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APPLICATIONS OF RAW TIME-CHANGES TO MARKOV PROCESSES

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The technique of raw time-change is applied to give another proof that the Knight-Pittenger procedure of deleting excursions of a strong Markov process from a set A which meet a disjoint set B yields a strong Markov process. A natural filtration is associated with the new process, and generalizations are given. Under natural hypotheses, the débuts of a class of nonadapted homogeneous sets are shown to be killing times of a strong Markov process. These are generalized (i.e. raw) terminal times. Let A_t be an increasing nonadapted continuous process, and let T_t be its right continuous inverse satisfying a hypothesis which ensures that the collection of σ-fields $\mathscr{F}_{T(t)}$ is increasing. The optional times of $\mathscr{F}_{T(t)}$ are characterized in terms of killing operators and the points of increase of A, and it is shown that $\mathscr{F}_{T(t)} = \mathscr{F}_{T(t+)}$.

0. Introduction. Let X be a right continuous strong Markov process defined on the canonical space of right continuous paths taking values in a Lusin topological space E equipped with its Borel field $\mathscr{E}$, and let A_t be a raw (i.e. nonadapted) continuous additive functional of X having the property

$$A_u \circ k_{T(t)} = A_u \quad \text{for all} \quad u \quad \text{in} \quad [0, T_t], \tag{0.1}$$

where T_t denotes the right continuous inverse of A_t, and where k_t is the killing operator associated to X_t. We gave general sufficient conditions in [5] for the raw time-changed process $X_{T(t)}$ to be a strong Markov process. This paper is devoted to variations on that theme: we present several applications of raw time-changes which are of independent interest.

Let A and B be two Borel sets in E with disjoint closures. Perform the following surgical operation: simply excise the excursions from A which meet B. This procedure (cleanly done by a change of time) was used by Knight [6] in investigating certain local times, and Knight and Pittenger [7] proved that the postoperative process is again a strong Markov process. This procedure is one of the most interesting nonadapted transformations of a Markov process produced so far, and we present herein another approach to proving that the excised process is strong Markov based on the method of raw time-change. The proof we present is perhaps a bit more technical than the one given by Knight and Pittenger, but we feel that our approach brings the essential ingredients of the transformation into clear view. The proof of this statement is, perhaps, that interesting generalizations of the procedure become apparent (Section 2). In Section 1, we present the Knight and Pittenger transformation. We first use a time-change by an increasing collection of *optional* times to eliminate pieces of the excursions traveling from B to A. We follow this with a raw time-change to remove the other half of the excursions traveling from A to B. A natural time-changed filtration is associated with the process during this procedure.

In Section 3, we characterize in terms of killing operators the optional times of a filtration $(\mathscr{F}_{T(t)+})$. An important corollary of this characterization settles a point left open in [5]: the filtration $(\mathscr{F}_{T(t)})$ is right continuous when constructed as in [5].

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In Section 4, we present a new class of killing times. Recall that if N is an optional homogeneous set, then the debut D of N is a *terminal time*. That is, we may kill the process X at D and the result is again a strong Markov process. If N is homogeneous, but no longer adapted, this is not always true. However, if the raw continuous additive functional supported by N^c has property (0.1), then the following is true. Let $R = \inf\{t > 0 : t \in N\}$ and let $D = \inf\{t \geq 0 : R \circ \theta_t = 0\}$. Then X killed at D is a strong Markov process. This result seems to be connected via heuristic time-reversal arguments to a recent result of Pittenger [9] characterizing the regular birth times of a Markov process. However, it seems difficult to make the connection precise, and we therefore simply indicate it to the reader here. One additional point of interest is that the optional projection of the process $1_{\{t<D\}}$ is an adapted multiplicative functional.

The remainder of this section is devoted to laying out the notation and hypotheses of the paper. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be the canonical realization of a normal right continuous strong Markov process with a Borel semigroup on a Lusin topological space E together with its Borel field $\mathscr{E}$ on the space Ω of right continuous paths in E with lifetime ζ. We assume $\mathscr{F}$ and $\mathscr{F}_t$ are the usual completions of the fields $\mathscr{F}^o$ and $\mathscr{F}_t^o$ generated by the coordinate maps [3]. Let $\mathscr{F}^e = \sigma\{f(X_s) : f \text{ is 1-excessive}, s \geq 0\}$.

We make the conventions that $X_\infty = \Delta$ and that $F \circ \theta_\infty = 0$ for all $F \in \mathscr{F}^e$. When we refer to an optional process or to an optional time without specifying the filtration, we mean relative to the filtration $(\mathscr{F}_t)$.

We define a raw continuous additive functional $(A_t)_{t \geq 0}$ to be an increasing process in $\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^e$ satisfying $A_0 = 0$; $A_{t+s} = A_t + A_s \circ \theta_t$; $A_t = A_\zeta$ for all $t > \zeta$; $t \to A_t$ is continuous. Usually, one requires A_t to be only $\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}$-measurable, but we show in [5] that one can choose a version in $\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^e$ (subject to a mild integrability hypothesis which is satisfied in all examples discussed herein).

Recall the canonical killing operators $k_t : \Omega \to \Omega$ defined by

$$X_s(k_t \omega) = X_s(\omega) \quad \text{if} \quad s < t$$
$$= \Delta \quad \text{if} \quad s \geq t.$$

Finally, we introduce a bit of nonstandard notation. If $\Gamma \subset R^+$, define

$$\sup{}^1 \Gamma = \sup \Gamma \quad \text{if} \quad \Gamma \neq \varnothing$$
$$= -1 \quad \text{if} \quad \Gamma = \varnothing$$

and

$$\sup{}^2 \Gamma = \sup \Gamma \quad \text{if} \quad \Gamma \neq \varnothing$$
$$= -2 \quad \text{if} \quad \Gamma = \varnothing.$$

1. The Surgery of Knight and Pittenger. Let A and B be two sets in $\mathscr{E}$ such that $\bar{A} \cap \bar{B} = \varnothing$. Assume $\Delta \in A$. We first construct a time-change by an increasing collection of *optional times* which deletes portions of the paths traveling from the set B to the set A (plus portions of the paths spending time in B).

Set $\Gamma_A = \{(t, \omega) : X_t(\omega) \in A\}$ and $\Gamma_B = \{(t, \omega) : X_t(\omega) \in B\}$. The two processes C_t' and D_t' defined by $C_t'(\omega) = \sup^1(\Gamma_A(\omega) \cap [0, t])$ and $D_t'(\omega) = \sup^2(\Gamma_B(\omega) \cap [0, t])$ are adapted increasing processes, whence the right continuous processes $C_t = C_{t+}'$ and $D_t = D_{t+}'$ are optional processes. Let H_s be the indicator of the set $\{C_s \geq D_s\}$, and let

$$A_t = \int_0^t H_s \, ds.$$

Then A_t is an adapted, increasing and continuous process. Notice that the process H_s is right continuous. For if not, either H_s oscillates as s decreases to t, or $\lim_{s \downarrow\downarrow t} H_s \neq H_t$. In the first case, we may find two positive sequences (r_n) and (q_n) decreasing to 0 so that $X_{s+r(n)} \in A$ and $X_{s+q(n)} \in B$. By right continuity, $X_s \in \bar{A} \cap \bar{B} = \varnothing$, so this case is ruled out.

If $\lim_{s\downarrow\downarrow t} H_s = \lim_{s\downarrow\downarrow t} 1_{\{C_s \geq D_s\}} = 1$, then $H_t = 1$ since C_s and D_s are right continuous. If $\lim_{s\downarrow\downarrow t} H_s = 0$ and $H_t = 1$, we conclude (again by right continuity of C_s and D_s) that $C_t = D_t$, which again implies that $X_s \in \bar{A} \cap \bar{B} = \varnothing$. Thus we conclude that H_s is right continuous.

LEMMA 1.1. *A_t is additive on the set consisting of its points of right increase.*

PROOF. Let $\Lambda = \{(t, \omega) : A_t(\omega) < A_{t+s}(\omega)$ for all $s > 0\}$, and let T be an optional time with $[T] \subset \Lambda$. Set $R(s, e) = \{u : T \leq u \leq s + T + e, X_u \in A\}$ and $S(s, e) = \{u : T \leq u \leq s + T + e, X_u \in B\}$. Then $H_s \circ \theta_t$ is the indicator of the set {there exists $n > 0$ so that $\sup^1 R(s, e) \geq \sup^2 S(s, e)$ for all $e < n^{-1}$}.

By the right continuity of H_s, $H_T = 1$. We need to show $H_s \circ \theta_T = H_{s+T}$. If there is a sequence (e_n) decreasing to 0 so that $R(s, e_n) = \varnothing$, then $\sup^1 R(s, e_n) = -1$. If in addition, $H_s \circ \theta_T = 1$, the right continuity of H_s forces $\sup^2 S(s, e_n) = -2$ for sufficiently large n. In other words, $S(s, e_n) = \varnothing$ for sufficiently large n. Therefore, $H_{s+T} = H_T$ and since $[T] \subset \Lambda$, $H_T = 1$. So $H_s \circ \theta_T = H_{s+T}$ in this case. On the other hand, if $H_s \circ \theta_T = 0$, then $S(s, e_n) \neq \varnothing$ for all sufficiently large n, and we get $H_{s+T} = 0$ from the definition of H_s.

The other case to consider is when there is a sequence (e_n) decreasing to 0 so that $R(s, e_n) \neq \varnothing$. If in addition, $H_s \circ \theta_T = 1$, then $H_{s+T} = H_s \circ \theta_T = 1$. If $H_s \circ \theta_T = 0$, then there must be a sequence (e_n) decreasing to 0 so that $\sup^1 R(s, e_n) < \sup^2 S(s, e_n)$. Therefore, $H_{s+T} = 0$. Therefore, we have shown that $A_{t+T} = A_T + A_t \circ \theta_T$. □

Define a strictly increasing right continuous collection of optional times by setting $T_t = T(t) = \inf\{s > 0 : A_s > t\}$. The standard proof (Lemma (1.1) in [5]) shows that $T_{t+s} = T_t + T_s \circ \theta_{T(t)}$ on $\{T_t < \infty\}$ by making use of Lemma (1.1). It is easy to conclude the following by a simple modification of the standard time-change theorem ([3], page 212).

THEOREM. *Let A_t be an adapted continuous increasing process which is additive on the set consisting of its points of right increase. Let T_t denote the right continuous inverse of A_t. Then $X_T = (\Omega, \mathscr{F}, \mathscr{F}_{T(t)}, X_{T(t)}, \theta_{T(t)}, P^x)$ is a right continuous strong Markov process.*

In an attempt to avoid notational intricacies, we rename this process $Y = (\Omega, \mathscr{F}, \mathscr{G}_t, Y_t, \tilde{\Theta}_t, P^x)$. Notice that the appropriate killing operator for Y_t is $j_t = k_{T(t)}$:

$$Y_t(j_{s\omega}) = Y_t(\omega) \quad \text{if} \quad t < s$$
$$= \Delta \quad \text{if} \quad t \geq s.$$

LEMMA 1.2. $P^x(Y_t \in B$ *for some* $t \geq 0) = 0$.

PROOF. Suppose $Y_t(\omega) = X_{T(t)}(\omega) \in B$. Since T_t is a point of right increase of A_t, $\sup^1(\Gamma_A \cap [0, T_t + e]) > \sup^2(\Gamma_B \cap [0, T_t + e])$ for all sufficiently small positive e. By the right continuity of $X_{T(s)}$, this would imply that $X_{T(t)} \in \bar{A} \cap \bar{B} = \varnothing$. □

What we have left to do, roughly speaking, is to remove portions of the path traveling from A to B. Let $T_A = \inf\{t > 0 : X_t \in A\}$, and let $T_B = \inf\{t > 0 : X_t \in B\}$. Let J be the indicator of the set $\{T_A < T_B\}$, and let $J_s = J(s) = J \circ \tilde{\Theta}_s$. Define a raw continuous additive functional of the process Y by setting

$$B_t = \int_0^t J(s)\, ds.$$

Note that B_t is a raw additive functional of $(Y_t, \mathscr{G}_t, \tilde{\theta}_t)$ (*not* $(Y_t, \sigma(Y_s : s \leq t), \theta_t)$). If we let $S_t = S(t) = \inf\{u : B_u > t\}$, then $S_{t+s} = S_t + S_s \circ \tilde{\Theta}_{S(t)}$. The reader may wish to convince himself of the truth of the following result with a picture before confronting the proof.

LEMMA 1.3. $B_u \circ j_{S(t)} = B_u$ *for all* u *in* $[0, S_t]$ *for each* $t \geq 0$.

PROOF. Let $V = \sup\{s \le T(S_t) : X_s \in B\}$. Suppose $T_v \in [0, V)$. We need to show $J_v \circ j_{S(t)} = J_v$. If $J_v = 1$, $T_A \circ \tilde{\Theta}_v < T_B \circ \tilde{\Theta}_v$. Since $T_v + T_B \circ \tilde{\Theta}_v \le T(S_t)$, $T_v + T_A \circ \tilde{\Theta}_v < T(S_t)$, so $T_A \circ \tilde{\Theta}_v \circ j_{S(t)} = T_A \circ \tilde{\Theta}_v$ and $T_B \circ \tilde{\Theta}_v \circ j_{S(t)} \ge T_B \circ \tilde{\Theta}_v$. Thus $J_v \circ j_{S(t)} = 1$.

On the other hand, if $J_v = 0$, we have $T_A \circ \tilde{\Theta}_v > T_B \circ \tilde{\Theta}_v$. Therefore (since $\Delta \in A$), $T_A \circ \tilde{\Theta}_v \circ j_{S(t)} = T_A \circ \tilde{\Theta}_v \wedge (T(S_t) - T_v)$. Notice that $T_v + T_B \circ \tilde{\Theta}_v < T(S_t)$. For if $T_v + T_B \circ \tilde{\Theta}_v = T(S_t)$ either $Y_{S(t)} \in B$ (which is impossible by Lemma (1.2)) or $T_B \circ \tilde{\Theta}_{S(t)} = 0 \le T_A \circ \tilde{\Theta}_{S(t)}$ (which cannot happen by definition of S_t and the fact that J_s is right continuous). Therefore, $T_B \circ \tilde{\Theta}_v \circ j_{S(t)} = T_B \circ \tilde{\Theta}_v < T_A \circ \tilde{\Theta}_v \wedge (T(S_t) - T_v)$, whence $J_v \circ j_{S(t)} = J_v$ whenever $T_v \in [0, V)$.

Now suppose $T_v \in (V, T(S_t))$. Since S_t is a point of right increase of B_u, $T_A \circ \tilde{\Theta}_{S(t)+e} < T_B \circ \tilde{\Theta}_{S(t)+e}$ for all sufficiently small positive e. Since $X_t \notin B$ for all $t \in (V, T(S_t)]$, it follows that $T_A \circ \tilde{\Theta}_v < T_B \circ \tilde{\Theta}_v$: i.e., $J_v = 1$. But $T_A \circ \tilde{\Theta}_v \circ j_{S(t)} \le T(S_t)$ (since $\Delta \in A$) and $T_B \circ \tilde{\Theta}_v \circ j_{S(t)} = \infty$, so $J_v \circ j_{S(t)} = 1$ whenever $T_v \in (V, T(S_t))$. □

Define a collection of σ-algebras as follows. A random variable $F \in \mathscr{F}$ is in $\mathscr{H}_t$ if and only if there is a $(\mathscr{G}_t)$-optional process Z so that $F = Z_{S(t)}$ on $\{S_t < \infty\}$. The condition in the statement of Lemma (1.3) is exactly the condition needed to guarantee that the collection $(\mathscr{H}_t)_{t\ge 0}$ is a filtration: see Proposition (1.3) in [5]. (Note: In [5], we assumed that $(\mathscr{G}_t)$ is the canonical filtration of a Markov process. The proof requires only minor changes to extend to this situation since $(\mathscr{G}_t)$ is an adapted time-change of the canonical filtration).

We now show that the process $(Y_{S(t)}, \mathscr{H}_{t+})$ is a strong Markov process. Since B_t has a homogeneous density, we may simplify the discussion in [5] a bit as follows. Recall J is the indicator function of $\{T_A < T_B\}$. for $F \in \mathscr{F}^{e+}$, define a kernel as follows.

$$K(x, F) = E^x[FJ]/P^x(J) \qquad \text{if } P^x(J) > 0.$$
$$= 0 \qquad \text{if } P^x(J) = 0.$$

Let Z_t be a positive $(\mathscr{G}_t)$-optional process. Then

$$(1.4) \qquad E^x \int Z_{S(t)} F \circ \tilde{\Theta}_{S(t)}\, dt = E^x \int Z_t\, F \circ \tilde{\Theta}_t J(t)\, dt = E^x \int Z_t K(Y_t, F) J(t)\, dt$$

$$(1.5) \qquad = E^x \int Z_{S(t)} K(Y_{S(t)}, F)\, dt.$$

Let F be of the form $f(Y_{S(r)})$ with f a bounded continuous function on E. Take Z to be the product of two $(\mathscr{G}_t)$-optional processes V and W so that $V_{S(t)} 1_{\{S(t)<\infty\}} = e^{-\alpha t} 1_{\{S(t)<\infty\}}$ (see Lemma (1.4) in [5] for the existence of such a process) and so that W is bounded, positive and right continuous. Substituting in (1.4) and (1.5) and applying Fubini's theorem we find

$$(1.6) \qquad \int e^{-\alpha t} E^x[W_{S(t)} f(Y_{S(r)}) \circ \tilde{\Theta}_{S(t)}]\, dt = \int e^{-\alpha t} E^x[W_{S(t)} K(Y_{S(t)}, f(Y_{S(r)}))]\, dt.$$

LEMMA 1.7 *$t \to K(Y_{S(t)}, f(Y_{S(r)}))$ is* a.s. *right continuous whenever f is a bounded continuous function.*

PROOF. Let $F = f(Y_{S(r)})$. It is simple to check that $E^x E^{Y(R(n))}[FJ]$ converges to $E^x E^{Y(R)}[FJ]$ whenever $(R(n))$ is a sequence of $(\mathscr{G}_t)$-optional times decreasing to R, and therefore $t \to E^{Y(t)}[FJ]$ is a.s. right continuous. It is just as easy to check that $t \to P^{Y(t)}(J)$ is right continuous. Let $N = \{x : P^x(J) = 0\}$. It follows that $K(Y_{S(t)}, F)$ is right continuous if $P^x(Y_{S(t)} \in N$ for some $t \ge 0) = 0$. Let $M = \{s : Y_s \in N\}$, and let Z_s be the optional process $s \cdot 1_M(s)$. Any optional process is optionally separable. That is, we may find a sequence (R_n) of $(\mathscr{G}_t)$-optional times so that the graph of the trajectory $t \to Z_t$ is contained in the closure of the graph $n \to Z_{R(n)}$ (see [2]). Moreover, $[R_n] \subset M$ for each n. Therefore, $0 = E^x[P^{Y(R(n))}(J)] = P^x[T_A \circ \tilde{\Theta}_{R(n)} < T_B \circ \tilde{\Theta}_{R_n}]$. Therefore $J_{R(n)} = 0$ for all n a.s. (P^x). We may eliminate the exceptional set. So if $t \in M$ is a limit point from the right of M, then $J_t = 0$

by the right continuity of J_s and the fact that $\cup_n [R_n]$ must be dense in M. If t is an isolated point of M, then there is an n so that $R_n = t$, and again $J_t = 0$. The last case to consider is when $t \in M$ is only a limit point from the left of points in M. That is, $t \in M$ is a left endpoint of an interval of M^c.

So let $\bar{M} = \cup_{e>0} \{t \in M : t + s \in M^c \ \forall\ s \in (0, e)\}$. We show $\bar{M}$ is empty. Since $\bar{M}$ is $(\mathscr{G}_t)$-progressive, its debut D is a $(\mathscr{G}_t)$-optional time and $D \in M$. Thus $0 = E^x P^{Y(D)}(J)$ implies $T_A \circ \tilde{\Theta}_D > T_B \circ \tilde{\Theta}_D$ a.s. But then $T_A \circ \tilde{\Theta}_t > T_B \circ \tilde{\Theta}_t$ for all t in $[D, V)$, where $V = \inf\{t > D : Y_t \in A\} > D$. It follows that $P^{Y(t)}(J) = 0$ for all $t \in [D, V)$, whence we conclude that $\bar{M}$ is, in fact, empty. □

Therefore both integrands in (1.6) are right continuous, and we conclude that $E^x[W_{S(t)} f(Y_{S(r)}) \circ \tilde{\Theta}_{S(t)}] = E^x[W_{S(t)} K(Y_{S(t)}, f(Y_{S(r)}))]$ for every bounded continuous function f and for every bounded right continuous $(\mathscr{G}_t)$-optional process W. It follows that

$$E^x[f(Y_{S(r)}) \circ \tilde{\Theta}_{S(t)} \mid \mathscr{H}_t] = K(Y_{S(t)}, f(Y_{S(r)})) \text{ a.s.} \tag{1.8}$$

Since $t \to K(Y_{S(t)}, f(Y_{S(r)}))$ is right continuous, an application of the standard argument ([5], Theorem (1.5); [3], page 42) shows that $Y_{S(t)}$ is strong Markov with respect to the filtration $(\mathscr{H}_{t+})$. (See Corollary (3.4)).

Finally, the reader may wish to check that the net effect of the adapted time-change followed by the raw time-change has been to delete the excursions of X from A which meet B.

2. More in the same vein. The purpose of this section is to abstract the essential points of the procedure in Section 1 in order to present several examples in a general framework. We first discuss the source of the adapted process A_t in Section 1.

The following discussion is intended to be motivation (until further notice): we assume $\zeta < \infty$ (so that we may reverse time easily), and we ignore all questions of measurability (although all operations can be justified: see the appendix of [5]).

One of the interesting facets of the Knight-Pittenger procedure is its symmetry in time. That is, if one removes the excursions from A which intersect B, it makes no difference if time is running forward or if it is running backwards during the procedure: the end result is the same. In fact, instead of using the two time-changes in Section 1, one might wish to try the following procedure, which makes the symmetry apparent.

STEP 1. Reverse X to get a process Y, and remove from Y (via a raw time-change) pieces of path traveling from A to B to get a new process $\bar{Y}$. Reverse $\bar{Y}$ to get a process $\bar{X}$.

STEP 2. With the same raw time-change, remove pieces of path of $\bar{X}$ traveling from A to B to get the desired process Z.

This approach is fraught with technical difficulties, due to all of the reversing (i.e. it is difficult to verify that the end result is a strong Markov process). Fortunately, Step 1 can be accomplished (and was accomplished in Section 1) by leaving out all reversals and using an adapted time-change of the process X, which we now construct. Let C_t be the raw additive functional of Y whose left continuous inverse S_t has the desired effect in Step 1. That is, starting with X_t, we reverse to get Y_t, then we time-change to get $Y_{S(t)} = \bar{Y}_t$; and we finally obtain $\bar{X}_t$ upon reversing one last time. We may obtain $\bar{X}_t$ in another manner as follows. Let $A_t = C_\zeta - C_{(\zeta-t)+}$, and let T_t denote the right continuous inverse of A_t. Then $\bar{X}_t$ and $X_{T(t)}$ are identical. Now what are the relationships between A_t and X_t? Azéma's work [1] on the duality of shift operators and killing operators indicates the following properties of A_t.

If X has shift operator θ_t and killing operator k_t, then Y has shift operator $\tilde{\theta}_t = k_{(\zeta-t)+}$ and killing operator $\tilde{k}_t = \theta_{(\zeta-t)+}$, respectively. So $A_t \circ k_t = A_t \circ \tilde{\theta}_{(\zeta-t)+} = (C_\zeta - C_{(\zeta-t)+}) \circ \tilde{\theta}_{(\zeta-t)+} = A_t$ since C_t is a raw additive functional of Y. This shows that A_t is adapted to the filtration of X. Recall that C_t satisfies $C_u \circ \tilde{k}_{S(t)} = C_u$ on $[0, S_t]$. Let T be an $\mathscr{F}^e$-measurable random variable contained in the points of right increase of A_t. Then for each t,

$$A_t \circ \theta_T = (C_\zeta - C_{(\zeta-t)+}) \circ \tilde{k}_{\zeta-\zeta\circ\theta_T} = C_{(\zeta-\zeta\circ\theta_T)+} - C_{(\zeta-T-t)+} = A_{t+T} - A_T.$$

Thus, A_t is additive on its points of right increase. As mentioned in Section 1, these two properties (adaptedness and additivity on points of right increase) are sufficient to guarantee that the process $X_T = (\Omega, \mathscr{F}, \mathscr{F}_{T(t)}, X_{T(t)}, \theta_{T(t)}, P^x)$ is a strong Markov process. One may also check that if one starts with C_t as described above and forms A_t, then the process A_t obtained here is *exactly* the process we called A_t in Section 1. Therefore, using Steps 1 and 2 makes apparent the symmetry in time, but to actually obtain $X_{T(t)}$, it is much easier to use A_t.

The discussion above is general in the following sense. Let X and Y be as above, and let C_t be a raw continuous additive functional of Y with left continuous inverse S_t satisfying $C_u \circ \tilde{k}_{S(t)} = C_u$ on $[0, S_t]$. Set $A_t = C_\zeta - C_{(\zeta-t)+}$. Then A_t is adapted to the filtration of X and is homogeneous on its points of increase. This observation has been used implicitly below to give other examples of transformations which are symmetric with respect to time-reversal.

EXAMPLE 1. Let A be a set in $\mathscr{E}$. We produce a transformation by deleting parts of the path of X. Roughly speaking, $X_t(\omega)$ is deleted if $X_s(\omega)$ visits A for some s in $[t-1, t+1)$. Let $\Gamma_A = \{(t, \omega): X_t(\omega) \in A\}$, and set $D_t(\omega) = \sup^1(\Gamma_A(\omega) \cap [0, t])$. Let H_t be the indicator of the set $\{t - D_{t+} \geq 1\}$, and set $A_t = \int_0^t H_s\, ds$. Time-change X_t by the inverse of A_t, T_t, to get a process Y with shift operator $\tilde{\theta}_t = \theta_{T(t)}$. Define a raw continuous additive functional B_t for Y as follows: let J be the indicator of $\{T_A \geq 1\}$, and set $B_t = \int_0^t J \circ \tilde{\theta}_s\, ds$. Time-change Y by the inverse of B_t, and the result is a strong Markov process.

EXAMPLE 2. This example generalizes Section 1 a bit. We shall not attempt to describe in prose the effect of the transformation; the reader is encouraged to indulge in drawing a few pictures to see the effect. Let A, B, and C be sets in $\mathscr{E}$ with mutually disjoint closures. Define an adapted process A_t, homogeneous on its points of right increase, as follows: Let

$$\Gamma_A(\omega) = \{(t, \omega): X_t(\omega) \in A\}, \quad \Gamma_B = \{(t, \omega): X_t(\omega) \in B\}, \quad \text{and} \quad \Gamma_C = \{(t, \omega): X_t(\omega) \in C\},$$

and define F'_t, C'_t and D'_t by setting

$$F'_t = \sup{}^1(\Gamma_A \cap [0, t]), \quad C'_t = \sup{}^2(\Gamma_B \cap [0, t]), \quad \text{and} \quad D'_t = \sup{}^2(\Gamma_C \cap [0, t]).$$

Set

$$F_t = F'_{t+}, \quad C_t = C'_{t+}, \quad \text{and} \quad D_t = D'_{t+},$$

and let H_s be the indicator of the set $\{F_s \geq C_s\} \cup \{F_s \geq D_s\}$. Then $A_t = \int_0^t H_s\, ds$ is the desired process. Time-change X_t by the inverse of A_t to get a process Y_t. Define a raw continuous additive functional of Y as follows: let J be the indicator of $\{T_A < T_B\} \cup \{T_A < T_C\}$, and define $B_t = \int_0^t J \circ \tilde{\theta}_s\, ds$. Time-change Y by the inverse of B_t to get a strong Markov process.

In each example above, it is easy to check that A_t is adapted and additive on its points of increase. If we let T_t denote the right continuous inverse of A_t, then $Y_t = X_{T(t)}$ is a strong Markov process with shift operator $\tilde{\theta}_t = \theta_{T(t)}$ and killing operator $\tilde{k}_t = k_{T(t)}$. One should then check that if S_t denotes the right continuous inverse of B_t, then $B_u \circ \tilde{k}_{S(t)} = B_u$ on $[0, S_t]$ for each $t \geq 0$. One may then follow the argument in Section 1 to observe that in each example the strong Markov property is obtained. We consider one final example which does *not* yield a strong Markov process in general. Let A, B, and C be sets with disjoint closures, and suppose we try to remove all excursions from A which hit B before hitting C. Consider the process Y in Step 1. What is the additive functional to use to delete portions of the path of Y traveling from A to B without hitting C? A moment's thought will convince the reader that there is no raw additive functional which will do that. Additive functionals are adapted to the future, while here you need to know something about the past. For example, let $[S, T]$ be the time-interval of an excursion from A which first hits B at time R. One cannot decide whether or not to keep each time $t \in [S, R]$ by looking at events after t. For if $Y_u \notin C$ for all u in $[S, R]$, you want to keep t, and if $Y_u \in C$ for some $u \in [S, T)$, you want to delete t. Since we cannot find an additive functional, the preceding discussion does not apply.

While the symmetry in time of the transformations of X heretofore discussed is attractive, it is not at all necessary. We now outline a general strategy for combining adapted and nonadapted time-changes. Let A_t be an increasing adapted continuous process which is additive on its points of increase, and let T_t be its inverse. Then $\tilde{X}_t = X_{T(t)}$ is a strong Markov process with shift operator $\tilde{\theta}_t = \theta_{T(t)}$ and killing operator $\tilde{k} = k_{T(t)}$. Let B_t be a continuous raw additive functional of $\tilde{X}_t$. That is, B_t is a continuous increasing process satisfying $B_{t+s} = B_t + B_s \circ \tilde{\theta}_t$. Let S_t denote the right continuous inverse of B_t, and assume that $B_u \circ \tilde{k}_{S(t)} = B_u$ on $[0, S_t]$ for each $t \geq 0$. An application of Motoo's theorem and a kernel argument (see Section 1 of [5]) shows that

$$E^x \int Z_{S(t)} F \circ \tilde{\theta}_{S(t)}\, dt = E^x \int Z_{S(t)} K(\tilde{X}_{S(t)}, F)\, dt,$$

where $K(x, d\omega)$ is a kernel from Ω to E, Z is $(\mathscr{F}_{T(t)})$-optional, and $F \in \mathscr{F}^{e+}$. If $t \to K(\tilde{X}_{S(t)}, f(X_{S(u)}))$ is right continuous a.s. for every u and for every bounded continuous function f, then $\tilde{X}_{S(t)}$ is a strong Markov process with associated filtration $\tilde{\mathscr{F}}_{S(t)}$ obtained in the same manner as the filtration $\mathscr{H}_{t+}$ in Section 1. If B_t has a density of the form $J \circ \tilde{\theta}_t$, the kernel is given by

$$\begin{aligned} K(x, F) &= E^x[FJ]/P^x(J) \quad && \text{if} \quad P^x(J) > 0 \\ &= 0 && \text{if} \quad P^x(J) = 0. \end{aligned}$$

3. A Characterization of Splitting Times. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be either the canonical realization of a right continuous strong Markov process as described in the introduction, or an adapted time-change of such a canonically defined process. In both cases, X has an associated killing operator k_t satisfying

$$\begin{aligned} X_s(k_t\omega) &= X_s(\omega) \quad && \text{if} \quad t > s \\ &= \Delta && \text{if} \quad t \leq s. \end{aligned}$$

Let $B_t \in \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^e$ be an increasing continuous process satisfying $B_o = 0$, $B_t = B_\zeta$ for all $t > \zeta$, and $B_t \leq t$. Let $S_t = \inf\{u : B_u > t\}$, and assume

(3.1) $$B_u \circ k_{S(t)} = B_u \quad \text{for all} \quad u \in [0, S_t] \quad \text{for each} \quad t.$$

Define a collection of σ-algebras $(\mathscr{H}_t)$ as follows. A random variable $F \in \mathscr{F}$ is in $\mathscr{H}_t$ if there is an $(\mathscr{F}_t)$-optional process Z_t so that $F = Z_{S(t)}$ on $\{S_t < \infty\}$. The proof of Proposition (1.3) in [5] applies here with no change to give the result:

PROPOSITION 3.2. *$(\mathscr{H}_t)_{t \geq 0}$ is a filtration.*

Let $\mathscr{I}_t = \mathscr{H}_{t+}$, and let $\tilde{\zeta} = \sup\{t : X_{S(t)} \in E\}$. We now characterize the collection of times

$$\Lambda = \{S_T : T \in \mathscr{F}^e \text{ is an } (\mathscr{I}_t)\text{-optional time with } [T] \subset [0, \tilde{\zeta})\}.$$

Notice that if B_t is a raw additive functional so that $(X_{S(t)}, \mathscr{I}_t)$ is a strong Markov process, then each $S_R \in \Lambda$ is a *splitting time* for the original process X_t. That is, for each $F \in b\mathscr{F}^e$,

$$E^x[F \circ \theta_{S(R)} \mid \mathscr{I}_R] = K(X_{S(R)}, F) \text{ a.s.}$$

for some kernel K. Such splitting times have received considerable attention recently [4, 8, 9, 10].

Let $\tilde{S}_t = \sup\{u \leq \zeta : B_u = t\}$. Notice that $\tilde{S}_t = S_t$ on $[0, \tilde{\zeta})$. A random variable $F \in \mathscr{F}$ is in $\tilde{\mathscr{H}}_t$ if there is an $(\mathscr{F}_t)$-optional process Z_t so that $F = Z_{\tilde{S}(t)}$ on $\{t < \tilde{\zeta}\}$. We leave it to the reader to check that in fact $\tilde{\mathscr{H}}_t = \mathscr{H}_t$. Before stating the characterization of Λ, we introduce two more classes of times. We abuse notation somewhat by letting $[B]$ denote $\{(t, \omega): B_t(\omega) < B_{t+s}(\omega) \ \forall\ s > 0\}$. Let

$$\Gamma = \{T \in \mathscr{F}^e : [T] \subset [B];\ T\circ k_s = T \quad \text{whenever} \quad s \in [B] \quad \text{and} \quad s \geq T;$$
$$T\circ k_s = \infty \quad \text{whenever} \quad s \in [B] \quad \text{and} \quad s < T\}.$$

and $\tilde{\Lambda} = \{S_T : T < \tilde{\zeta}$ is an $(\mathscr{H}_t)$-optional time, $T \in \mathscr{F}^e\}$. One might think of Γ as a class of "predictable times relative to $[B]$." Recall Azéma's characterization of a predictable time V (1):

$$V\circ k_s = V \qquad \text{whenever} \quad s \geq V$$
$$V\circ k_s = \infty \qquad \text{whenever} \quad s < V.$$

THEOREM 3.3. $\Lambda = \Gamma = \tilde{\Lambda}$.

PROOF. Let $T \in \Gamma$, and let J be the indicator of $\{T \leq t\}$. Then $Z_s = J\circ k_s$ is an $(\mathscr{F}_s)$-optional process. Therefore $Z_{S(t)} = \{T\circ k_{S(t)} \leq t\} \in \mathscr{H}_t$. But $\{T\circ k_{S(t)} \leq t\} = \{T \leq t\} \cap \{S_t \geq T\}$ and $\{T \leq t\} = \{T \leq t\} \cap \{S_t \geq T\}$, since $\{S_t < T\} = \{t < B_T\} \subset \{t < T\}$ because $B_u \leq u$. We point out that the last equality in the preceding line holds since S_t is a point of right increase of B_t. Therefore $Z_{S(t)} = \{T \leq t\} \in \mathscr{H}_t$, and T is an $(\mathscr{H}_t)$-optional time. Now let $R = \inf\{t : S_t = T\}$. Since $[T] \subset [B] \subset [0, \zeta)$, $S_R = T$. We show that R is also an $(\mathscr{H}_t)$-optional time. Since $\{R \leq t\} = \{S_t \geq T\}$, we need to show $\{S_t \geq T\} \in \mathscr{H}_t$ for each t. Let J be the indicator of $\{\tilde{S}_t \geq T\}$. Define an $(\mathscr{F}_t)$-optional process $Z_s = J\circ k_s$. Then $Z_{S(t)} = \{\tilde{S}_t\circ k_{S(t)} \geq T\circ k_{S(t)}\} = \{\tilde{S}_t \geq T\} \cap \{S_t \geq T\} = \{S_t \geq T\}$, the last equality holding since $T < \zeta$. Therefore R is an $(\mathscr{H}_t)$-optional time and $\Gamma \subset \tilde{\Lambda}$. Since $\tilde{\Lambda}$ is clearly contained in Λ, we have left to show only that $\Lambda \subset \Gamma$.

Let T be an $(\mathscr{I}_t)$-optional time so that $S_T \in \Lambda$. Then $[S_T] \subset [B]$. Since $T < \tilde{\zeta}$, $S_T = \tilde{S}_T$. Then $\tilde{S}_T\circ k_s = \tilde{S}_t$ whenever $s \in [B]$, $s \geq \tilde{S}_T$. Now suppose $s \in [B]$, $s < \tilde{S}_T$. Then $B_s < B_{s+e}$ for all $e > 0$. Therefore $B_u\circ k_s < T$ for all $u \geq 0$, so $\tilde{S}_T\circ k_s = \infty$, and we conclude $\tilde{S}_T = S_T \in \Gamma$. □

From this result we conclude the following important corollary which decides a point left open in [5].

COROLLARY 3.4. $\mathscr{H}_t = \mathscr{H}_{t+}$.

PROOF. Let $T < \tilde{\zeta}$ be an $(\mathscr{H}_{t+})$-optional time. There is an $(\mathscr{H}_t)$-optional time R so that $S_T = S_R$. Since S is strictly increasing, $T = R$, and T is an $(\mathscr{H}_t)$-optional time. Therefore $\mathscr{H}_t = \mathscr{H}_{t+}$. □

There are two important points in this section. The first is contained in the statement of Corollary (3.4) (it is always useful to know that the filtration in hand is right continuous). Second, given an $(\mathscr{I}_t)$-optional time $T \in \mathscr{F}^e$ with $[T] \subset [0, \tilde{\zeta})$, S_T is a splitting time for X_t. Theorem (3.3) provides a more algebraic means of characterizing these splitting times in terms of $[B]$. This may provide some means of understanding the filtrations produced in Section 1.

4. A Class of Killing Times. Let X be a Markov process as described in the introduction, and let $M \in \mathscr{B}(R^+) \times \mathscr{F}^e$ be a random set which is homogeneous on $(0, \infty)$. That is, $1_{M\circ\theta_t}(s) = 1_M(s + t)$ for all $s > 0$, $t \geq 0$. We assume:

(4.1) $\qquad$ if $\ t \in M(\omega)$, $\ $ then $\ M(\omega) \cap [0, t) = M(k_t\omega) \cap [0, t)$.

Let $R = \inf\{t > 0 : t \notin M\}$ and let $D = \inf\{t \geq 0 : R\circ\theta_t = 0\}$. We shall prove the following result.

THEOREM 4.2. *The process X_t killed at D is a strong Markov process.*

Define a raw continuous additive functional by setting

$$A_t = \int_0^t 1_{\{R\circ\theta_s>0\}}\, ds,$$

and let T_t denote the right continuous inverse of A_t. Now $\{R\circ\theta_s\circ k_t > 0\} = \{\inf\{u > 0 : u + s \notin M\circ k_t\} > 0\}$. If we let $s < t$ and use (4.1), this is $\{\inf\{u > 0 : u + s \notin M\} > 0\}$, so that A_t has the (by now) familiar property:

$$A_u\circ k_{T(t)} = A_u \qquad \text{for all} \quad u \quad \text{in} \quad [0, T_t].$$

Let Z be a positive optional process, and let $F \in \mathscr{F}^{e+}$. As in (1.4) and (1.5), a simple computation shows that

$$E^x \int Z_{T(t)} F\circ\theta_{T(t)}\, dt = E^x \int Z_{T(t)} K(X_{T(t)}, F)\, dt, \tag{4.3}$$

where

$$\begin{aligned} K(x, F) &= E^x[F; R > 0]/P^x(R > 0) &&\text{if} \quad P^x(R > 0) > 0 \\ &= 0 &&\text{if} \quad P^x(R > 0) = 0. \end{aligned}$$

Now if it is the case that $K(X_{T(t)}, F)$ is a.s. right continuous for an appropriately large collection of random variables F, then $X_{T(t)}$ is once again a strong Markov process ([5]). This is not always true, however, and so we shall show that $K(X_{T(t)}, F)$ is a.s. right continuous on $[0, D)$, for an appropriately large collection of functions F, and that we can always extract the section of the process on $[0, D)$ to obtain Theorem (4.2). We give some examples at the end of the section.

We now examine the continuity properties of the kernel K. For the remainder of this discussion, we set

$$F = \Pi_{i=1}^n \int_0^\infty e^{-a(i)t} f_1(X_{t_1+t}) \cdots f_n(X_{t_n+t})\, dt,$$

where each function f_i is bounded, positive continuous on E, each $a(i)$ is positive, and $0 \le t_1 < \cdots\cdots < t_n$. Note that random variables of this form generate $\mathscr{F}^o$. Let (T_n) be a sequence of optional times decreasing to T, and look at

$$E^x[E^{X(T(n))}[F;\ R > 0]] = E^x[F\circ\theta_{T(n)};\ R\circ\theta_{T(n)} > 0].$$

It is easy to check from the definitions of F and R that as n increases to infinity, this expression converges to

$$E^x[F\circ\theta_T;\ R\circ\theta_T > 0] = E^x[E^{X(T)}[F; R > 0]].$$

If $t \to E^{X(t)}[F; R > 0]$ is an optional process, this is enough to imply that $E^{X(t)}[F; R > 0]$ is right continuous. But it is easy to check that $G(b, x) = E^x[F \exp(-bR)]$ is $(b + \Sigma a(i))$-excessive, $E^x[F]$ is $(\Sigma a(i))$-excessive, and that $\lim_{b\to\infty} G(b, x) = E^x[F; R = 0]$, so $E^{X(t)}[F; R > 0]$ is optional. Therefore, $K(X_t, F)$ is right continuous on $A = \{(t, \omega) : X_t \notin \{x : P^x(R > 0) = 0\}\}$. Note that if $A = \varnothing$ a.s., one may follow the discussion in Section 1 of [5] to show that $X_{T(t)}$ is a strong Markov process. In general, however, let T be an optional time with $[T] \subset \{(t, \omega) : P^{X(t,\omega)}(R > 0) = 0\}$. Then $0 = E^x[P^{X(T)}(R > 0)] = P^x(R\circ\theta_T > 0)$. Therefore, $R\circ\theta_T = 0$, so $T \ge D$ a.s. Thus we conclude that $K(X_t, F)1_{\{t<D\}}$ is a.s. right continuous.

LEMMA 4.4. *There is an optional process W so that $W_{T(t)} = 1$ if $T_t \le D$ and $W_{T(t)} = 0$ if $T_t > D$.*

PROOF. We first show that D can be chosen to be $\mathscr{F}^e$-measurable. Let $C = \inf\{t > 0 : R\circ\theta_t = 0\}$; C satisfies $C\circ\theta_s = C - s$ on $\{s < C\}$. Let F denote a random variable of the form given above. It is easy to check that $E^x[\exp(-C)F]$ is $(1 + \Sigma a(i))$-excessive. Thus if we define $Q^x(F) = E^x[\exp(-C)F]$, we have $Q^x \ll P^x$ for each x. Since $\mathscr{F}^o$ is generated by random variables of the form F given above, $Q^x(G) \in \mathscr{E}^e$ for all $G \in \mathscr{F}^o$. Since $P^x(G) \in \mathscr{E}^e$ for all $G \in \mathscr{F}^o$, Doob's lemma yields the existence of a density $p(x, \omega) \in \mathscr{E}^e \times \mathscr{F}^o$ so that $Q^x(G) = E^x[p(x, \omega)G] = E^x[p(X_0(\omega), \omega)G]$ for all $G \in \mathscr{F}^o$. Thus $p(X_0(\omega), \omega) = \exp(-C)$ a.s. and $p(X_0(\omega), \omega) \in \mathscr{F}^e$. A similar procedure yields a version of $R \in \mathscr{F}^e$. Since $D = C1_{\{R>0\}}$, we have shown that D can be chosen in $\mathscr{F}^e$. Now let $Z_t = D\circ k_t$: this is optional since $D \in \mathscr{F}^e$. It is easy to check that $Z_{T(t)} = \infty$ if $T_t \leq D$ and $Z_{T(t)} = D$ if $T_t > D$ (by using (4.1)). Finally, we let $W_t = 1_{\{Z_t=\infty\}}$. □

Let $V_t = \exp(-aA_t\circ k_t)$ so that $V_{T(t)} = e^{-at}$ if $T_t < \infty$. Note that V is optional. We replace Z in (4.3) with $W\cdot V\cdot Z$, where Z is a bounded right continuous optional process. Applying Fubini's theorem, (4.3) becomes

$$\int e^{-at}E^x[W_{T(t)}Z_{T(t)}F\circ\theta_{T(t)}]\,dt = \int e^{-at}E^x[W_{T(t)}Z_{T(t)}K(X_{T(t)}, F)]\,dt.$$

Taking into account $W_{T(t)}$, this may be rewritten as

$$\int e^{-at}E^x[Z_tF\circ\theta_t;\, t \leq D]\,dt = \int e^{-at}E^x[Z_tK(X_t, F);\, t \leq D]\,dt.$$

It follows that

$$\int e^{-at}E^x[Z_tF\circ\theta_t;\, t < D]\,dt = \int e^{-at}E^x[Z_tK(X_t, F);\, t < D]\,dt.$$

Since both integrands are right continuous, we conclude that

$$E^x[Z_tF\circ\theta_t;\, t < D] = E^x[Z_tK(X_t, F);\, t < D].$$

This implies that

$$E^x[F\circ\theta_t 1_{\{t<D\}} \mid \mathscr{G}_t] = E^x[K(X_t, F)1_{\{t<D\}} \mid \mathscr{G}_t] = K(X_t, F)1_{\{t<D\}},$$

where $\mathscr{G}_t = \sigma\{Z_t : Z \text{ is an optional process}\} \vee \sigma\{1_{\{t<D\}}\}$. Therefore, the process X_t killed at D is *Markov*, and the *strong Markov* property follows from the right continuity of $K(X_t, F)1_{\{t<D\}}$ by applying the standard argument ([5], Theorem (1.5); [3], page 42). This concludes the proof of Theorem (4.2).

Before discussing examples, we make one more observation.

PROPOSITION 4.5. *The optional projection of* $Z_t = 1_{\{t<D\}}$ *is a multiplicative functional.*

PROOF. The process X killed at D is clearly subordinate to X. Therefore, there is a multiplicative functional m_t so that if $t_1 < t_2 < \cdots < t_n = t$,

$$E^x[f_1(X_{t(1)}) \cdot \cdots \cdot f_n(X_{t(n)})Z_t] = E^x[f_1(X_{t(1)}) \cdot \cdots \cdot f_n(X_{t(n)})m_t]. \quad \square$$

EXAMPLE. Here is a familiar one. Let L be a cooptional time, and let $M = \{(t, \omega) : t < L\}$. Assumption (4.1) as written does not apply to this set M, but by adding an extra death point to the state space, one can adjust Ω and the killing operators slightly so that (4.1) holds: see Section 3 of [5]. Then the process X killed at D is exactly X killed at L.

EXAMPLE. Let A_t be the difference of two adapted continuous additive functionals of X, and set $M = \{(t, \omega) : A_s\circ\theta_t(\omega) > 0 \text{ for all } s > 0\} = \{(t, \omega) : A_{s+t}(\omega) > A_t(\omega) \text{ for all } s > 0\}$. It is easy to check that (4.1) holds, so that X killed at D is strong Markov. It does not seem to be the case that $(X_{T(t)};\, 0 \leq t < \infty)$ is always strong Markov.

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REFERENCES

[1] AZÉMA, J. (1973). Théorie générale des processus et retournement du temps. *Ann. Sci. Ecole Norm. Sup.* **6** 459–519.

[2] BENVENISTE, A. (1976). Separabilite optionelle, d'apres Doob. Seminarie de Probabilities X. *Lect. Notes in Math.* **511** 521–531. Springer, Berlin.

[3] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic, New York.

[4] GETOOR, R. K. and SHARPE, M. J. (1979). The Markov property at cooptional times. *Z. Wahrsch. verw. Gebiete* **48** 201–211.

[5] GLOVER, J. Raw time changes of Markov processes. *Ann. of Probability* **9** 90–102.

[6] KNIGHT, F. B. (1971). The Local Time at Zero of the Reflected Symmetric Stable Process. *Z. Wahrsch. verw. Gebiete* **19** 180–190.

[7] KNIGHT, F. B. and PITTENGER, A. O. (1972). Excision of a strong Markov process. *Z. Wahrsch. verw. Gebiete* **23** 114–120.

[8] MILLAR, P. W. (1976). Random times and decomposition theorems. *Proc. of the Symposium in Pure Mathematics* **31** 91–104. Amer. Math Soc.

[9] PITTENGER, A. O. Regular birth times for Markov processes. *Ann. of Probability* **9** 769–780.

[10] WILLIAMS, D. (1974). Path decomposition and continuity of local time for one-dimensional diffusions I. Proc. London Math. Soc. **28** 738–768.

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