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A CONVERSE TO THE SPITZER-ROSÉN THEOREM

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Let S_n be the sum of n independent and identically distributed random variables with zero means and unit variances. The central limit theorem implies that $P(S_n \leq 0) \to \frac{1}{2}$, and the Spitzer-Rosén theorem (with refinements by Baum and Katz, Heyde, and Koopmans) provides a rate of convergence in this limit law. In the present paper we investigate the converse of this result. Given a certain rate of convergence of $P(S_n \leq 0)$ to $\frac{1}{2}$, what does this imply about the common distribution of the summands?

1. Introduction. Let $X, X_1, X_2, \cdots$ be independent and identically distributed random variables with $E(X^2) = 1$ and $E(X) = 0$. Then $S_n = \sum_1^n X_i$ is asymptotically normally distributed, and $P(S_n \leq 0) \to \frac{1}{2}$. Spitzer (1960) estimated the rate of convergence in this limit theorem by showing that the series

$$\sum_1^\infty n^{-1}[P(S_n \leq 0) - \tfrac{1}{2}]$$

converges. Rosén (1961) showed that the series is absolutely convergent, and Baum and Katz (1963) that if

(1) $$E|X|^{2+2\alpha} < \infty$$

for some $0 \leq \alpha < \frac{1}{2}$ then

(2) $$\sum_1^n n^{\alpha-1}|P(S_n \leq 0) - \tfrac{1}{2}| < \infty.$$

This problem is of course closely related to the question of rates of convergence in the central limit theorem, for which if $0 < \alpha < \frac{1}{2}$,

$$\sum_1^n n^{\alpha-1} \sup_{-\infty<x<\infty} |P(S_n \leq n^{1/2}x) - \Phi(x)| < \infty$$

if and only if (1) holds (see Heyde (1967); Φ denotes the standard normal distribution function). Koopmans (1963) and Heyde (1966) refined Rosén's techniques and considered the case of nonidentically distributed summands.

Our principal aim in this paper is to establish necessary conditions for the convergence of series of the type (2), and to determine when the conditions (1) and (2) are equivalent. We generalize the Spitzer-Rosén problem by studying series of the form

$$\sum_1^\infty n^{-1}\lambda(n)|P(S_n \leq 0) - \tfrac{1}{2}|,$$

where λ is a nonnegative, measurable function satisfying mild regularity conditions.

Central to our discussion of necessary conditions is the class $\mathscr{C}$ of distributions F for which

$$\int_{-\infty}^{\infty} \sin tx \, dF(x)$$

does not change sign in some interval $(0, \epsilon]$, $\epsilon > 0$. Trivially, $\mathscr{C}$ contains all the symmetric distributions. In Section 2 we discuss properties of the distributions in $\mathscr{C}$, and give an

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example of a distribution not in $\mathscr{C}$. The Spitzer-Rosén problem is treated in Section 3, and the proofs of our main results are placed together in Section 4.

The symbol C, with or without subscripts, denotes a positive generic constant. The imaginary part of the complex number z is denoted by $\operatorname{Im} z$.

2. The class $\mathscr{C}$. The class $\mathscr{C}$ contains most of the distributions commonly encountered in statistics.

THEOREM 1. *Any distribution uniquely determined by its moments (all assumed finite) is in $\mathscr{C}$.*

PROOF. Let $\phi(t) = E(e^{itX})$ be the characteristic function of such a distribution. If $\operatorname{Im} \phi(t_k) = 0$ for a sequence $0 \neq t_k \to 0$ then

$$E(X) = \lim_{k\to\infty} \operatorname{Im} \phi(t_k)/t_k = 0.$$

Proceeding by induction we deduce that $E(X^{2n+1}) = 0$ for all n, and so $\psi(t) = \tfrac{1}{2}[\phi(t) + \phi(-t)]$ is the characteristic function of a distribution with the same moments as X. Hence $\psi \equiv \phi$, and $\operatorname{Im} \phi$ vanishes everywhere.

Therefore the distribution of any variable X satisfying Carleman's criterion (that is, $\sum_1^\infty (EX^{2n})^{-1/2n} = \infty$) is in $\mathscr{C}$. However the class is much wider.

THEOREM 2. (i) *The class $\mathscr{C}$ contains all distributions with $E|X| < \infty$ and bounded above or below.* (ii) *If $n \geq 0$, $E|X|^{2n+1} < \infty$ and $E(X^{2n+1}) \neq 0$ then the distribution of X is in $\mathscr{C}$.* (iii) *If $E|X| < \infty$, $E|X|^\alpha = \infty$ for some $\alpha > 1$, and either*

$$\begin{aligned} &\liminf_{x\to\infty}[P(X > x)/P(X \leq -x) - 1] > 0 \\ \textit{or}\quad &\liminf_{x\to\infty}[P(X \leq -x)/P(X > x) - 1] > 0, \end{aligned} \tag{3}$$

then the distribution of X is in $\mathscr{C}$.

Part (i) covers all scale and location changes of the lognormal distribution, which are excluded from consideration in Theorem 1 (see Heyde (1963)). Part (iii) is particularly relevant to the work of Section 3.

PROOF. (i) If the distribution of X is bounded below then in view of Theorem 1 we may suppose that $P(X \leq x) < 1$ for all x. In this case there exists a smallest integer $n \geq 0$ such that $0 \neq E(X^{2n+1}) \leq \infty$, and then

$$t^{-(2n+1)}E(\sin tX) \to (-1)^n E(X^{2n+1})$$

as $t \downarrow 0$. Part (ii) is proved similarly. (iii) Let $n \geq 0$ be the largest integer such that $E|X|^{2n+1} < \infty$. We may suppose that $E(X^{2j+1}) = 0$ for $0 \leq j \leq n$, for otherwise the result follows from (ii). Let $F_1(x) = P(0 \leq X \leq x)$ and $F_2(x) = P(0 \leq -X \leq x)$ for $x > 0$, and define

$$f(x) = \sin x - \textstyle\sum_0^n (-1)^j x^{2j+1}/(2j+1)!.$$

Note that f and f' do not change sign on $(0, \infty)$, and $|f(x)| \leq |x|^{2n+3}$. For $t > 0$ and $\lambda > 0$,

$$\begin{aligned} t^{-(2n+3)}|E \sin tX| &= t^{-(2n+3)}\left|\int_0^\infty f(tx)d[F_1(x) - F_2(x)]\right| \\ &\geq t^{-(2n+3)}\left|\int_\lambda^\infty f(tx)d[F_1(x) - F_2(x)]\right| - \lambda^{2n+3} \\ &\geq t^{-(2n+2)}\left|\int_\lambda^\infty f'(tx)[F_1(x) - F_2(x)]dx\right| - 2\lambda^{2n+3}. \end{aligned}$$

In view of condition (3) we may assume that for some $\epsilon > 0$,

$$F_2(x) - F_1(x) = 1 - F_1(x) - [1 - F_2(x)] \geq \epsilon[1 - F_1(x)]$$

for all $x > \lambda$. Therefore for all $\mu > \lambda$,

$$t^{-(2n+3)}\,|E \sin tX| \geq t^{-(2n+2)}\epsilon \left| \int_\lambda^\infty f'(tx)[1 - F_1(x)]dx \right| - 2\lambda^{2n+3}$$

$$\geq t^{-(2n+3)}\epsilon \int_\lambda^\mu |f(tx)|\,dF_1(x) - (2 + \epsilon)\lambda^{2n+3}.$$

Choose $\delta > 0$ so small that

$$|f(x)| > \tfrac{1}{2}\, x^{2n+3}/(2n + 3)!$$

for $0 < x < \delta$. Then if $t < \delta/\mu$,

$$t^{-(2n+3)}\,|E \sin tX| \geq \tfrac{1}{2}\, \epsilon[(2n + 3)!]^{-1} \int_\lambda^\mu x^{2n+3} dF_1(x) - (2 + \epsilon)\lambda^{2n+3}.$$

Our assumptions imply that $E(X^+)^{2n+3} = \infty$, and so by choosing μ sufficiently large we deduce that

$$\liminf_{t\downarrow 0} t^{-(2n+3)}\,|E \sin tX| > 0,$$

from which the result follows.

EXAMPLE. We exhibit a distribution not belonging to $\mathscr{C}$. The lognormal distribution, with density

$$p(x) = (2\pi)^{-1/2}x^{-1} \exp[-\tfrac{1}{2} (\log x)^2], \qquad x > 0,$$

has the same moments as a distribution with density

$$q(x) = p(x)[1 - \sin(2\pi \log x)], \qquad x > 0$$

(Heyde (1963)). Let X be an absolutely continuous variable with density $r(x) = \tfrac{1}{2}\, p(x)$ if $x > 0$, and $r(x) = \tfrac{1}{2}\, q(-x)$ if $x < 0$. Then

$$E(\sin tX) = \tfrac{1}{2} \int_0^\infty p(x) \sin(2\pi \log x) \sin(tx)dx = \tfrac{1}{2}\, (2\pi)^{-1/2}h(t),$$

say. We shall show that there exists a sequence $0 < t_k \to 0$ such that $h(t_k) = 0$ for all k. If this is false then

$$h(t) = \int_{-\infty}^\infty e^{-x^2/2} \sin(2\pi x) \sin(te^x)\, dx$$

is of the one sign for all $t \in (0, \epsilon]$, some $\epsilon > 0$. Without loss of generality it is positive. Now, $h''(t) = -e^2h(te^2)$, and so h' is strictly decreasing on $(0, \epsilon e^{-2}]$. Since $h'(0) = 0$ then h' must be negative on $(0, \epsilon e^{-2}]$, and since $h(0) = 0$ then h must be negative on $(0, \epsilon e^{-2}]$. This contradicts our assumption. Similarly it can be shown that h does not vanish identically in a neighbourhood of the origin.

The variable X has all moments finite and all odd order moments zero, but is asymmetrically distributed, a curious pathology indeed.

3. The Spitzer-Rosén theorem. Let $X, X_1, X_2, \cdots$ be independent and identically distributed variables with characteristic function ϕ, $E(X^2) = 1$ and $E(X) = 0$, and set $S_n = \sum_1^n X_i$. Let $\lambda: \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function with the properties

(i) $\lambda(x) = 0$ for $0 \le x < 1$;

(ii) for any $0 < a < b < \infty$, $\sup_{a<y<b} \lambda(xy)/\lambda(x)$ is bounded as $x \to \infty$;

and

(iii) for some $k > 0$, $\int_0^\infty \lambda(x)x^{-k}\,dx < \infty$.

With each λ satisfying (i)–(iii) we associate a function Λ defined by

$$\Lambda(t) = \int_0^\infty \lambda(x/t^2)e^{-x}\,dx.$$

THEOREM 3. *If*

$$\int_0^\infty \lambda(x)x^{-3/2}\log x\,dx < \infty \tag{4}$$

then condition (5) *implies* (6) *and* (7):

$$\int_0^1 |\arg\phi(t)|\,t^{-3}\Lambda(t)\,dt < \infty; \tag{5}$$

$$\sup_m \left|\textstyle\sum_1^m n^{-1}\lambda(n)[P(S_n \le 0) - \tfrac12]\right| < \infty; \tag{6}$$

$$\textstyle\sum_1^\infty n^{-1}\lambda(n)\,|P(S_n \le 0) - \tfrac12| < \infty. \tag{7}$$

Conversely, if the distribution of X is in $\mathscr{C}$ then (5), (6) *and* (7) *are equivalent.*

(We impose the restriction $-\pi < \arg\phi \le \pi$.)

The first part of Theorem 3 represents a generalization of the results of Spitzer (1960), Rosén (1961), Baum and Katz (1963), Koopmans (1963) and Heyde (1966). The converse is entirely new.

By imposing a "smoothness" condition on the distribution of X we may obtain faster rates of convergence than are possible under the condition (4). We adopt Cramér's continuity condition,

$$\limsup_{|t|\to\infty} |\phi(t)| < 1. \tag{C}$$

Condition (C) implies that $\sum_x P(X = x) < 1$, and (C) holds if the absolutely continuous part of the distribution of X does not vanish.

THEOREM 4. *Let λ be any function satisfying* (i)–(iii), *and suppose that* (C) *holds. Then* (5) *implies* (6) *and* (7). *Conversely, if the distribution of X is in $\mathscr{C}$ then* (5), (6) *and* (7) *are equivalent.*

One rationale behind studying a limit theorem for the quantity $P(S_n \le 0) - 1/2$ is that zero is not very distant from the median of S_n. Indeed, the central limit theorem implies that med $(S_n)/n^{1/2} \to 0$ as $n \to \infty$. If $E|X|^3 < \infty$ and $E(X^3) = \tau$, and if X does not have a lattice distribution, then med $(S_n) \to -(1/6)\tau$ as $n \to \infty$ (Haldane (1942), Hall (1980)). In this case it is appropriate to seek a limit theorem for the quantity $P(S_n \le -(1/6)\tau) - 1/2$.

We introduce the class $\mathscr{C}'$ of distributions with $E|X|^3 < \infty$ and such that $E[\sin tX - tX + (1/6)(t/X)^3]$ does not change sign in some interval $(0, \epsilon]$, $\epsilon > 0$. Sufficient conditions for a distribution to belong to $\mathscr{C}'$ may be established using the techniques of Section 2.

THEOREM 5. *Assume that $E|X|^3 < \infty$ and $E(X)^3 = \tau$, $E(X^2) = 1$, $E(X) = 0$, condition* (C) *holds, and λ is a function satisfying* (i)–(iii) *and*

$$\int_0^\infty \lambda(x)x^{-2}\,dx < \infty. \tag{8}$$

Then condition (9) *implies* (10) *and* (11), *and if the distribution of* X *is in* $\mathscr{C}'$ *then* (9), (10) *and* (11) *are equivalent*:

$$\int_0^1 |\operatorname{Im}\phi(t) + \tfrac{1}{6}\tau t^3|t^{-3}\Lambda(t)\,dt < \infty; \tag{9}$$

$$\sup_m |\textstyle\sum_1^m n^{-1}\lambda(n)[P(S_n \le -\tfrac{1}{6}\tau) - \tfrac{1}{2}]| < \infty; \tag{10}$$

$$\textstyle\sum_1^\infty n^{-1}\lambda(n)|P(S_n \le -\tfrac{1}{6}\tau) - \tfrac{1}{2}| < \infty. \tag{11}$$

We restrict our attention now to the special case $\lambda(x) = x^\alpha$, $x \ge 1$, where $\alpha > 0$. Our aim is to replace the condition (5) by an equivalent condition on the moments of X.

THEOREM 6. *Suppose* $E(X^2) = 1$, $E(X) = 0$ *and the distribution of* X *is in* $\mathscr{C}$. *If* $0 < \alpha < 1/2$ *and* $E(X^-)^{2+2\alpha} < \infty$ *then the following four conditions are equivalent.*

$$E(X^+)^{2+2\alpha} < \infty; \tag{12}$$

$$\sup_m |\textstyle\sum_1^m n^{\alpha-1}[P(S_n \le 0) - \tfrac{1}{2}]| < \infty; \tag{13}$$

$$\textstyle\sum_1^\infty n^{\alpha-1}|P(S_n \le 0) - \tfrac{1}{2}| < \infty; \tag{14}$$

$$\textstyle\sum_1^\infty n^{\alpha-1}\sup_{-\infty<x<\infty}|P(S_n \le n^{1/2}x) - \Phi(x)| < \infty. \tag{15}$$

The roles of X^+ and X^- above may of course be reversed. Theorem 6 is largely a corollary of Theorem 3, and a similar corollary of Theorem 4 may be deduced by the same argument.

4. The proofs.

PROOF OF THEOREM 3. As in Rosén (1961, Theorem 2) we have for any $\delta > 0$,

$$P(S_n \le 0) - \tfrac{1}{2} = \int_0^\delta (2\pi i t)^{-1}[\phi^n(-t) - \phi^n(t)]\,dt + R(n\delta) - \tfrac{1}{2}P(S_n = 0) \tag{16}$$

where

$$R(n, \delta) = -\pi^{-1}\int_{-\infty}^\infty dF_n(x)\int_\delta^\infty t^{-1}\sin tx\,dt,$$

F_n denoting the distribution function of S_n. If Q denotes the concentration function it is easily seen from Petrov (1975, Lemma 3, page 38) that for any $n \ge 2$,

$$Q(|S_n|; \log n) \le C(\log n)/n^{1/2}. \tag{17}$$

From this fact and the inequality

$$\left|\int_\delta^\infty t^{-1}\sin tx\,dt\right| \le C\min(1, 1/\delta|x|) \tag{18}$$

we find that for $0 < \delta \le 1$,

$$\begin{aligned}\pi|R(n,\delta)| &\le \left\{\int_{|x|\le\log n} + \textstyle\sum_{j=1}^{n-1}\int_{j\log n<|x|\le(j+1)\log n}\right.\\ &\qquad \left.+\int_{|x|>n\log n}\right\}\left|\int_\delta^\infty t^{-1}\sin tx\,dt\right| dF_n(x)\\ &\le C_1\{n^{-1/2}\log n + \delta^{-1}\textstyle\sum_{j=1}^{n-1} n^{-1/2}(\log n)(j\log n)^{-1} + n^{-1}\}\\ &\le C_2\,\delta^{-1}n^{-1/2}\log n,\end{aligned} \tag{19}$$

where C_2 depends on neither δ nor n. Conditions (ii) and (4) imply that

$$\sum_1^\infty \lambda(n) n^{-3/2} \log n < \infty.$$

Choose $\delta_n \downarrow 0$ so slowly that $\delta_n n^{1/2}/\log n \uparrow \infty$ and

$$\sum_1^\infty \delta_n^{-1} \lambda(n) n^{-3/2} \log n < \infty.$$

Set $\delta = \delta_n$ in (16); in view of (17) and (19) it suffices to prove Theorem 3 with the quantity $P(S_n \le 0) - 1/2$ appearing in (6) and (7) replaced by

$$\begin{aligned}\Delta_n &= -\int_0^{\delta_n} (2it)^{-1}[\phi^n(-t) - \phi^n(t)]\, dt \\ &= \int_0^{n^{1/2}\delta_n} t^{-1} |\phi(t/n^{1/2})|^n \sin[n \arg \phi(t/n^{1/2})]\, dt.\end{aligned}$$

For sufficiently large n we have $|\phi(t/n^{1/2})| \le e^{-t^2/4}$ for all $|t| \le n^{1/2}\delta_n$, and so if each $\delta_n \le 1$,

$$\begin{aligned}\sum_1^\infty n^{-1}\lambda(n)|\Delta_n| &\le C \sum_1^\infty \lambda(n) \int_0^{n^{1/2}\delta_n} t^{-1} |\arg \phi(t/n^{1/2})| e^{-t^2/4}\, dt \\ &\le C \int_0^1 t^{-1} |\arg \phi(t)| \left[\sum_1^\infty \lambda(n) e^{-nt^2/4}\right] dt.\end{aligned}$$

Using condition (ii),

$$\sum_1^\infty \lambda(n) e^{-nt^2/4} \le C_1 \int_0^\infty \lambda(x) e^{-xt^2/4}\, dx \le C_2 \int_0^\infty \lambda(x) e^{-xt^2}\, dx = C_2 t^{-2} \Lambda(t),$$

and so (6) and (7) follow from (5).

Conversely, suppose that the distribution of X is in $\mathscr{C}$, and condition (6) holds. We may neglect the case where $\arg \phi$ vanishes in a neighbourhood of the origin, for then (5) holds trivially.

Without loss of generality, $\arg \phi(t) \ge 0$ in $(0, \epsilon]$. Since $|\sin z - z| \le |z|^3/6$,

$$\begin{aligned}\Delta_n &= \int_0^{n^{1/2}\delta_n} t^{-1} |\phi(t/n^{1/2})|^n\, n \arg \phi(t/n^{1/2})\, dt \\ &\quad + A_n \int_0^{n^{1/2}\delta_n} t^{-1} |\phi(t/n^{1/2})|^n [n \arg \phi(t/n^{1/2})]^3\, dt \\ &= \Delta_{n1} + A_n \Delta_{n2},\end{aligned}$$

say, where $|A_n| \le 1/6$. We may write $\arg \phi(t) = t^2 a(t)$ where $a(t) \to 0$ as $t \to 0$. Let $\{\epsilon_n\}$ be a sequence of positive numbers chosen such that

$$\{(1 - \epsilon_n)/(1 + \epsilon_n)\}^{1/2} = ([n/2]/n)^{1/2} = k_n,$$

say. (On this occasion $[x]$ denotes the integer part of x.) Then $\{\epsilon_n\}$ is bounded away from zero, and for large n,

$$\begin{aligned}\Delta_{n2} &\le \int_0^{n^{1/2}\delta_n} t^5 \exp[-\tfrac{1}{2}(1 - \tfrac{1}{2}\epsilon_n)t^2] a^3(t/n^{1/2})\, dt \\ &\le C_1 \int_0^{n^{1/2}\delta_n} t \exp[-\tfrac{1}{2}(1 - \epsilon_n)t^2] a^3(t/n^{1/2})\, dt \\ &\le C_2 \int_0^{n^{1/2}\delta_n k_n} t \exp[-\tfrac{1}{2}(1 + \epsilon_n)t^2] a^3(t/n^{1/2} k_n)\, dt\end{aligned}$$

$$= o(1) \int_0^{[n/2]^{1/2}\delta_{[n/2]}} t \exp[-\tfrac{1}{2}(1 + \epsilon_n)t^2] a(t/[n/2]^{1/2})\, dt$$

$$= o(\Delta_{[n/2],1}).$$

(Note that $\{\delta_n\}$ is decreasing.) Since $\lambda(n)/\lambda([n/2])$ is bounded as $n \to \infty$ then

$$\sum n^{-1}\lambda(n)\Delta_n = \sum n^{-1}\lambda(n)(1 + \eta_n)\Delta_{n1}$$

where $\eta_n \to 0$ as $n \to \infty$. Therefore (6) implies that

$$\sum_1^\infty n^{-1}\lambda(n)\,|\,\Delta_{n1}\,| < \infty. \tag{20}$$

Choose $\delta > 0$ so small that $e^{-t^2} \le |\,\phi(t)\,| \le e^{-t^2/4}$ for $|\,t\,| \le \delta$. For large n we have $n^{1/2}\delta_n > \log n$ and

$$n^{-1}\Delta_{n1} = \int_0^{\delta_n} t^{-1}\,|\,\phi(t)\,|^n \arg \phi(t)\, dt$$

$$\ge \int_0^{\delta} t^{-1}e^{-nt^2} \arg\phi(t)\,dt - \pi \int_{n^{-1/2}\log n}^{\delta} t^{-1}e^{-nt^2/4}\, dt.$$

The last written term is $0(n^{-k})$ for all $k > 0$, and so in view of (iii) and (20),

$$\infty > \sum_1^\infty \lambda(n) \int_0^\delta t^{-1}e^{-nt^2} \arg \phi(t)\, dt$$

$$= \int_0^\delta t^{-1}|\arg \phi(t)\,|\,[\textstyle\sum_1^\infty \lambda(n)e^{-nt^2}]\, dt$$

$$\ge C \int_0^\delta t^{-3}|\arg \phi(t)\,|\Lambda(t)\, dt,$$

from which follows (5).

PROOF OF THEOREM 4. Using condition (C) we improve on two estimates used in the proof of Theorem 3, and then the proof can be completed as before. In place of (17) we note from Petrov (1975, Lemma 3, page 38) that for any $\epsilon > 0$,

$$\begin{aligned} Q(\,|\,S_n\,|;\, e^{-\epsilon n}) &\le Ce^{-\epsilon n} \int_{-e^{\epsilon n}}^{e^{\epsilon n}} |\,\phi(t)\,|^n\, dt \\ &\le 2C\{e^{-\epsilon n} + [\sup_{|t|>1}|\,\phi(t)\,|]^n\} = O(e^{-\epsilon' n}) \end{aligned} \tag{21}$$

where $0 < \epsilon' \le \epsilon$. To improve on the estimate (19) we observe that for $0 < \delta \le 1$,

$$\pi|\,R(n, \delta)\,| \le \tfrac{1}{2} \int_\delta^{2^n} t^{-1}[\,|\,\phi(t)\,|^n + |\,\phi(-t)\,|^n]\, dt + \int_{-\infty}^{\infty} \left| \int_{2^n}^{\infty} t^{-1} \sin tx\, dt \right| dF_n(x)$$

$$\le [\sup_{|t|>\delta}|\,\phi(t)\,|]^n \log(2^n/\delta) + CP(\,|\,S_n\,| \le 2^{-n} \quad \text{or} \quad > 2^n)$$

$$+ \left[\sum_1^{2^n-1} \int_{j2^{-n}<|x|\le(j+1)2^{-n}} + \sum_1^{2^n-1} \int_{j<|x|\le j+1} \right] \left| \int_{2^n}^{\infty} t^{-1} \sin tx\, dt \right| dF_n(x).$$

For each $\delta > 0$, $[\sup_{|t|>\delta}|\,\phi(t)\,|]^n = O(e^{-\epsilon n})$ for some $\epsilon = \epsilon(\delta) > 0$. Choose $\delta_n \downarrow 0$ so slowly that $[\sup_{|t|>\delta_n}|\,\phi(t)\,|]^n = O(n^{-k})$ for all $k > 0$, and such that $\delta_n > 2^{-n}$. Then the first term on the right hand side is $O(n^{-k})$ for all $k > 0$. Using (18) and (21) we see that the two series on the right are respectively dominated by

$$C_1 \sum_1^{2^n-1} P(j2^{-n} < |S_n| \le (j+1)2^{-n})/2^n \cdot j2^{-n} \le C_2 Q(|S_n|; 2^{-n}) \log(2^n) = O(e^{-\epsilon n})$$

for some $\epsilon > 0$, and

$$C_1 \sum_1^{2^n-1} (j2^n)^{-1} \le C_2 2^{-n} \log(2^n).$$

Finally,

$$P(|S_n| \le 2^{-n} \quad \text{or} \quad > 2^n) \le Q(|S_n|; 2^{-n}) + 2^{-2n} E(S_n^2),$$

and combining these estimates we find that

$$|R(n, \delta_n)| = O(n^{-k}) \quad \text{for all} \quad k > 0.$$

In view of this result, (iii), (16) and (21) it suffices to prove Theorem 4 with $P(S_n \le 0) - \frac{1}{2}$ replaced by Δ_n, and the proof goes through as before.

Proof of Theorem 5. Let $m = -\frac{1}{6}\tau$. From Gil Pelaez' (1951) inversion formula and the techniques above we deduce that there exists a sequence $\delta_n \downarrow 0$ such that for any $k > 0$,

$$P(S_n \le m) - \tfrac{1}{2} = \int_0^{\delta_n} (2\pi i t)^{-1} [e^{itm} \phi^n(-t) - e^{-itm} \phi^n(t)]\, dt + O(n^{-k})$$

$$= \pi^{-1} \int_0^{\delta_n} t^{-1} |\phi(t)|^n \{\sin(tm)\cos[n \arg \phi(t)] - \cos(tm)\sin[n \arg \phi(t)]\}\, dt$$

$$= \pi^{-1} \int_0^{n^{1/2}\delta_n} t^{-1} |\phi(t/n^{1/2})|^n \{(tm/n^{1/2}) - n \arg \phi(t/n^{1/2})\}\, dt + O(n^{-1}).$$

(Note that $\arg \phi(t) = O(|t|^3)$ as $t \to 0$.) Since $(d/dt)|\phi(t)| = -t + O(t^2)$ as $t \to 0$ then

$$(d/dt)|\phi(t/n^{1/2})|^n = -t|\phi(t/n^{1/2})|^{n-1} + r_{n1}(t) = -t|\phi(t/n^{1/2})|^n + r_{n2}(t),$$

where for large n, $|t| \le n^{1/2}\delta_n$ and $j = 1, 2$, $|r_{nj}(t)| \le Cn^{-1/2}[1 + |t|^3]e^{-t^2/4}$. Using this estimate and integrating by parts we see that if $\delta_n \to 0$ slowly,

$$\int_0^{n^{1/2}\delta_n} t^{-1} |\phi(t/n^{1/2})|^n (tm/n^{1/2})\, dt = mn \int_0^{n^{1/2}\delta_n} t^{-1} |\phi(t/n^{1/2})|^n (t/n^{1/2})^3\, dt + O(n^{-1}).$$

Therefore

$$P(S_n \le m) - \tfrac{1}{2} = n\pi^{-1} \int_0^{n^{1/2}\delta_n} t^{-1} |\phi(t/n^{1/2})|^n \{m(t/n^{1/2})^3 - \arg \phi(t/n^{1/2})\}\, dt + O(n^{-1}),$$

and if each $\delta_n \le 1$ we see from (8) that

$$\sum_1^\infty n^{-1}\lambda(n) |P(S_n \le m) - \tfrac{1}{2}| \le \pi^{-1} \sum_1^\infty \lambda(n) \int_0^1 t^{-1} |\phi(t)|^n |mt^3 - \arg \phi(t)|\, dt + C_1$$

$$\le C_2 \int_0^1 t^{-1} |\arg \phi(t) + \tfrac{1}{6}\tau t^3| [\textstyle\sum_1^\infty \lambda(n) e^{-nt^2/4}]\, dt + C_1$$

$$\le C_3 \int_0^1 |\arg \phi(t) + \tfrac{1}{6}\tau t^3| t^{-3} \Lambda(t)\, dt + C_1.$$

Therefore (10) and (11) follow from (9). The converse is proved as before. Note that arg $\phi(t) = \operatorname{Im} \phi(t) + O(t^4)$ as $t \to 0$.

PROOF OF THEOREM 6. Condition (15) is equivalent to $E|X|^{2+2\alpha} < \infty$; see Heyde (1967). Therefore it suffices to prove that (5) and (12) are equivalent. Now,

$$\operatorname{Im} \phi(t) = |\phi(t)| \sin[\arg \phi(t)] \sim \arg \phi(t)$$

as $t \to 0$, and since the distribution of X is in $\mathscr{C}$ it suffices to prove that (12) is equivalent to

$$\lim_{\epsilon \to 0} \left| \int_\epsilon^\infty E(\sin tX) t^{-(3+2\alpha)}\, dt \right| < \infty. \tag{22}$$

Define

$$I(\epsilon) = \left| \int_\epsilon^\infty E(\sin tX) t^{-(3+2\alpha)}\, dt \right| = \left| E \int_\epsilon^\infty (tX - \sin tX) t^{-(3+2\alpha)}\, dt \right|,$$

$$I_+(\epsilon) = E \int_\epsilon^\infty (tX^+ - \sin tX^+) t^{-(3+2\alpha)}\, dt \quad \text{and} \quad I_-(\epsilon) = E \int_\epsilon^\infty (tX^- - \sin tX^-) t^{-(3+2\alpha)}\, dt.$$

The functions I_+ and I_- are nonnegative, and $I_+(\epsilon) - I_-(\epsilon) \le I(\epsilon) \le I_+(\epsilon) + I_-(\epsilon)$. Since

$$\int_0^\epsilon (tx - \sin tx) t^{-(3+2\alpha)}\, dt = C_\alpha x^{2+2\alpha}$$

where the positive constant C_α depends only on α, then

$$\lim_{\epsilon \to 0} I_-(\epsilon) = C_\alpha E(X^-)^{2+2\alpha} < \infty,$$

and therefore $\lim_{\epsilon \to 0} I(\epsilon) < \infty$ if and only if (12) holds.

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