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CONVERGENCE TO A STABLE DISTRIBUTION VIA ORDER STATISTICS[1]

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Let $X_1, X_2, \cdots$ be i.i.d. random variables whose common distribution function F is in the domain of attraction of a nonnormal stable distribution. A simple, probabilistic proof of the convergence of the normalized partial sums to the stable distribution is given. The proof makes use of an elementary property of order statistics and clarifies the manner in which the largest few summands determine the limiting distribution. The method is applied to determine the limiting distribution of self-norming sums and deduce a representation for the limiting distribution. The representation affords an explanation of the infinite discontinuities of the limiting densities which occur in some cases. Application of the technique to prove weak convergence in a separable Hilbert space is explored.

1. Introduction. Let X denote a random variable; let F denote its distribution function; and let $1 - G$ denote the distribution function of $Y = |X|$, so that $G(y) = P\{Y > y\} = F(-y -) + 1 - F(y)$, $y > 0$. It is well known that X is in the domain of attraction of a stable law with index α, $0 < \alpha < 2$, if and only if there is a function L which varies slowly at ∞ and a p, $0 \le p \le 1$, for which

$$G(y) = P\{Y > y\} = y^{-\alpha}L(y), \qquad y > 0, \tag{1}$$

and

$$[1 - F(y)]/G(y) \to p \qquad \text{and} \qquad F(-y)/G(y) \to q, \qquad \text{as} \quad y \to \infty, \tag{2}$$

where $q = 1 - p$. That is, if $X, X_1, X_2, \cdots$ are i.i.d., then (1) and (2) are necessary and sufficient for the existence of normalizing constants $a_n > 0$, $n \ge 1$, and b_n, $n \ge 1$, for which the normalized partial sums

$$S_n^{\#} = a_n^{-1}(X_1 + \cdots + X_n) \qquad \text{or} \qquad S_n^* = a_n^{-1}(X_1 + \cdots + X_n - nb_n)$$

converge in distribution to a stable law with index α. In this case, the normalizing constants are determined by

$$nG(a_n y) \to y^{-\alpha}, \qquad \text{as} \quad n \to \infty, \quad y > 0, \tag{3}$$

and

$$b_n = \int_{-a_n}^{a_n} x\,dF(x), \qquad n \ge 1. \tag{4}$$

Classical proofs of these assertions are analytical, using either semigroups of convolution operators or characteristic functions. See, for example, Feller (1966, Sections 9.6 and 17.5). Recently, Simon and Stout (1978) have given a probabilistic proof of sufficiency.

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Here we present an alternative proof of the sufficiency of (1) and (2). Our proof makes extensive use of the following elementary property of order statistics: let $Y_1, Y_2, \cdots$ be i.i.d., with common distribution function denoted by $1 - G$, and let $Y_{n1} \geq \cdots \geq Y_{nn}$ denote the ordered values of $Y_1, \cdots, Y_n$; then there is a sequence $E_1, E_2, \cdots$ of exponential random variables with unit mean for which

$$(5) \qquad (Y_{n1}, \cdots, Y_{nn}) =_d [G^{-1}(\Gamma_1/\Gamma_{n+1}), \cdots, G^{-1}(\Gamma_n/\Gamma_{n+1})], \qquad n \geq 1,$$

where

$$\Gamma_k = E_1 + \cdots + E_k, \qquad k \geq 1,$$

and

$$G^{-1}(u) = \inf\{y : G(y) \leq u\}, \qquad 0 < u < 1,$$

and $=_d$ denotes equality in distribution. See, for example, Breiman (1968, Section 13.6).

Our approach applies (5) to the sequence $Y_k = |X_k|$, $k \geq 1$, where $X_1, X_2, \cdots$ are i.i.d. with common distribution function F which satisfies (1) and (2). We illustrate it in the special case that F is symmetric and continuous. Then the partial sums $S_n = X_1 + \cdots + X_n$ may be written in the form

$$(6) \qquad S_n = \sum_{j=1}^n \delta_{nj} Y_{nj},$$

where δ_{nj} is the sign of the variable X_i for which $i \leq n$ and $Y_i = Y_{nj}$. It is easily seen that $\delta_{n1}, \cdots, \delta_{nn}$ are i.i.d. random variables taking the values $+1$ and -1 with probability ½ each and that $\delta_{n1}, \cdots, \delta_{nn}$ are independent of $Y_{n1}, \cdots, Y_{nn}$. Thus, $S_n^{\#} = a_n^{-1} S_n$ has the same distribution as

$$\tilde{S}_n = \sum_{j=1}^n \delta_j \cdot a_n^{-1} G^{-1}(\Gamma_j/\Gamma_{n+1}),$$

where $\delta_1, \delta_2, \cdots$ are i.i.d. random variables taking the values $+1$ and -1 with probabilities ½ each and $\delta_1, \delta_2, \cdots$ are independent of $E_1, E_2, \cdots$. By (3),

$$a_n^{-1} G^{-1}(\Gamma_j/\Gamma_{n+1}) \to \Gamma_j^{-1/\alpha}, \qquad \text{w.p.1}, \ j \geq 1,$$

so

$$(7) \qquad S_n^{\#} \Rightarrow S^{\#} = \sum_{j=1}^\infty \delta_j \Gamma_j^{-1/\alpha},$$

is strongly suggested. Here $\Rightarrow$ denotes convergence in distribution.

In Section 2 we modify the above argument slightly to give a rigorous determination of the limiting distribution of $S_n^* = (S_n - nb_n)/a_n$, assuming (1) and (2). The limiting distribution has a representation similar to that in (7), but including centering constants. The representation provides insight into the limiting distribution of S_n^*. That stable distributions must admit such representations may also be deduced from Ferguson and Klass' (1972) general representation theorem for infinitely divisible processes without Gaussian components.

In Section 3 we use (5) and (6) to give an alternative derivation of the limiting distribution of self-norming sums, which were introduced by Darling (1952) and have been studied by Logan, Mallows, Rice, and Shepp (1973) and Darling (1975); and we obtain a representation for the limiting distribution in terms of the random variables $\Gamma_k = E_1 + \cdots + E_k$, $k \geq 1$, and random signs. The representation allows us to explain the apparent infinite discontinuities of limiting densities at ± 1.

In Section 4 we extend our technique to more general spaces. We show that if $\delta_1, \delta_2, \cdots$ are i.i.d. random vectors, of possibly infinite dimension, then any series of the form (7) must have a strictly stable distribution, provided only that it is convergent w.p.1; and we explore the use of (6) and (7) to prove weak convergence of normalized partial sums of symmetric i.i.d. random vectors in a separable Hilbert space.

2. Stable distributions. In this section we let $X, X_1, X_2, \cdots$ denote i.i.d. random variables whose common distribution function F satisfies (1) and (2). Further, we let $Y_k = |X_k|$, $k \geq 1$, and we define the random element Z_n by

$$Z^n = a_n^{-1}(Y_{n1}, \cdots, Y_{nn}, 0, 0, \cdots)$$

where $Y_{n1} \geq \cdots \geq Y_{nn}$ denote the ordered values of $Y_1, \cdots, Y_n$ and $a_n, n \geq 1$, satisfy (3). Thus, Z^n takes values in the subspace Z of R^∞ consisting of all $z = (z_1, z_2, \cdots) \in R^\infty$ for which $z_1 \geq z_2 \geq \cdots \geq 0$. Observe that Z is a topologically complete separable metric space when endowed with the relative topology which it inherits from the product topology on R^∞.

We continue to let $E_1, E_2, \cdots$ denote i.i.d. standard exponential random variables with partial sums $\Gamma_k = E_1 + \cdots + E_k$, $k \geq 1$. Our first lemma then follows directly from (3) and (5), since convergence in distribution in Z is equivalent to the convergence of finite dimensional distributions.

LEMMA 1. *Z^n converges in distribution to $Z = (Z_1, Z_2, \cdots)$, where $Z_k = \Gamma_k^{-1/\alpha}$, $k \geq 1$.*

The analysis of the signs is more complicated. First, there are random permutations $\sigma = (\sigma_{n1}, \cdots, \sigma_{nn})$ of the integers $1, 2, \cdots, n$ for which

$$Y_{nk} = Y_{\sigma_{nk}}, \qquad 1 \leq k \leq n.$$

Let

$$\delta_{nk} = \text{sign}(X_{\sigma_{nk}}), \qquad 1 \leq k \leq n,$$

and

$$\delta^n = (\delta_{n1}, \cdots, \delta_{nn}, 1, 1, \cdots), \qquad n \geq 1.$$

Thus, δ^n is a random element taking values in the space $\{-1, +1\}^\infty$.

LEMMA 2. *δ^n converges in distribution to $\delta = (\delta_1, \delta_2, \cdots)$, where $\delta_1, \delta_2, \cdots$ are i.i.d. random variables taking the values ± 1 with probabilities p and q. Moreover, δ^n is asymptotically independent of Z^n.*

PROOF. Let X^+ and X^- denote the positive and negative parts of X; and let

$$H_n^\pm\{A\} = nP\{a_n^{-1}X^\pm \in A\}, \qquad A \in \mathscr{B}(0, \infty).$$

Next, let $k \geq 1$, let B be a compact subset of $(0, \infty)^k$, and let $C = \{x \in B : x_1 > \cdots > x_k\}$. Then

$$\begin{aligned} &P\{(Z_{n1}, \cdots, Z_{nk}) \in C, Z_{nk} > Z_{nk+1}, \delta_{n1} = \epsilon_1, \cdots, \delta_{nk} = \epsilon_k\} \\ &\qquad = n^{-k}(n)_k \int_C (1 - G)^{n-k}(a_n z_{k^-}) H_n^{\epsilon_k}\{dz_k\} \cdots H_n^{\epsilon_1}\{dz_1\} \end{aligned} \tag{8}$$

for $\epsilon_1, \cdots, \epsilon_k \in \{-1, +1\}^k$, where $(n)_k = n(n-1) \cdots (n-k+1)$ and $1 - G$ denotes the distribution function of $Y = |X|$. As $n \to \infty$, the distributions H_n^+ and H_n^- converge weakly to H^+ and H^-, where

$$H^+\{dz\} = \alpha p z^{-\alpha-1}\,dz \qquad \text{and} \qquad H^-\{dz\} = \alpha q z^{-\alpha-1}\,dz$$

by a simple application of (3). It then follows easily that the integral on the right side of (8) converges to

$$p^s q^{k-s} \int_C \exp\{-z_k^{-\alpha}\} \alpha^k z_k^{-\alpha-1} \cdots z_1^{-\alpha-1}\,dz_k \cdots dz_1 = p^s q^{k-s} p\{(Z_1, \cdots, Z_k) \in C\},$$

where S is the number of positive $\epsilon_1, \cdots, \epsilon_k$. The lemma follows directly.

In our first theorem, $Z_1, Z_2, \cdots$ and $\delta_1, \delta_2, \cdots$ denote independent sequences of random variables with distributions as described in Lemmas 1 and 2: that is, $\delta_1, \delta_2, \cdots$ are i.i.d. random variables taking the values $+1$ and -1 with probabilities p and q; and $Z_k = (E_1 + \cdots + E_k)^{-1/\alpha}$, where $E_1, E_2, \cdots$ are i.i.d. standard exponential random variables. We let a_n, $n \geq 1$, and b_n, $n \geq 1$, be as in (3) and (4). And, if A is a Borel set, we let

$$xA = xI_A(x)$$

where I_A denotes the indicator of the set A.

THEOREM 1. *As $n \to \infty$, the normalized partial sums $S_n^* = a_n^{-1}(S_n - nb_n)$ converge in distribution to*

$$S^* = \sum_{k=1}^{\infty} \{\delta_k Z_k - (p-q)E[Z_k(0, 1]]\}. \tag{9}$$

PROOF. For $0 < \epsilon < \lambda \leq \infty$, we let

$$K_{n,\epsilon} = \sup\{j : Z_{nj} > \epsilon\}$$

and

$$S_n(\epsilon, \lambda) = \sum_{j=1}^{\infty} \delta_{nj} \cdot Z_{nj}(\epsilon, \lambda] = \sum_{K_{n,\lambda}<j\leq K_{n,\epsilon}} \delta_{nj} Z_{nj};$$

and we define K_ϵ and $S(\epsilon, \lambda)$ by the same formulas, with Z_{nj} replaced by Z_j for all $j \geq 1$. Then $S(\epsilon,\lambda)$ is well-defined for all $\epsilon > 0$, since $Z_j \to 0$ w.p.1 as $j \to \infty$. Observe that

$$S_n^* = S_n(0, \infty) - E[S_n(0, 1)] = S_n(0, \epsilon] - E[S_n(0, \epsilon]] + S_n(\epsilon, \infty) - E[S_n(\epsilon, 1)], \qquad \epsilon > 0,$$

and

$$\text{Var}[S_n(0, \epsilon]] = \text{Var}[a_n^{-1} \textstyle\sum_{j=1}^{n} X_j[-\epsilon a_n, \epsilon a_n]\} \leq n a_n^{-2} \displaystyle\int_{-\epsilon a_n}^{\epsilon a_n} x^2\, dF(x),$$

which tends to zero as $n \to \infty$ and then $\epsilon \to 0$. See, for example, Feller (1966, Section 8.9). Thus, it suffices to show: for $\epsilon > 0$,

$$S_n(\epsilon, \infty) - E[S_n(\epsilon, 1)] \Rightarrow S(\epsilon, \infty) - E[S(\epsilon, 1)], \qquad \text{as} \quad n \to \infty; \tag{10}$$

and

$$S(\epsilon, \infty) - E[S(\epsilon, 1)] \Rightarrow S^* \qquad \text{as} \quad \epsilon \to 0. \tag{11}$$

where $\Rightarrow$ denotes convergence in distribution. See Billingsley (1968, page 25).

Relation (10) follows easily from Lemmas 1 and 2. To see this, define a function ϕ on $Zx\{-1, +1]^\infty$ by

$$\phi(z, d) = \sum_{j=1}^{\infty} d_j \cdot z_j(\epsilon, \infty) \tag{12}$$

for $d = (d_1, d_2, \cdots)\epsilon\{-1, +1\}^\infty$ and $z = (z_1, z_2, \cdots)\epsilon Z$ for which $z_j \to 0$ as $j \to \infty$, and $\phi(z, d) = 0$ for other values of z. Then ϕ is continuous, in the product topology of $Zx\{-1, +1\}^\infty$ at a.e. (z, d) with respect to the distribution of Z and δ. Thus,

$$S_n(\epsilon, \infty) = \phi(Z^n, \delta^n) \Rightarrow \phi(Z, \delta) = S(\epsilon, \infty)$$

as $n \to \infty$ for $\epsilon > 0$. A similar argument shows that $S_n(\epsilon, 1) \Rightarrow S(\epsilon, 1)$ as $n \to \infty$ for $\epsilon > 0$, so (10) would follow from the uniform integrability of $S_n(\epsilon, 1)$. This uniform integrability follows from the inequality,

$$|S_n(\epsilon, 1)| \leq \sum_{j=1}^{K_{n,\epsilon}} 1 \leq K_{n,\epsilon},$$

and the fact that $K_{n,\epsilon}$ has the binomial distribution with parameters n and $p_n(\epsilon) = P\{|X| > \epsilon a_n\} \sim 1/n\epsilon^\alpha$.

To establish (11) we first observe that the series defining S^* is convergent w.p.1, since

$$S^* = \sum_{j=1}^{\infty} [\delta_j - (p-q)]Z_j + (p-q) \sum_{j=1}^{\infty} \{Z_j - E[Z_j(0, 1]]\}.$$

In fact, the first of these series converges by the three series theorem, applied conditionally given $Z_1, Z_2, \cdots$, since $Z_j \sim 1/j^{1/\alpha}$ w.p.1 as $j \to \infty$; and a simple Taylor series expansion and the law of the integrated logarithm show that

$$Z_j - E[Z_j(0, 1]] = 0[(1/j^{1/2+1/\alpha})\sqrt{(\log \log j)}]$$

w.p.1 as $j \to \infty$, so the second series converges absolutely w.p.1. The convergence of $S_n(\epsilon, \infty) - E[S_n(\epsilon, 1)]$ to S^* now follows easily, since

$$S^* - \{S(\epsilon, \infty) - E[S(\epsilon, 1)]\} = fs_{j=1}^{\infty} \{\delta_j \cdot Z_j(0, \epsilon] - E[\delta_j \cdot Z_j(0, \epsilon]]\};$$

for the latter series has expectation 0, and its variance is at most

$$\sum_{j=1}^{\infty} E[Z_j(0, \epsilon]^2] \leq \sum_{j=1}^{\infty} \min\{\epsilon^2, E(Z_j^2)\},$$

which tends to zero as $\epsilon \to 0$.

REMARKS 1. As a corollary, we see that any stable distribution is the distribution of a linear function of a random variable of the form (9). This fact may also be deduced, with some effort, from Ferguson and Klass' (1972) representation theorem for infinitely divisible processes.

2. It is known that the limiting distribution of sums of independent, uniformly asymptotically negligible random variables is normal if and only if the largest summand (in absolute value) contributes negligibly to the sum. The proof of Theorem 1 and the representation (9) clarify the manner in which the largest summands affect the limiting distribution in the case of convergence to a stable distribution.

3. The centering constants may be eliminated in some cases. For example, if F is symmetric, then $b_n = 0$ for all n and $p = q$, so $S_n^{\#} = a_n^{-1} S_n$ converges in distribution to $S^{\#} = \delta_1 Z_1 + \delta_2 Z_2 + \cdots$, as in (7). Similarly, if $0 < \alpha < 1$, then the series defining S converges w.p.1, without centering, and $a_n^{-1} n b_n$ converges to a finite limit, so $S_n^{\#}$ converges in distribution to $S^{\#}$ in this case too. Finally, if $1 < \alpha < 2$ and $E(X) = 0$, then $a_n^{-1} n b_n \to 0$, so $S_n^{\#}$ converses in distribution to S^*.

4. The series representation (7) affords a simple derivation of a result of Cressie (1975). If W_α denotes a stable random variable with index α, $0 < \alpha < 1$, so normalized that

$$W_\alpha = c \sum_{j=1}^{\infty} \delta_j \Gamma_j^{-1/\alpha}, \qquad 0 < \alpha < 1,$$

then

$$|W_\alpha|^\alpha \Rightarrow \max_{j \geq 1} \Gamma_j^{-1} = 1/E_1,$$

the reciprocal of a standard exponential random variable, as $\alpha \to 0$. The convergence in distribution is uniform in p, $0 \leq p \leq 1$, and uniform in c for which $c^\alpha \to 1$.

In the next section we require a minor extension of Theorem 1. Let

$$S_{n,r}^{\#} = a_n^{-1} \{\sum_{j=1}^{n} |X_j|^r\}^{1/r}, \qquad 1 \leq r < \infty,$$

$$S_r^{\#} = \{\sum_{j=1}^{\infty} Z_j^r\}^{1/r} \qquad \alpha < r < \infty,$$

$$S_{n,\infty}^{\#} = a_n^{-1} \max\{|X_1|, \cdots, |X_n|\}, \qquad \text{and} \qquad S_\infty^{\#} = Z_1.$$

Thus, $S_{n,r}^{\#} \Rightarrow S_r^{\#}$ as $n \to \infty$ for all r, $\alpha < r \leq \infty$, by Remark 3, applied to $|X|^r$ for $r < \infty$, and Lemma 1 when $r = \infty$.

THEOREM 1′. *As $n \to \infty$, $(Z^n, S_n^*, S_{n,r}^{\#}) \Rightarrow (Z, S^*, S_r^{\#})$ for all r, $\alpha < r \leq \infty$.*

PROOF. The proof of Theorem 1′ is essentially the same as that of Theorem 1. The major change is that the function ϕ of (12) is replaced by a vector valued function. When $\alpha < r < \infty$, ϕ is replaced by (z, ϕ, ψ), where

$$\psi(z, d) = \{\sum_{j=1}^{\infty} z_j(\epsilon, \infty)^r\}^{1/r}$$

if $z_j \to 0$ as $j \to \infty$ and $\psi(z, d) = 0$ otherwise; and when $r = \infty$, ϕ is replaced by (z, ϕ). In either case the vector valued function is continuous a.e. with respect to the distribution of Z and δ.

3. Self norming sums. In this section we study random variables of the form

$$T_{n,r} = (\sum_{j=1}^{n} X_j)/\{\sum_{j=1}^{n} |X_j|^r\}^{1/r}, \qquad r > 0, \quad n \geq 1, \tag{13}$$

where $X_1, X_2, \cdots$ are i.i.d. with common distribution function F satisfying (1) and (2). When $r = \infty$, the denominator is to be interpreted as $\max\{|X_1|, \cdots, |X_n|\}$.

The random variable $T_{n,\infty}$ describes the influence of the maximal term Y_{n1} on the sum S_n. The characteristic function of its asymptotic distribution was found by Darling (1952), under the assumption that X is attracted to a positive stable distribution (that is, $q = 0$ and $0 < \alpha < 1$). More recently, Logan, Mallows, Rice, and Shepp (1973) have found the characteristic function of the asymptotic distribution of $T_{n,r}$, when one exists, for $1 \leq r < \infty$, assuming only (1) and (2); and Darling (1975) has given an alternative proof which is valid when $r = 1$ and $0 < \alpha < 1$. Logan et al. (1973) also inverted the limiting characteristic functions, exactly when $r = 1$ and numerically when $r = 2$, and presented graphs of the limiting densities. The case $r = 2$ is especially interesting, since $T_{n,2}$ has the same limiting distribution, if any, as the t-statistic; so the asymptotic distribution of $T_{n,2}$ provides insight into the properties of the t-statistic when sampling from a population with an infinite variance.

Logan et al. (1973) remark that Darling's (1952) methods fail when $1 \leq r < \infty$, and that their methods fail when $r = \infty$. By contrast Theorem 1′ offers a unified approach and provides a simple representation of the limiting distribution in terms of the random variables $Z_1, Z_2, \cdots$ and $\delta_1, \delta_2, \cdots$.

COROLLARY 1. *If either $0 < \alpha < 1$ or F is symmetric, then $T_{n,r}$ converges in distribution to*

$$T_r = (\sum_{k=1}^{\infty} \delta_k Z_k)/\{\sum_{k=1}^{\infty} Z_k^r\}^{1/r} \tag{14}$$

as $n \to \infty$ for all r, $\alpha < r \leq \infty$. If $1 < \alpha < 2$ and $E(X) = 0$, then $T_{n,r}$ converges in distribution to

$$T_r^* = \{\sum_{k=1}^{\infty} [\delta_k Z_k - (p - q)E[Z_k(0, 1)]]\}/\{\sum_{k=1}^{\infty} Z_k^r\}^{1/r}$$

as $n \to \infty$ for all r, $\alpha < r \leq \infty$.

The Corollary follows directly from Theorem 1′, since $T_{n,r} = S_n^{\#}/S_{n,r}^{\#}$ and division is continuous a.e. with respect to the limiting distribution of $S_n^{\#}$ and $S_{n,r}^{\#}$. See also Remark 3.

The simple representation (14) allows us to explain the apparent infinite discontinuities at ± 1 in the graphs of Logan et al. (1973). We show that they arise from the influence of the maximal term Z_1. In the proof we use the following easily verified facts: the distribution of

$$T_r(y) = \frac{1 + y^{1/\alpha} \sum_{k=1}^{\infty} \delta_k (y + \Gamma_k)^{-1/\alpha}}{\{1 + y^{r/\alpha} \sum_{k=1}^{\infty} (y + \Gamma_k)^{-r/\alpha}\}^{1/r}} \tag{15}$$

is a version of the conditional distribution of T_r, given $E_1 = y$ and $\delta_1 = 1$, when $\alpha < r < \infty$; and

$$R(y) = y^{-1/\alpha}[T_r(y) - 1] \to S^{\#} = \sum_{k=1}^{\infty} \delta_k Z_k \tag{16}$$

w.p.1 as $y \to 0$, when $1 < r < \infty$. When $r = \infty$, the denominator in (15) should be interpreted as 1, and (16) is valid; but if $r = 1 > \alpha$, (16) fails and $R(y) \to S^{\#} - S_1^{\#} \leq 0$ w.p.1 as $y \to 0$.

COROLLARY 2. *Suppose $0 < \alpha < 1$ and $0 < p < 1$; and let $K_r = K_r(\cdot \mid \alpha, p)$ denote the distribution function of T_r. If $1 < r \leq \infty$, then $K_r'(\pm 1) = \infty$; and $K_1'(1-) = \infty = K_1'(-1+)$, where ′ denotes differentiation.*

PROOF. Suppose first that $1 < r \leq \infty$. By conditioning on E_1 and δ_1, we find that

$$P\{1 < T < 1 + h\} \geq \tfrac{1}{2} \int_0^\infty P\{1 < T < 1 + h \mid E_1 = y, \delta_1 = 1\} e^{-y}\, dy$$
$$= \tfrac{1}{2} h^\alpha \int_0^\infty P\{0 < T - 1 < h \mid E_1 = h^\alpha y, \delta_1 = 1\} e^{-h^\alpha y}\, dy$$
$$= \tfrac{1}{2} h^\alpha \int_0^\infty P\{0 < R(h^\alpha y) < y^{-1/\alpha}\} e^{-h^\alpha y}\, dy,$$

so

$$\liminf_{h\to 0} h^{-\alpha}[K(1 + h) - K(1)] \geq \tfrac{1}{2} \int_0^\infty P\{0 < S^\# < y^{-1/\alpha}\}\, dy,$$

which is positive, if $p > 0$. Similarly, one may show that $\liminf_{h\to 0} h^{-\alpha}[K(1) - K(1 - h)]$ is positive, if $q > 0$. This shows that $K'(1) = \infty$, if $0 < p < 1$; and $K'(-1) = \infty$ may be established by a similar argument.

The case $r = 1$ may also be handled similarly.

4. Higher dimensions. Let $\mathscr{X}$ denote a separable Banach space, and let $X, X_1, X_2, \cdots$ denote i.i.d. random elements in $\mathscr{X}$. Then the distribution of X is said to be strictly stable, with index α, $0 < \alpha < 2$, if and only if $X_1 + \cdots + X_n =_d n^{1/\alpha} X$ for all $n \geq 1$.

THEOREM 2. *Let $V_1, V_2, \cdots$ be* i.i.d. *random elements in $\mathscr{X}$; let $E_1, E_2, \cdots$ denote* i.i.d. *standard exponential random variables which are independent of $V_1, V_2, \cdots$; and let $Z_k = \{E_1 + \cdots + E_k\}^{-1/\alpha}$, $k \geq 1$. If $0 < \alpha < 2$, and if the series*

$$S^\# = \textstyle\sum_{k=1}^\infty V_k Z_k \tag{17}$$

converges w.p.1, *then the distribution of $S^\#$ is strictly stable with index α.*

PROOF. We show that if $S_1^\#, \cdots, S_n^\#$ are i.i.d. as $S^\#$, then $S_1^\# + \cdots + S_n^\# =_d n^{1/\alpha} S^\#$. We may suppose that each $S_i^\#$ is of the form (17) with V_k and Z_k replaced by V_{ik} and Z_{ik}, where the sequences $V_{i1}, V_{i2}, \cdots$ and $Z_{i1}, Z_{i2}, \cdots$ are mutually independent with the same distributions as $V_1, V_2, \cdots$ and $Z_1, Z_2, \cdots$ of (17). Then $\Gamma_{ik} = Z_{ik}^{-\alpha}$, $k \geq 1$, $i = 1, \cdots, n$, may be regarded as the arrival times of events in n independent Poisson processes, each with unit arrival rate. Let $\Gamma_1', \Gamma_2', \cdots$ denote Γ_{ik}, $k \geq 1$, $i = 1, \cdots, n$ arranged in increasing order. Then $\Gamma_1', \Gamma_2', \cdots$ are the arrival times of a Poisson process with arrival rate n, so that $Z_k' = (\Gamma_k'/n)^{-1/\alpha}$, $k \geq 1$, have the same distribution as $Z_1, Z_2, \cdots$ of (17). Next, let $V_1', V_2', \cdots$ denote V_{ik}, $k \geq 1$, $i = 1, \cdots, n$ arranged in the same order as Z_{ik}, $k \geq 1$, $i = 1, \cdots, n$. Then $V_1', V_2', \cdots$ are i.i.d. as $V_1, V_2, \cdots$ $V_1', V_2', \cdots$ are independent of $Z_1', Z_2', \cdots$. Next, let $K_\epsilon = \sup\{k : Z_k' \geq n^{-1/\alpha}\epsilon\}$ and $K_{i,\epsilon} = \sup\{k : Z_{ik} \geq \epsilon\}$ for $i = 1, \cdots, n$ and $\epsilon > 0$. Then

$$\textstyle\sum_{j=1}^{K_\epsilon} V_j' Z_j' = n^{-1/\alpha}\{\sum_{j=1}^{K_{1,\epsilon}} V_{1j} Z_{1j} + \cdots + \sum_{j=1}^{K_{n,\epsilon}} V_{nj} Z_{nj}\} \tag{18}$$

for $\epsilon > 0$. As $\epsilon \to 0$, $K_\epsilon \to \infty$ and $K_{i,\epsilon} \to \infty$ w.p.1 for all $i = 1, \cdots, n$, so the right side of (18) converges to $n^{-1/\alpha}(S_1^\# + \cdots + S_n^\#)$; and the left side of (18) converges to a random element having the same distribution as $S^\#$. The theorem follows.

It is also possible to prove limit theorems for sums of i.i.d. random vectors. We illustrate this in the special case that $\mathscr{X}$ is a separable Hilbert space. Let X denote a random element in a separable Hilbert space $\mathscr{X}$; let $Y = \|X\|$, where $\|\cdot\|$ denotes the norm in $\mathscr{X}$; and let $U = X/Y$ denote the unit vector, when $Y > 0$. By analogy with (1) and (2) we require the existence of an α, $0 < \alpha < 2$, and a slowly varying function L for which

$$G(y) = P\{Y > y\} = y^{-\alpha} L(y), \qquad y > 0, \tag{19}$$

and

$$H(y, A) = P\{U \in A \mid Y > y\} \Rightarrow H(A), \qquad \text{as } y \to \infty, \tag{20}$$

where H is a proper probability distribution on the unit sphere of $\mathscr{X}$.

Now let $X_1, X_2, \cdots$ be i.i.d. as X, where X satisfies (19) and (20), and let a_n, $n \geq 1$, be as in (3)—that is, $nG(a_n y) \to y^{-\alpha}$ as $n \to \infty$, for $y > 0$. Let $Y_k = \| X_k \|$, $k \geq 1$; let $Y_{n1} \geq \cdots \geq Y_{nn}$ denote the order statistics; and let

$$Z_{nk} = a_n^{-1} Y_{nk} \quad \text{and} \quad X_k = V_{nk} Y_{nk}, \qquad 1 \leq k \leq n,$$

for $n \geq 1$. Then $Z^n = (Z_{n1}, \cdots, Z_{nn}, 0, 0, \cdots)$ converges in distribution to $Z = (Z_1, Z_2, \cdots)$, as in Lemma 1. Observe that $V^n = (V_{n1}, \cdots, V_{nn}, 0, 0, \cdots)$ is a random element taking values in $\mathscr{X}^\infty$.

LEMMA 3. *As $n \to \infty$, V^n converges in distribution to $V = (V_1, V_2, \cdots)$, where $V_1, V_2, \cdots$ are* i.i.d. *random elements with common distribution H, where H is as in* (20). *Moreover, V^n and Z^n are asymptotically independent.*

PROOF. The proof is similar to that of Lemma 2. Let

$$K_n(y, A) = nP\{U \in A, Y > y\}$$

for $y > 0$, Borel sets A of the unit sphere of $\mathscr{X}$, and $n \geq 1$; and observe that $K_n(y, A) \to y^{-\alpha} H(A)$, whenever A is a continuity set of H. If $k \geq 1$, $A_1, \cdots, A_k$ are continuity sets of H, B is a compact subset of $(0, \infty)^k$, and $C = \{z \in B : z_1 > \cdots > z_k\}$, then

$$\begin{aligned} P\{V_{n1} \in A_1, \cdots, V_{nk} \in A_k, (Z_{n1}, \cdots, Z_{nk}) \in C\} \\ &= n^{-k}(n)_k \int_C (1 - G)^{n-k}(a_n z_k) K_n(dz_k, A_k) \cdots K_n(dz_1, A_1) + o(1) \\ &\to H(A_1) \cdots H(A_k) \int_C \exp(-z_k^{-\alpha}) \alpha^k z_k^{-\alpha-1} \cdots z_1^{-\alpha-1}\, dz_k \cdots dz_1 \\ &= H(A_1) \cdots H(A_k) P\{(Z_1, \cdots, Z_k) \in C\}. \end{aligned}$$

In our final theorem, we let $V_1, V_2, \cdots$ and $Z_1, Z_2, \cdots$ denote independent sequences of random elements with distributions as described in Lemmas 1 and 3. Observe that if either $\alpha < 1$ or V_1 has a symmetric distribution, then

$$S^{\#} = \sum_{k=1}^{\infty} V_k Z_k$$

is norm convergent w.p.1, by the three series theorem applied conditionally given $Z_1, Z_2, \cdots$. See Kahane (1968, page 27).

THEOREM 3. *Let X satisfy* (19) *and* (20) *and let $X_1, X_2, \cdots$ be* i.i.d. *as X. If either X is symmetric or $\alpha < 1$, then the normalized partial sums $S_n^{\#} = a_n^{-1} S_n$ converge in distribution to $S^{\#}$* as $n \to \infty$.

PROOF. Let $k \geq 1$ and write

$$S_n^{\#} = \sum_{j=1}^{k} V_{nj} Z_{nj} + \sum_{j=k+1}^{n} V_{nj} Z_{nj}. \tag{21}$$

Clearly, the first summation in (21) converges in distribution to $V_1 Z_1 + \cdots + V_k Z_k$ as $n \to \infty$ for each fixed k; and $V_1 Z_1 + \cdots + V_k Z_k \to S^{\#}$ w.p.1 as $k \to \infty$. Thus it suffices to show that the second summation in (21) converges to zero as $n \to \infty$ and then $k \to \infty$. When $0 < \alpha < 1$, the norm of the second term is at most $Z_{nk+1} + \cdots + Z_{nn}$, which converges in distribution to $Z_{k+1} + Z_{k+2} + \cdots$ as $n \to \infty$, by Theorem 1′; and $Z_{k+1} + Z_{k+2} + \cdots \to 0$ w.p.1 as $k \to \infty$. When X has a symmetric distribution, the conditional distributions of $V_{n1}, \cdots, V_{nn}$, given $Z_{n1}, \cdots, Z_{nn}$, are symmetric; and, since $V_{n1}, \cdots, V_{nn}$ are conditionally independent, in any case, we have

$$E\{\| \textstyle\sum_{j=k+1}^{n} V_{nj} Z_{nj} \|^2 \mid Z^n\} = \sum_{j=k+1}^{n} Z_{nj}^2. \tag{22}$$

The right side of (22) tends to zero as $n \to \infty$ and then $k \to \infty$, by the argument just given. The theorem then follows from Markov's inequality, applied conditionally.

Kuelbs and Mandrekar (1974) have shown that the conditions (19) and (20) are necessary and sufficient for a random element X to be in the domain of attraction of a stable distribution, when $0 < \alpha < 2$. From this result and the observation that a stable random element is in its own domain of attraction, we easily deduce the following corollary.

COROLLARY 3. *Suppose that X has a stable distribution and that either X is symmetric or $0 < \alpha < 1$. Then there are $a > 0$, $b \in \mathscr{X}$, and* i.i.d. *random elements $V, V_1, V_2, \cdots$ in $\mathscr{X}$ for which $\| V \| = 1$ with probability one and*

$$X = aS^{\#} + b,$$

where $S^{\#}$ is as in (17).

We note that Kuelbs and Mandrekar make extensive use of characteristic functionals, so the proof of the final corollary is not as elementary as that of the theorem.

REFERENCES

BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

BREIMAN, L. (1968). *Probability.* Addison-Wesley, Reading, Massachusetts.

CRESSIE, N. (1975). A note on the behavior of the stable distribution for small index. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **33** 65–88.

DARLING, D. (1952). The influence of the maximal term in the addition of independent random variables. *Trans. Amer. Math. Soc.* **83** 95–107.

DARLING, D. (1975). A note on the limit theorem. *Ann. Probability* **3** 876–888.

FELLER, W. (1966). *An Introduction to Probability Theory and its Applications, vol.* 2. Wiley, New York.

FERGUSON, T. and KLASS, M. (1972). A representation of independent increments processes without Gaussian components. *Ann. Math. Statist.* **43** 1634–1643.

KUELBS, J. and MANDREKAR, V. (1974). Domains of attraction of stable measures on a Hilbert Space. *Studia Math.* **50** 149–162.

LOGAN, B., MALLOWS, C., RICE, S. and SHEPP, L. (1973). Limit distributions of self-normalized sums. *Ann. Probability* **1** 788–809.

SIMONS, G. and STOUT, W. (1978). A weak invariance principle with applications to domains of attraction. *Ann. Probability* **6** 294–315.

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