_{t/4\mu} \ne 0]$. It is shown in [6] that $\varphi_n \to \varphi$ a proper, nondegenerate L.S. transform and therefore $\Psi_n(u) \to \exp[-b(1 - \varphi(u))]$ which is also proper and nondegenerate. Hence if $\delta > 0$ is selected suitably small, then for all large n

$$\begin{aligned}
\frac{3}{4} &\le P_{a_n}[X_{t/4\mu} > \delta a_n] \\
&\le P_{a_n}[T_n > t/4\mu] + P_{a_n}[T_n \le t/4\mu, X_{t/4\mu} > \delta a_n] \\
&= P_{a_n}[T_n > t/4\mu] + \int_{\{T_n \le t/4\mu, X_{t/4\mu} > \delta a_n\}} 1 \, dP_{a_n} \\
&\le P_{a_n}[T_n > t/4\mu] + \frac{1}{\delta a_n} \int_{\{T_n \le t/4\mu\}} X_{t/4\mu} \, dP_{a_n} \\
&= P_{a_n}[T_n > t/4\mu] + \frac{1}{\delta a_n} \int_{\{T_n \le t/4\mu\}} X_{T_n} \, dP_{a_n} \\
&\le P_{a_n}[T_n > t/4\mu] + d/\delta.
\end{aligned}$$

If we let $d < \delta/4$ we have the bound

$$P_{a_n}[T_n > t/4\mu] > ½.$$

In a similar vein

$$\begin{aligned}
P_{Ma_n}[S_n \le t/4\mu] &\le \frac{1}{Da_n} \int_{\{S_n \le t/4\mu\}} X_{S_n} \, dP_{Ma_n} \\
&= \frac{1}{Da_n} \int_{\{S_n \le t/4\mu\}} X_{t/4\mu} \, dP_{Ma_n} \\
&\le \frac{M}{D} < \frac{1}{4} \text{ if } D > 4M,
\end{aligned}$$

using the criticality of $\{X_k\}$. Combining the two bounds we have

$$C_n > 1/4 \text{ for all large } n.$$

Finally,

$$B_n = [f_{n-t/2\mu}(0)]^{Da_n} = [1 - a_{n-t/2\mu}^{-1}]^{Da_n}$$

and this approaches $\exp(-KD) > 0$ where $K = [1 - (\mu + \gamma)/2\mu]^{1/\alpha}$ is the limit of $a_n/a_{n-t/2\mu}$.

(We require $\gamma < \mu$ in this argument but by changing the intermediate points $t/4\mu$ and $t/2\mu$ the result holds for all $\gamma > 0$).

As mentioned above this gives $A_n \to 0$ as $n \to \infty$. But $A_n = 1 - G(t)^{a_n(t/4\mu)}$ implying that $a_n t \log G(t) \to 0$ which is (D).

(D) $\Rightarrow$ (B). This proof follows along lines identical to [4], Section 2, except that in place of the Baum-Katz convergence rate used there we invoke an extension due to Heyde and Rohatgi ([5], Theorem 1a, b). Although their more general result requires the monotonicity of a certain slowly varying function, these theorems apply to our case and we omit the details.

(B) $\Rightarrow$ (A). This is identical to [4], Section 3.

(A) $\Rightarrow$ (B). Let Ψ be the L.S. transform of H and set $y_t = \exp[-u(1 - F(0, t))]$ for $u > 0$ obtaining from (A)

$$\lim_{t\to\infty} \frac{1 - F(y_t, t)}{1 - F(0, t)} = 1 - \Psi(u).$$

From Goldstein's comparison inequalities [3],

$$(4) \qquad 1 - f_j(s) - (1 - s)G^{*j}(t) \le 1 - F(s, t) \le 1 - f_j(s) + (1 - s)(1 - G^{*j}(t))$$

substituting $s = y_t$, then $j = n'$ in the left-hand side and $j = n$ on the right and finally defining the integer $k \equiv k(t)$ by the sandwich

$$f_k(0) \le y_t < f_{k+1}(0)$$

we get using the finiteness of μ that

$$\frac{1 - \Psi(u)}{u} \le \liminf_t \frac{1 - f_{n+k}(0)}{1 - f_k(0)}$$

and

$$\limsup_t \frac{1 - f_{n'+k}(0)}{1 - f_k(0)} \le \frac{1 - \Psi(u)}{u}.$$

Let $0 < \delta < ½$ be arbitrary. Then if γ and γ' are suitably small there is a $t_0(\gamma, \gamma', \delta)$ such that if $t \ge t_0$ then

$$1 - \delta \le \frac{1 - f_{(t/\mu)+k}(0)}{1 - f_{n+k}(0)} \le 1 \le \frac{1 - f_{(t/\mu)+k}(0)}{1 - f_{n'+k}(0)} \le 1 + \delta.$$

Only the extreme inequalities are not immediate. For the left one note that

$$\frac{1 - f_{(t/\mu)+k}(0)}{1 - f_{n+k}(0)} = \prod_{i=n+k}^{(t/\mu)+k-1} \frac{1 - f_{i+1}(0)}{1 - f_i(0)}$$

and by the form of $f(s)$ and (1)

$$\frac{1 - f_{i+1}(0)}{1 - f_i(0)} = 1 - (1 - f_i(0))^\alpha L(1 - f_i(0)) > 1 - \frac{2}{\alpha i}$$

if i is sufficiently large. The product therefore exceeds $(1 - 2/\alpha n)^{(t/\mu)-n}$ which converges to $\exp[-2\gamma/\alpha\mu]$ as $t \to \infty$. Choosing γ suitably small we get the desired inequality. The right-hand inequality is handled similarly. It now follows that

$$\lim_{t\to\infty} \frac{1 - f_{(t/\mu)+k}(0)}{1 - f_k(0)} = \frac{1 - \Psi(u)}{u}$$

so from (1)

$$\lim_{t\to\infty} \frac{(t/\mu) + k}{k} = \left[\frac{u}{1 - \Psi(u)}\right]^{\alpha}$$

or

$$\lim_{t\to\infty} t/k = \mu\beta \qquad \text{where} \quad \beta = \left[\frac{u}{1 - \Psi(u)}\right]^{\alpha} - 1.$$

But because $[1 - f_k(0)]/[1 - y_t] \to 1$ we get, using (2) and $k \sim t/\mu\beta$, that

$$\lim_{t\to\infty} \frac{1 - f_{t/\mu}(0)}{1 - y_t} = (1/\beta)^{1/\alpha}$$

and therefore

$$\lim_{t\to\infty} \frac{1 - F(0, t)}{1 - f_{t/\mu}(0)} = \frac{\beta^{1/\alpha}}{\mu}. \tag{5}$$

The left side is, and hence the right side must be, independent of u. Call it K. To evaluate K, from (4)

$$\frac{1 - f_{n'}(y_t)}{1 - F(0, t)} - \frac{1 - y_t}{1 - F(0, t)} G^{*n'}(t) \le \frac{1 - F(y_t, t)}{1 - F(0, t)} \to 1 - \Psi(u). \tag{6}$$

If $\delta > 0$ is arbitrary and if t is sufficiently large and γ' sufficiently small then from (2) and (5)

$$(1 - \delta)K(1 - f_{n'}(0)) \le 1 - F(0, t) \le (1 + \delta)K(1 - f_{n'}(0)).$$

Substituting this into the left side of (6) and taking limits on $t \to \infty$ and then on $\delta, \gamma' \to 0$ gives

$$K^{-1}[1 + \beta^{-1}]^{-1/\alpha} \le 1 - \Psi(u).$$

To obtain this expression we have made use of the definition of y_t, the convergence of K/n', and (2) in the form

$$\frac{1 - f_{n'}(y_t)}{1 - f_{n'}(0)} \sim \frac{1 - f_{n'+k}(0)}{1 - f_{n'}(0)} \to \left[1 + \frac{\mu - \gamma'}{\mu\beta}\right]^{-1/\alpha}.$$

There is a similar inequality in the other direction and we finally get

$$\Psi(u) = 1 - K^{-1}[1 + \beta^{-1}]^{-1/\alpha} = 1 - u(1 + K^{\alpha}u^{\alpha})^{-1/\alpha}.$$

Let $u \to \infty$. $\Psi(\infty) = H(0^+) = 0$ and $\beta^{-1} \to 0$. Hence $K = 1$ and we get (B) as well as the corollary.

Remark. $\Psi(u)$ is the same L.S. transform as obtained by Slack [6] in the Bienaymé-Galton-Watson case.

4. Concluding remarks and suggestions for further work. The analysis (A) $\Rightarrow$ (B) suggests some possibilities for the age-dependent process which do not arise in the Galton-Watson case, related to behavior when the lifetime distribution has a long tail.

Suppose that in condition (A) we permit $H(0^+) = q_0$ and $H(\infty) = q_\infty$ where possibly $q_0 > 0$ and $q_\infty < 1$. The limiting L.S. transform Ψ will be defective at the origin with $\Psi(0^+) = q_\infty$ and also $\Psi(\infty) = q_0$. In place of (B) we have

$$\lim_{t\to\infty} \frac{1 - F(0, t)}{1 - f_{t/\mu}(0)} = K$$

where K is no longer necessarily 1. We still have, however,

$$\Psi(u) = 1 - \frac{u}{(1 + K^\alpha u^\alpha)^{1/\alpha}}.$$

It follows that $\Psi(0^+) = 1$ so $q_\infty = 1$ and the limit distribution is not defective. However

$$\Psi(\infty) = 1 - \frac{1}{K}$$

giving

$$K = (1 - q_0)^{-1}.$$

There seems to be no reason a priori for excluding the case $q_0 > 0$. In fact Vatutin [7] has considered some cases when the process $\{Z(t) \mid Z(t) \neq 0\}$ converges to a discrete limit. It follows that

$$\{(1 - F(0, t))Z(t) \mid Z(t) \neq 0\}$$

converges to a limit degenerate at 0 with $q_0 = 1$ and $K = \infty$, that is

$$\lim_{t\to\infty} \frac{1 - F(0, t)}{1 - f_{t/\mu}(0)} = \infty.$$

One of his assumptions is that

$$\lim_{n\to\infty} \frac{n(1 - G(n))}{1 - f_n(0)} = \infty$$

and it is reasonable to conjecture that there is a general equivalence between the asymptotic behavior of

$$\frac{1 - F(0, t)}{1 - f_{t/\mu}(0)}$$

and $n(1 - G(n))/[1 - f_n(0)]$ which subsumes (B) $\Leftrightarrow$ (C).

If we look at $K < \infty$, then since

$$1 - f_{\lambda t/\mu}(0) \sim \lambda^{-1/\alpha}(1 - f_{t/\mu}(0))$$

we get

$$\lim_{t\to\infty} \frac{1 - F(0, t)}{1 - f_{\lambda t/\mu}(0)} = K\lambda^{1/\alpha}$$

and defining λ to make $K\lambda^{1/\alpha} = 1$ then heuristically the dominant generation at time t is $\lambda t/\mu$ which is smaller than t/μ suggesting that the long lifetimes depress the rate at which the dominant generation grows. In a future paper we plan to study the distribution of generations when the lifetime distribution has a long tail.

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DÉPARTMENT DE MATHÉMATIQUES
UNIVERSITÉ DE MONTRÉAL
CASE POSTALE 6128
MONTRÉAL 101
QUÉBEC, CANADA

DEPARTMENT OF STATISTICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN 48104