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SELECTING UNIVERSAL PARTITIONS IN ERGODIC THEORY[1]

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Let $\mathscr{P}$ be the set of all k-atom measurable partitions of a standard measurable space $(\Omega, \mathscr{F})$, and let T be an isomorphism of $(\Omega, \mathscr{F})$ onto itself. Given $P \in \mathscr{P}$, each probability measure μ on $\mathscr{F}$ stationary and ergodic with respect to T determines a joint distribution under μ of the k-state stochastic process (P, T). We say that P is universal for a property S (depending on μ) if the distribution of (P, T) satisfies S for all μ. Theorems are given which assure the existence of a universal $P \in \mathscr{P}$.

Introduction. Fix a standard measurable space $(\Omega, \mathscr{F})$ and an isomorphism $T:\Omega \to \Omega$. Let $\mathscr{M}_s(T)$ ($\mathscr{M}_e(T)$) be the set of all probability measures on $\mathscr{F}$ stationary (stationary and ergodic) with respect to T. Fix a finite index set A and let $\mathscr{P}$ be the set of all partitions $P = \{P^j \in \mathscr{F} : j \in A\}$. For $\mathscr{D} \subset \mathscr{M}_e(T)$ and each $\mu \in \mathscr{D}$, let $\mathscr{P}_\mu \subset \mathscr{P}$ be specified. This paper gives sufficient conditions on the $\{\mathscr{P}_\mu\}$ so that $\cap_{\mu\in\mathscr{D}} \mathscr{P}_\mu \neq \varnothing$.

Intuitively, a partition $P \in \cap_{\mu\in\mathscr{D}} \mathscr{P}_\mu$ is "universal" in the sense that if each $\mathscr{P}_\mu$ represents the partitions satisfying a certain property (depending on μ), then P satisfies the property for all μ. One application of "universal" partitions, treated at the end of the paper, is the extension of theorems of ergodic theory (such as the Sinai Theorem and the Ornstein isomorphism theorem) from the ergodic to the nonergodic case. A second application has an information-theoretic flavor. As is well known, each $P \in \mathscr{P}$ induces under T a stochastic process (P, T) with state space A (see [3] or the end of this paper). Suppose an individual called the "sender" selects a $P \in \mathscr{P}$ and then transmits the resulting process (P, T) to another individual called the "receiver." We suppose that the receiver will be satisfied provided the joint distribution of (P, T) is always λ, say. However the joint distribution of (P, T) depends on the measure μ being used on $\mathscr{F}$, over which the sender may have no control. Thus the sender needs to find a "universal" P so that the distribution of (P, T) is λ no matter what μ is.

In the next section we present selection theorems which give the existence of "universal" partitions.

Some selection theorems. If $P, Q \in \mathscr{P}$ and $\mu \in \mathscr{M}_s(T)$, $|P - Q|_\mu$, the partition distance between P and Q relative to μ, is defined to be $\frac{1}{2}\sum_{j\in A} \mu(P^j \Delta Q^j)$; for each μ, ρ_μ denotes the pseudometric on $\mathscr{P}$ such that $\rho_\mu(P, Q) = |P - Q|_\mu$, $P, Q \in \mathscr{P}$.

We make $\mathscr{M}_e(T)$ a measurable space by adjoining to $\mathscr{M}_e(T)$ the smallest σ-field of subsets of $\mathscr{M}_e(T)$ such that for each $E \in \mathscr{F}$, the map $\mu \to \mu(E)$ from $\mathscr{M}_e(T) \to [0, 1]$ is measurable. Hereafter we use the notation $\mathscr{D} \subset \mathscr{M}_e(T)$ to indicate that $\mathscr{D}$ belongs to this σ-field. If $\mathscr{D} \subset \mathscr{M}_e(T)$, let $\mathscr{F}(\mathscr{D})$ denote the σ-field of subsets of $\mathscr{D}$ induced by the σ-field on $\mathscr{M}_e(T)$. Let N denote the set of positive integers.

THEOREM 1. *Let $\mathscr{D} \subset \mathscr{M}_e(T)$ and let $\tilde{\mathscr{P}} \subset \mathscr{P}$ be countable. Let $\{\mathscr{P}_\mu \subset \mathscr{P} : \mu \in \mathscr{D}\}$ satisfy* (a) *for any $P \in \mathscr{P}$, $\{\mu \in \mathscr{D} : P \in \mathscr{P}_\mu\} \in \mathscr{F}(\mathscr{D})$;* (b) *for any $\mu \in \mathscr{D}$, $P \in \mathscr{P}_\mu$, and $Q \in \mathscr{P}$, if $|Q - P|_\mu = 0$ then $Q \in \mathscr{P}_\mu$; and* (c) *for any $\mu \in \mathscr{D}$, $\mathscr{P}_\mu \cap \tilde{\mathscr{P}} \neq \varnothing$.*

Then $\cap_{\mu\in\mathscr{D}} \mathscr{P}_\mu \neq \varnothing$.

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PROOF. By [2, Lemma 5] pick $\{\mu_\omega:\omega \in \Omega\} \subset \mathscr{D}$ such that (d) $\mu\{\omega:\mu_\omega = \mu\} = 1$, $\mu \in \mathscr{D}$, and (e) for each $E \in \mathscr{F}$, the map $\omega \to \mu_\omega(E)$ from $\Omega \to [0, 1]$ is measurable. For each $\mu \in \mathscr{D}$, define $A_\mu = \{\omega:\mu_\omega = \mu\}$. Then $\{A_\mu:\mu \in \mathscr{D}\}$ is a partition of Ω, each $A_\mu \in \mathscr{F}$, and $\mu(A_\mu) = 1$, $\mu \in \mathscr{D}$. Let $P_1, P_2, \cdots$ be an enumeration of the elements of $\tilde{\mathscr{P}}$. For $n = 1, 2, \cdots$, set $B_n = \{\omega:P_n \in \mathscr{P}_{\mu_\omega}\}$. By (c), $\cup_n B_n = \Omega$. Also each B_n is in $\mathscr{F}$ and is a union of "ergodic sets" A_μ. Form the disjoint sequence $C_n = B_n \cap [\cup_{j=1}^{n-1} B_j]^c$, $n = 1, 2, \cdots$. *If* $P_n = \{P_n^j: j \in A\}$ define $P = \{P^j: j \in A\}$ by $P^j = \cup_{n=1}^\infty (P_n^j \cap C_n)$, $j \in A$. For $\mu \in \mathscr{D}$, $A_\mu \subset C_n$ for a unique n, so by definition of B_n, $P_n \in \mathscr{P}_\mu$. But $| P - P_n |_\mu = 0$ since $\mu(C_n) = 1$. Hence, by (b), $P \in \mathscr{P}_\mu$.

THEOREM 2. *Let $\mathscr{D} \subset \mathscr{M}_e(T)$, and let $\{\mathscr{P}_\mu \subset \mathscr{P}:\mu \in \mathscr{D}\}$ satisfy*: (a) *for each $\mu \in \mathscr{D}$, $\mathscr{P}_\mu$ is nonempty and ρ_μ-open, and* (b) *for each $P \in \mathscr{P}$, $\{\mu \in \mathscr{D}:P \in \mathscr{P}_\mu\} \in \mathscr{F}(\mathscr{D})$.*
Then $\cap_{\mu\in\mathscr{D}} \mathscr{P}_\mu \neq \varnothing$.

PROOF. In Theorem 1, take $\tilde{\mathscr{P}}$ to be ρ_μ-dense in $\mathscr{P}$ for each $\mu \in \mathscr{D}$.

DEFINITION. Let $\mathscr{D} \subset \mathscr{M}_e(T)$. We say a map $\phi:\mathscr{D} \times \mathscr{P} \to [0, \infty)$ is *admissible* if for each $P \in \mathscr{P}$, $\phi(\cdot, P)$ is measurable and for each $\mu \in \mathscr{D}$, $\phi(\mu, \cdot)$ is ρ_μ-continuous.

LEMMA 1. *Let $\mathscr{D} \subset \mathscr{M}_e(T)$ and let $\phi:\mathscr{D} \times \mathscr{P} \to [0, \infty)$ be admissible. Then for each $\varepsilon > 0$, there exists $\delta = \delta^{(\phi,\varepsilon)}$ from $\mathscr{D} \times \mathscr{P} \to (0, 1]$ such that*: (a) *$\delta(\cdot, P)$ is measurable for each $P \in \mathscr{P}$*; (b) *if $\mu \in \mathscr{D}$, $P \in \mathscr{P}$, $\{P_n\} \subset \mathscr{P}$, and $| P_n - P |_\mu \to 0$, then $\liminf_{n\to\infty} \delta(\mu, P_n) > 0$*; (c) *if $\mu \in \mathscr{D}$, $P, Q \in \mathscr{P}$, and $| Q - P |_\mu < \delta(\mu, P)$, then $| \phi(\mu, Q) - \phi(\mu, P) | \leq \varepsilon$; and* (d) *$\delta(\mu, Q) = \delta(\mu, P)$ if $| P - Q |_\mu = 0$.*

PROOF. Fix $\varepsilon > 0$. For each $P \in \mathscr{P}$, $\delta > 0$, let $E_\delta(P)$ be the set of all $\mu \in \mathscr{D}$ such that $| Q - P |_\mu < \delta$ implies $| \phi(\mu, Q) - \phi(\mu, P) | \leq \varepsilon$. $E_\delta(P)$ is measurable; for if we let $\{Q_1, Q_2, \cdots\}$ be partitions ρ_μ-dense in $\mathscr{P}$ for every μ, then

$$E_\delta(P) = \cap_{i=1}^\infty \{\mu \in \mathscr{D}: | Q_i - P |_\mu \geq \delta \quad \text{or} \quad | \phi(\mu, Q_i) - \phi(\mu, P) | \leq \varepsilon\}.$$

Define the function $\delta = \delta^{(\phi,\varepsilon)}$ by $\delta(\mu, P) = \sup\{\delta \text{ rational}:\mu \in E_\delta(P), 0 \leq \delta \leq 1\}$. Lemma 1 (a), (c), (d) easily follow. To see that (b) holds, fix P and μ. Fix a positive rational α such that $| Q - P |_\mu < \alpha$ implies $| \varnothing(\mu, Q) - \phi(\mu, P) | < \varepsilon/2$. Fix Q so that $| Q - P |_\mu < \alpha/2$. Then $\delta(\mu, Q) \geq \alpha/2$.

For the proof of the following lemma, the reader may consult [2], proof of Theorem 3.

LEMMA 2. *Let $\mathscr{D} \subset \mathscr{M}_e(T)$ and let $\{P_n \in \mathscr{P}:n \in N\}$ satisfy $\sum_{n=1}^\infty | P_n - P_{n+1} |_\mu < \infty$, $\mu \in \mathscr{D}$. Then there exists $P \in \mathscr{P}$ such that $| P_n - P |_\mu \to 0$, $\mu \in \mathscr{D}$.*

DEFINITION. If $\mu \in \mathscr{M}_s(T)$, $\mathscr{Q} \subset \mathscr{P}$ is nonempty, and $P \in \mathscr{P}$, let $\rho_\mu(P, \mathscr{Q}) = \inf\{ | P - Q |_\mu:Q \in \mathscr{Q}\}$.

THEOREM 3. *Let $\mathscr{D} \subset \mathscr{M}_e(T)$ and let $\{\mathscr{P}_\mu:\mu \in \mathscr{D}\}$ be a family of nonempty subsets of $\mathscr{P}$ such that*: (a) *for each $P \in \mathscr{P}$, the map $\mu \to \rho_\mu(P, \mathscr{P}_\mu)$ is measurable, and* (b) *there exists a sequence $\{\phi_n\}$ of admissible functions from $\mathscr{D} \times \mathscr{P} \to [0, \infty)$ such that $P \in \mathscr{P}_\mu$ if and only if $\inf_n \phi_n(\mu, P) = 0$, $\mu \in \mathscr{D}$, $P \in \mathscr{P}$.*
Then $\cap_{\mu\in\mathscr{D}} \mathscr{P}_\mu \neq \varnothing$.

PROOF. Let $\{\alpha_i\}_1^\infty \downarrow 0$. For each $i, j \in N$, let $\delta_i^j = \delta^{(\phi_i,\alpha_j)}$. Inductively, we will construct partitions $\{P_n:n \in N\}$ and subsets $\{\mathscr{D}_{i_1\cdots i_n}:(i_1, \cdots, i_n) \in N^n, n = 1, 2, \cdots\}$ of $\mathscr{D}$ such that (a) $\{\mathscr{D}_i:i \in N\}$ is a partition of $\mathscr{D}$; (b) for each $n \in N$ and $(i_1, \cdots, i_n) \in N^n$, $\{\mathscr{D}_{i_1\cdots i_n i_{n+1}}:i_{n+1} \in N\}$ is a partition of $\mathscr{D}_{i_1\cdots i_n}$; (c) $\mu \in \mathscr{D}_{i_1\cdots i_n}$ implies $\phi_{i_n}(\mu, P_n) < \alpha_n$, $n \geq 1$; (d) $\mu \in \mathscr{D}_{i_1\cdots i_n}$ implies $| P_n - P_{n-1} |_\mu \leq \min_{1\leq s\leq n-1} \delta_{i_s}^s(\mu, P_s)/2^{n-s+1}$, $n \geq 2$; and (e) $\mu \in \mathscr{D}_{i_1\cdots i_n}$ implies $\rho_\mu(P_n, \mathscr{P}_\mu) < \min_{1\leq s\leq n} \delta_{i_s}^s(\mu, P_s)/2^{n-s+2}$, $n \geq 1$. First, we construct P_1 and $\{\mathscr{D}_i:i \in N\}$. Let $\tilde{\mathscr{P}}$ be a countable subset of $\mathscr{P}$ which is ρ_μ-dense in $\mathscr{P}$ for each $\mu \in \mathscr{D}$. For each $i \in N$, let

$$\mathscr{E}_i = \{\mu \in \mathscr{D}: \quad \text{for some} \quad Q \in \tilde{\mathscr{P}}, \rho_\mu(Q, \mathscr{P}_\mu) < \delta_i^1(\mu, Q)/2^2 \quad \text{and} \quad \phi_i(\mu, Q) < \alpha_1\}.$$

Now $\cup_{i=1}^{\infty} \mathscr{E}_i = \mathscr{D}$. For, if not, we may fix a $\mu \in \mathscr{D}$ such that μ is not in any $\mathscr{E}_i$. Since $\mathscr{P}_\mu \neq \varnothing$ we may select $P \in \mathscr{P}_\mu$. Fix i. Since for each $Q \in \tilde{\mathscr{P}}$ either $\rho_\mu(Q, \mathscr{P}_\mu) \geq \delta_i^1(\mu, Q)/2^2$ or $\phi_i(\mu, Q) \geq \alpha_1$, we obtain, by approaching P by a sequence from $\tilde{\mathscr{P}}$ and using Lemma 1(b), that either $\rho_\mu(P, \mathscr{P}_\mu) > 0$ or $\phi_i(\mu, P) \geq \alpha_1$. The former inequality is impossible and so $\phi_i(\mu, P) \geq \alpha_1$ for each i. This is impossible since $\inf_i \phi_i(\mu, P)$ must be 0. Consequently the $\{\mathscr{E}_i\}$ cover $\mathscr{D}$ and so we may choose a partition $\{\mathscr{D}_i: i \in N\}$ of $\mathscr{D}$ such that $\mathscr{D}_i \subset \mathscr{E}_i$, $i \in N$. Applying Theorem 1 we may obtain $P_1 \in \mathscr{P}$ such that if $\mu \in \mathscr{D}_i$ then $\rho_\mu(P_1, \mathscr{P}_\mu) < \delta_i^1(\mu, P_1)/2^2$ and $\phi_i(\mu, P_1) < \alpha_1$.

Now suppose $\{P_1, \cdots, P_n\}$ and $\{\mathscr{D}_{i_1\cdots i_k}: (i_1, \cdots, i_k) \in N^k, 1 \leq k \leq n\}$ have been constructed. We show how to construct P_{n+1} and $\{\mathscr{D}_{i_1\cdots i_{n+1}}: (i_1, \cdots, i_{n+1}) \in N^{n+1}\}$. Fix $(i_1, \cdots, i_n) \in N^n$. For each $i_{n+1} \in N$, define $\mathscr{E}_{i_1\cdots i_n i_{n+1}}$ to be the set of all $\mu \in \mathscr{D}_{i_1\cdots i_n}$ such that for some $Q \in \tilde{\mathscr{P}}$ we have (f) $|P_n - Q|_\mu < \min_{1\leq s\leq n} \delta_{i_s}^s(\mu, P_s)/2^{n-s+2}$; (g) $\phi_{i_{n+1}}(\mu, Q) < \alpha_{n+1}$; and (h) $\rho_\mu(Q, \mathscr{P}_\mu)$ is less than $\delta_{i_{n+1}}^{n+1}(\mu, Q)/2^2$ and less than $\min_{1\leq s\leq n}\delta_{i_s}^s(\mu, P_s)/2^{n-s+3}$.

Now $\mathscr{D}_{i_1\cdots i_n} = \cup_{i_{n+1}=1}^{\infty} \mathscr{E}_{i_1\cdots i_{n+1}}$. For, if not, fix $\mu \in \mathscr{D}_{i_1\cdots i_n}$ such that μ is not in any $\mathscr{E}_{i_1\cdots i_{n+1}}$. Fix i_{n+1}. For each $Q \in \tilde{\mathscr{P}}$, either (f), (g), or (h) fails. Hence by Lemma 1 (b), for each $Q \in \mathscr{P}$, either (f) fails, or (g) fails, or $\rho_\mu(Q, \mathscr{P}_\mu) > 0$. Now by (e), (f) holds for some $Q \in \mathscr{P}_\mu$. For this Q we must have then that (g) fails for every i_{n+1}, which is impossible since $\inf_{i_{n+1}} \phi_{i_{n+1}}(\mu, Q) = 0$. Consequently we may find a partition $\{\mathscr{D}_{i_1\cdots i_n i_{n+1}}: i_{n+1} \in N\}$ of $\mathscr{D}_{i_1\cdots i_n}$ such that $\mathscr{D}_{i_1\cdots i_{n+1}} \subset \mathscr{E}_{i_1\cdots i_{n+1}}$. Applying Theorem 1, we obtain $P_{n+1} \in \mathscr{P}$ such that if $\mu \in \mathscr{D}_{i_1\cdots i_{n+1}}$, then (f), (g), (h) hold for $Q = P_{n+1}$. This completes the inductive construction of $\{P_n\}_1^{\infty}$ and $\{\mathscr{D}_{i_1\cdots i_n}: (i_1, \cdots, i_n) \in N^n, n = 1, 2, \cdots\}$.

Now from (d) we have $\sum_{n=1}^{\infty} |P_n - P_{n+1}|_\mu < \infty$ for every $\mu \in \mathscr{D}$. Therefore by Lemma 2, find P such that $|P_n - P|_\mu \to 0$ for all μ. Let $\mu \in \mathscr{D}$. Pick $(i_1, i_2, \cdots)$ such that $\mu \in \mathscr{D}_{i_1\cdots i_n}$, $n \in N$. Then by (d), $|P_n - P|_\mu \leq \sum_{i=n}^{\infty} |P_i - P_{i+1}|_\mu \leq \sum_{i=n}^{\infty} \delta_{i_n}^n(\mu, P_n)/2^{i-n+2} \leq 2^{-1}\delta_{i_n}^n(\mu, P_n)$. Applying Lemma 1 (c), $\phi_{i_n}(\mu, P) \leq \phi_{i_n}(\mu, P_n) + \alpha_n < 2\alpha_n$. We see that $\inf_i \phi(\mu, P) = 0$ and so $P \in \mathscr{P}_\mu$. Thus, $P \in \cap_\mu \mathscr{P}_\mu$.

LEMMA 3. *Let $\mathscr{D} \subset \mathscr{M}_e(T)$ and let $\{\mathscr{P}_\mu: \mu \in \mathscr{D}\}$ satisfy the hypothesis of Theorem 2. Then for each $P \in \mathscr{P}$, the map $\mu \to \rho_\mu(P, \mathscr{P}_\mu)$ is measurable.*

PROOF. Let $\tilde{\mathscr{P}}_1, \tilde{\mathscr{P}}_2, \cdots$ be an increasing sequence of finite subsets of $\mathscr{P}$ whose union $\tilde{\mathscr{P}}$ is ρ_μ-dense in $\mathscr{P}$ for every μ. For each μ, P, $\rho_\mu(P, \mathscr{P}_\mu) = \rho_\mu(P, \mathscr{P}_\mu \cap \tilde{\mathscr{P}}) = \lim_{n\to\infty} \rho_\mu(P, \mathscr{P}_\mu \cap \tilde{\mathscr{P}}_n)$. For each P, the map $\mu \to \rho_\mu(P, \mathscr{P}_\mu)$ is measurable because it is a limit of a sequence of measurable functions.

THEOREM 4. *Let $\mathscr{D} \subset \mathscr{M}_e(T)$ and $\phi: \mathscr{D} \times \mathscr{P} \to [0, \infty)$ be given. Suppose:* (a) *there exist admissible functions $\{\phi_n\}$ such that $\phi = \inf_n \phi_n$;* (b) *for each $\mu \in \mathscr{D}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $P \in \mathscr{P}$ and $\phi(\mu, P) < \delta$ then there exists Q such that $|P - Q|_\mu < \varepsilon$ and $\phi(\mu, Q) = 0$; and* (c) *for each $\mu \in \mathscr{D}$, there exists $P \in \mathscr{P}$ such that $\phi(\mu, P) = 0$.*

Then, there exists $P \in \mathscr{P}$ such that $\phi(\mu, P) = 0$ for all $\mu \in \mathscr{D}$.

PROOF. Set $\mathscr{P}_\mu = \{P \in \mathscr{P}: \phi(\mu, P) = 0\}$, $\mu \in \mathscr{D}$. By (c) each $\mathscr{P}_\mu \neq \varnothing$. By (a), assumption (b) of Theorem 3 holds. The result will follow if we can show $\mu \to \rho_\mu(P, \mathscr{P}_\mu)$ is measurable for each P. For each $n \in N$ and $\mu \in \mathscr{D}$, let $= \mathscr{P}_\mu^n$ $\{P: \phi(\mu, P) < n^{-1}\}$. The map $\mu \to \rho_\mu(P, \mathscr{P}_\mu^n)$ is measurable by Lemma 3. It is not hard to see that assumption (b) implies that $\rho_\mu(P, \mathscr{P}_\mu) = \lim_{n\to\infty} \rho_\mu(P, \mathscr{P}_\mu^n)$, giving the desired measurability of the left-hand side as a function of μ for fixed P.

THEOREM 5. *Let $\mathscr{D} \subset \mathscr{M}_e(T)$ and let $\{\mathscr{P}_\mu: \mu \in \mathscr{D}\}$ be nonempty subsets of $\mathscr{P}$ such that:* (a) *each $\mathscr{P}_\mu$ is ρ_μ-closed and* (b) *for each $P \in \mathscr{P}$, $\mu \to \rho_\mu(P, \mathscr{P}_\mu)$ is measurable.*

Then $\cap_{\mu\in\mathscr{D}} \mathscr{P}_\mu \neq \varnothing$.

PROOF. Note $\rho_\mu(P, \mathscr{P}_\mu) = 0$ if and only if $P \in \mathscr{P}_\mu$. Also the map $(\mu, P) \to \rho_\mu(P, \mathscr{P}_\mu)$ is admissible, so (a), (b) of Theorem 3 hold.

Applications. Let $(A^\infty, \mathscr{G})$ be the measurable space consisting of A^∞, the set of all doubly infinite sequences from A, and $\mathscr{G}$, the usual product σ-field of subsets of A^∞. $T_A : A^\infty \to A^\infty$ will denote the shift transformation. Define the families $\mathscr{M}_e(T_A)$, $\mathscr{M}_s(T_A)$ by analogy to the definition of $\mathscr{M}_e(T)$, $\mathscr{M}_s(T)$. Let $\mathscr{N}_s(T)$ (resp $\mathscr{N}_e(T)$, $\mathscr{N}_s(T_A)$, $\mathscr{N}_e(T_A)$) denote the nonatomic measures in $\mathscr{M}_s(T)$ (resp $\mathscr{M}_e(T)$, $\mathscr{M}_s(T_A)$, $\mathscr{M}_e(T_A)$). If $P \in \mathscr{P}$, by the process (P, T) we mean the map from $\Omega \to A^\infty$ such that $(P, T)(\omega)_i = j$ if and only if $T^i\omega \in P^j$. If $\mu \in \mathscr{M}_s(T)$, dist $_\mu(P, T) \in \mathscr{M}_s(T_A)$ denotes the distribution of (P, T) under μ.

For each $n \in N$, let $X^n : A^\infty \to A^n$ be the map $X^n(x) = (x_1, \cdots, x_n)$. If $\mu, \nu \in \mathscr{M}_s(T_A)$ define $|\mu - \nu| = \sum_{n=1}^\infty 2^{-n} \sum_{a \in A^n} |\mu[X^n = a] - \nu[X^n = a]|$.

THEOREM 6. *Let $\mathscr{D} \subset \mathscr{N}_e(T)$ and let $\{\nu_\mu : \mu \in \mathscr{D}\} \subset \mathscr{M}_s(T_A)$ be such that for each $E \in \mathscr{G}$, the map $\mu \to \nu_\mu(E)$ is a measurable map from $\mathscr{D} \to [0, 1]$. Given $\varepsilon > 0$, there exists $P \in \mathscr{P}$ such that* $|$ dist $_\mu(P, T) - \nu_\mu| < \varepsilon$, $\mu \in \mathscr{D}$.

PROOF. For each $\mu \in \mathscr{D}$, let $\mathscr{P}_\mu = \{P : |\text{dist}_\mu(P, T) - \nu_\mu| < \varepsilon\}$. By [3], Lemma 5, page 22, each $\mathscr{P}_\mu \neq \varnothing$. Apply Theorem 2.

The following is an immediate consequence of Theorem 6 and the ergodic decomposition theorem [2], Lemma 5.

COROLLARY 1. *Let $\mathscr{D} = \mathscr{N}_s(T)$, $\nu \in \mathscr{M}_s(T_A)$ and $\varepsilon > 0$. There exists $P \in \mathscr{P}$ such that* $|$ dist $_\mu(P, T) - \nu| < \varepsilon$, $\mu \in \mathscr{D}$.

The following strengthens Rohlin's Theorem [4], page 16.

COROLLARY 2. *Let $\mathscr{D} = \mathscr{N}_s(T)$, $n \in N$, and $\varepsilon > 0$. There exists $F \in \mathscr{F}$ such that $F, TF, \cdots, T^{n-1}F$ are disjoint and $\mu(F \cup \cdots \cup T^{n-1}F) > 1 - \varepsilon$, $\mu \in \mathscr{D}$.*

PROOF. Fix $\nu \in \mathscr{M}_s(T_A)$ and a finite dimensional cylinder set $\bar{F}$ in A^∞ such that $\bar{F}, T_A\bar{F}, \cdots, T_A^{n-1}\bar{F}$ are disjoint and the ν-measure of the union is 1. By Corollary 1, obtain $P \in \mathscr{P}$ such that $|\mu[(P, T)^{-1}\bar{F}] - \nu(\bar{F})| < \varepsilon/n$, $\mu \in \mathscr{D}$. Take $F = (P, T)^{-1}\bar{F}$.

DEFINITION. Call $\mu \in \mathscr{N}_e(T)(\mathscr{N}_e(T_A))$ *Bernoulli* if $T(T_A)$ is a Bernoulli automorphism of $(\Omega, \mathscr{F}, \mu)((A^\infty, \mathscr{G}, \mu))$. If $\mu \in \mathscr{M}_s(T)(\mathscr{M}_s(T_A))$, $H_\mu(T)(H_\mu(T_A))$ denotes the entropy of $T(T_A)$ on $(\Omega, \mathscr{F}, \mu)((A^\infty, \mathscr{G}, \mu))$.

THEOREM 7. *Let $\mathscr{D} \subset \mathscr{N}_e(T)$. Let $\{\nu_\mu : \mu \in \mathscr{D}\} \subset \mathscr{N}_e(T_A)$ be Bernoulli measures such that for each $E \in \mathscr{G}$, the map $\mu \to \nu_\mu(E)$ is measurable. Suppose $H_{\nu_\mu}(T_A) \leq H_\mu(T)$, $\mu \in \mathscr{D}$. Then there exists $P \in \mathscr{P}$ such that* dist $_\mu(P, T) = \nu_\mu$, $\mu \in \mathscr{D}$.

PROOF. Define $\phi(\mu, P) = \bar{d}(\text{dist}_\mu(P, T), \nu_\mu)$, $\mu \in \mathscr{D}$, $P \in \mathscr{P}$, where $\bar{d}$ is the $\bar{d}$-metric on $\mathscr{M}_s(T_A)$ [1]. Assumption (c) of Theorem 4 holds by the Sinai Theorem and (b) follows from [3], Proposition 8, page 26.

We can now extend the Sinai Theorem to nonergodic automorphisms.

COROLLARY 3. *Let $\nu \in \mathscr{N}_e(T_A)$ be Bernoulli and let $\mu \in \mathscr{N}_s(T)$. Suppose there exists a probability space $(S, \mathscr{S}, \lambda)$ and a family of measures $\{\mu_\theta : \theta \in S\}$ in $\mathscr{M}_e(T)$ such that:* (a) *the map $\theta \to \mu_\theta(E)$ is measurable for each $E \in \mathscr{F}$;* (b) $\mu(E) = \int_S \mu_\theta(E)\, d\lambda(\theta)$, $E \in \mathscr{F}$; *and* (c) $H_{\mu_\theta}(T) \geq H_\nu(T_A)$ *for λ-almost all $\theta \in S$. Then there exists $P \in \mathscr{P}$ such that* dist $_\mu(P, T) = \nu$.

DEFINITION. For $\mu \in \mathscr{M}_s(T)$ we say $P = \{P^j : j \in A\} \in \mathscr{P}$ is a μ-generator if given $E \in \mathscr{F}$ there exists E' in the σ-field generated by $\{T^iP^j : j \in A, i = 0, \pm 1, \pm 2, \cdots\}$ such that $\mu(E' \Delta E) = 0$. If $P \in \mathscr{P}$ and $C = \{C^j : j \in E\}$ is a partition of A^∞, define $C(P) = \{C(P)^j : j \in E\}$ by $C(P)^j = (P, T)^{-1}C^j$, $j \in E$.

THEOREM 8. *Let $\mathscr{D} \subset \mathscr{N}_e(T)$ be a set of Bernoulli measures and let $\{\nu_\mu : \mu \in \mathscr{D}\} \subset \mathscr{N}_e(T_A)$ be Bernoulli measures such that for each $E \in \mathscr{G}$, the map $\mu \to \nu_\mu(E)$ is measurable. Suppose $H_\mu(T) = H_{\nu_\mu}(T_A)$, $\mu \in \mathscr{D}$. Then there exists $P \in \mathscr{P}$ such that for every $\mu \in \mathscr{D}$,* (a) dist $_\mu(P, T) = \nu_\mu$, *and* (b) *P is a μ-generator.*

PROOF. For $j = 1, 2, \cdots$, we define finite measurable partitions $Q_j = \{Q_j^i : i \in E_j\}$ of Ω so that the smallest σ-field containing all the sets in all the Q_j's is $\mathscr{F}$. We also suppose Q_{j+1} refines Q_j, $j = 1, 2, \cdots$. For each j, let $C_j^{(1)}, C_j^{(2)}, \cdots$ be an enumeration of all the measurable partitions of A^∞ indexed by E_j whose atoms are all finite dimensional cylinder sets. If $\mu \in \mathscr{D}$ and $P \in \mathscr{P}$, define $\phi(\mu, P) = \sum_{j=1}^\infty 2^{-j} \inf_i |C_j^{(i)}(P) - Q_j|_\mu + \bar{d}(\text{dist}_\mu(P, T), \nu_\mu)$. Note that $\phi(\mu, P) = 0$ if and only if (a) $-$ (b) hold. Assumption (b) of Theorem 4 follows immediately from [3], Proposition 8, page 26, and Proposition 11, page 31. Assumption (c) is Ornstein's isomorphism theorem [3].

The following generalizes the Ornstein isomorphism result to nonergodic automorphisms.

COROLLARY 4. *Let $\mu_1 \in \mathscr{M}_s(T)$, $\mu_2 \in \mathscr{M}_s(T_A)$. For $i = 1, 2$, suppose there exists a probability space $(S_i, \mathscr{S}_i, \lambda_i)$ with $(S_i, \mathscr{S}_i)$ standard and a family $\mathscr{D}_i = \{\mu_i^\alpha : \alpha \in S_i\}$ of Bernoulli measures ($\mathscr{D}_1 \subset \mathscr{N}_e(T)$, $\mathscr{D}_2 \subset \mathscr{N}_e(T_A)$) such that:* (a) *the map $\alpha \to \mu_i^\alpha$ is measurable and one-to-one and* (b) *$\mu_i(E) = \int_{S_i} \mu_i^\alpha(E)\, d\lambda_i(\alpha)$, for each measurable set E. Suppose there is a one-to-one measurable map Φ of S_1 onto S_2 such that* (c) *$\lambda_2 = \lambda_1 \cdot \Phi^{-1}$ and* (d) *$H_{\mu_1^\alpha}(T) = H_{\mu_2^{\Phi(\alpha)}}(T_A)$, $\alpha \in S_1$. Then there exists $P \in \mathscr{P}$ such that* (e) dist $_{\mu_1}(P, T) = \mu_2$ *and* (f) *P is a μ_1-generator.*

PROOF. Applying Theorem 8, we may obtain P such that (g) $\text{dist}_{\mu_1^\alpha}(P, T) = \mu_2^{\Phi(\alpha)}$, $\alpha \in S_1$, and (h) P is a μ_1^α-generator, $\alpha \in S_1$. From (c), (g), we get (e). Let $g : \mathscr{D}_2 \to \mathscr{D}_1$ be the measurable map such that $g(\mu_2^{\Phi(\alpha)}) = \mu_1^\alpha$. Let $Q = \{Q^j : j \in E\}$ be a fixed partition of Ω. Let $\mathscr{C}$ be the set of all partitions $C = \{C^j : j \in E\}$ of A^∞ indexed by E. Fix $\varepsilon > 0$. For each $\nu \in \mathscr{D}_2$, let $\mathscr{C}_\nu = \{C \in \mathscr{C} : |C(P) - Q|_{g(\nu)} < \varepsilon\}$. Since P is a $g(\nu)$-generator, $\mathscr{C}_\nu \neq \varnothing$. Also $\mathscr{C}_\nu$ is open in the partition distance on $\mathscr{C}$ induced by ν. By Theorem 2, there exists $C \in \mathscr{C}$ such that $|C(P) - Q|_{g(\nu)} < \varepsilon$, $\nu \in \mathscr{D}_2$. Equivalently, $|C(P) - Q|_{\mu_1^\alpha} < \varepsilon$, $\alpha \in S_1$. Integrating, $|C(P) - Q|_{\mu_1} < \varepsilon$. Since Q and ε are arbitrary, (f) holds.

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