The Annals of Probability
1981, Vol. 9, No. 6, 1012–1018

A FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR A CLASS OF SUBORDINATORS

R. P. PAKSHIRAJAN AND R. VASUDEVA

University of Mysore, India

Let $\{X(t),\ t \in [0, \infty)\}$ be a subordinator whose Lévy spectral function $H(x)$ satisfies the inequality

$$c_1 x^{-\alpha} \le -H(x) \le c_2 x^{-\alpha},$$

for all $x > 0$, for a $\alpha \in (0, 1)$ and for certain constants c_1 and c_2, $0 < c_1 \le c_2 < \infty$. In this paper we obtain (in the M_1 topology) the set of all almost sure limit functions of the sequence

$$(n^{-1/\alpha} X(nt))^{\frac{1}{\log \log n}}, \qquad t \in [0, 1],\ n \ge 3.$$

Introduction. Let $\{X(t),\ t \in [0, \infty)\}$ be a subordinator and let at any fixed t, the characteristic function of $X(t)$ be given by

$$f_t(u) = \exp\left\{t\left(\int_0^\infty (e^{iux} - 1)\, dH(x)\right)\right\}, \qquad t \in [0, \infty),$$

where H is the corresponding Lévy spectral function. Assume that the process $\{X(t),\ t \in [0, \infty)\}$ has been defined over a probability triplet $(\Omega, \mathscr{B}, P)$ and that there exist an α, $0 < \alpha < 1$, c_1 and c_2, $0 < c_1 \le c_2 < \infty$, such that for all $x > 0$,

$$c_1 x^{-\alpha} \le -H(x) \le c_2 x^{-\alpha}. \tag{1}$$

Consider the version of $X(t)$ with its sample functions in $D\,[0, \infty)$ and define

$$Z_n(t) = (n^{-1/\alpha} X(nt))^{1/\log\log n}, \tag{2}$$

$t \in [0, 1]$, $n \ge 3$.

In the space $D = D\,[0, 1]$ of all real valued functions on $[0, 1]$ that are right continuous with finite left limits, let $D_1 = D_1\,[0, 1]$ be the space of all functions which are non-negative valued and non-decreasing and let $D_2 = D_2\,[0, 1]$ be the space of all elements of D_1 which are step functions with at most countably many jump points.

For an $\alpha \in (0, 1)$, which is specified, define

$$A = \{x \in D_1,\ 1 \le x(t) \le e^{1/\alpha} \quad \text{for} \quad 0 < t \le 1\} \tag{3}$$

and

$$K = \{x \in A \cap D_2;\ \textstyle\prod_{j=1}^{k} x(t_j) \le e^{1/\alpha} \quad \text{whenever} \tag{4}$$
$$x(t_1) < x(t_2) < \cdots < x(t_k), \qquad 0 < t_1 < t_2 \cdots < t_k \le 1,\ k = 1, 2 \cdots\}.$$

In this paper we establish, under M_1 convergence, that the sequence (Z_n) is relatively compact with probability one (w.p.1) and has K for the set of all its almost sure limit functions.

When $H(x) = c\, x^{-\alpha}$, $0 < \alpha < 1$, the process $X(t)$ turns out to be a stable subordinator. For such processes, M. J. Wichura (1974b) has presented a functional law of the iterated logarithm. A comparison of the result of Wichura is made in Remark 2 of our paper.

Received November 21, 1977; revised October 15, 1980.
AMS 1970 *subject classifications.* Primary, 60F15.
Key words and phrases. Iterated logarithm, subordinators, stable subordinators.

PRELIMINARIES, NOTATIONS AND LEMMAS. A sufficient condition for a sequence (x_n), $x_n \in D_1$ to be relatively compact in the M_1 topology is that

$$\limsup_{n\to\infty} x_n(1) < \infty. \tag{5}$$

For $x_n \in D_1$, $n \geq 1$, and $x \in D_1$, we say that $x_n \overset{M_1}{\to} x$ ($x_n \to x$ in M_1 topology) if and only if for each t in a dense subset of $(0, 1)$, $\lim_{n\to\infty} x_n(t) = x(t)$.

Notice that K is closed and that $\sup_{x\in K} x(1) < \infty$. Hence K is a compact set in D_1.

Let $\{Y(t), t \in [0, \infty)\}$ be a stable subordinator with the Lévy measure $\mu[x, \infty) = cx^{-\alpha}$, $c > 0$, $x > 0$. Set $J_\mu(t) = Y(t) - y(t-)$.

For a non-decreasing (non-increasing) function f define f^{-1} by $f^{-1}(y) = \inf_x \{f(x) \geq y\}$ $(f^{-1}(y) = \inf_x \{f(x) \leq y\})$.
Throughout the paper, let R and N (integers), c and ϵ, with or without a suffix, stand for positive constants; let i.o. and a.s. mean 'infinitely often' and 'almost surely.' For any positive number x, let $[x]$ denote the greatest integer $\leq x$.

LEMMA 1. *The process*

$$\phi(t) = \sum_{0\leq v\leq t} (-H)^{-1}(c(J_\mu(v))^{-\alpha}), \qquad t \in [0, \infty),$$

is a subordinator with the Lévy spectral function H.

PROOF. Define $g(x) = (cx^{-1})^{1/\alpha}$ and $h(x) = g(-H)(x)$, $x > 0$. Observe that $g^{-1}(x) = cx^{-\alpha}$, $h^{-1}(x) = (-H)^{-1}g^{-1}(x)$ and that $h(x)$ and $h^{-1}(x)$ are non-decreasing functions of x. With the h introduced, one can write

$$\phi(t) = \sum_{0\leq v\leq t} h^{-1}(J_\mu(v)), \qquad t \in [0, \infty).$$

As $h^{-1}(0) = 0$, $\phi(t)$ becomes a non-negative valued process in t with non-decreasing sample functions. From the fact that $Y(t)$ has stationary independent increments, $\phi(t)$ can now be claimed to be a subordinator. Let ν be the Lévy measure of $\phi(t)$; $A = A_1 \times A_2$ be a subset of the product space $[0, \infty) \times (0, \infty)$ and let $N(A)$ be the number of points t such that $\{(t, h^{-1}(J_\mu(t))) \in A\}$. Then $N(A)$ is a Poisson random variable (r.v.) with mean $\lambda(A_1) \times \nu(A_2)$, where $\lambda(A_1)$ is the Lebesgue measure of the set A_1.

The number of t with $\{(t, h^{-1}(J_\mu(t))) \in A\}$ is the same as the number of t with $\{(t, J_\mu(t)) \in A_1 \times h(A_2)\}$ where $h(A_2) = \{h(x); x \in A_2\}$. Thus $N(A)$ is a Poisson r.v. with mean $\lambda(A_1) \times \mu(h(A_2))$. On comparison one gets $\lambda(A_1) \times \nu(A_2) = \lambda(A_1) \times \mu(h(A_2))$. This in turn implies that $\nu[x, \infty) = \mu[h(x), \infty) = c(h(x))^{-\alpha} = -H(x)$, $x > 0$. The proof of the lemma is now complete.

LEMMA 2. *Let (y_n) be a sequence of positive numbers such that $y_n \to \infty$ as $n \to \infty$ and let $0 \leq t_1 < t_2 < \infty$. Then there exist constants $0 < c_3 \leq c_4 < \infty$ such that*

$$\begin{aligned} c_3(t_2 - t_1) &\leq \liminf_{n\to\infty} y_n^\alpha P(X(nt_2) - X(nt_1) \geq n^{1/\alpha} y_n) \\ &\leq \limsup_{n\to\infty} y_n^\alpha P(X(nt_2) - X(nt_1) \geq n^{1/\alpha} y_n) \leq c_4(t_2 - t_1). \end{aligned}$$

PROOF. Recall that the spectral function H satisfies the inequality $c_1 x^{-\alpha} \leq -H(x) \leq c_2 x^{-\alpha}$, $x > 0$.

Let $\{Y_1(t), t \in [0, \infty)\}$ be a stable subordinator with the Lévy spectral function $\mu_1(x) = -c_1 x^{-\alpha}$, $x > 0$. Then the fact that $-H(x) \leq -\mu_1(x)$ implies that $(-H)^{-1}(x) \geq (-\mu_1)^{-1}(x)$, which in turn implies that

$$(-H)^{-1}(c_1(J_{\mu_1}(x))^{-\alpha}) \geq (-\mu_1)^{-1}(c_1(J_{\mu_1}(x))^{-\alpha}) = J_{\mu_1}(x), \qquad x > 0,$$

where $J_{\mu_1}(x) = Y_1(x) - Y_1(x-)$. By Lemma 1 notice that the r.v. $\sum_{nt_1\leq v\leq nt_2}(-H)^{-1}\cdot (c_1(J\mu_1(v))^{-\alpha})$ has the same distribution as $X(nt_2) - X(nt_1)$. Hence for any $y > 0$,

(6) $$P(X(nt_2) - X(nt_1) \geq n^{1/\alpha}y) \geq P(Y_1(nt_2) - Y_1(nt_1) \geq n^{1/\alpha}y).$$

But

$$P(Y_1(nt_2) - Y_1(nt_1) \geq n^{1/\alpha}y_n) = P(Y(1) \geq (t_2 - t_1)^{-1/\alpha}y_n) \geq c_3(t_2 - t_1)y_n^{-\alpha},$$

for all $n \geq N_1$.

Consequently, for every $n \geq N_1$,

(7) $$P(X(nt_2) - X(nt_1) \geq n^{1/\alpha}y_n) \geq c_1(t_2 - t_1)y_n^{-\alpha}.$$

Similarly, let $\{Y_2(t), t \in [0, \infty)\}$ be a stable subordinator with the Lévy spectral function $\mu_2(x) = -c_2x^{-\alpha}$, $x > 0$. Noticing that $-H(x) \leq -\mu_2(x)$ and making slightly modified arguments in the steps used in obtaining (7), one can show that for any $y > 0$,

(8) $$P(X(nt_2) - X(nt_1) \geq y) \leq P(Y_2(nt_2) - Y_2(nt_1) \geq y).$$

Again, since $P(Y_2(nt_2) - Y_2(nt_1) \geq n^{1/\alpha}y_n) = P(Y_2(1) \geq (t_2 - t_1)^{-1/\alpha}y_n) \leq c_4(t_2 - t_1)y_n^{-\alpha}$, for all $n \geq N_2$ (for some $N_2 > 0$), along with (8) we have

(9) $$P(X(nt_2) - X(nt_1) \geq n^{1/\alpha}y_n) \leq c_4(t_2 - t_1)y_n^{-\alpha},$$

whenever $n \geq N_2$. The proof is complete by (7) and (9).

LEMMA 3. *For any* $\epsilon > 0$ *and for any* $t > 0$,

(10) $$P(Z_n(t) \leq e^{-\epsilon} \text{ i.o.}) = 0$$

and

(11) $$P(Z_n(t) \geq e^{(1+\epsilon)/\alpha} \text{ i.o.}) = 0.$$

PROOF. Define the integer sequence $n_r = [e^r]$, $r = 1, 2, \cdots$ and the events

$$A_n = \{X(nt) \leq n^{1/\alpha}(\log n)^{-\epsilon}\}$$

and

$$B_r = \{X(n_rt) \leq n_{r+1}^{1/\alpha}(\log n_r)^{-\epsilon}\}.$$

Observe that

(12) $$P(A_n \text{ i.o.}) \leq P(B_r \text{ i.o.}).$$

By (6) we have $P(B_r) \leq P\{Y_1(n_rt) \leq n_{r+1}^{1/\alpha}(\log n_r)^{-\epsilon}\} = P\left\{Y_1(1) \leq \dfrac{n_{r+1}^{1/\alpha}(\log n_r)^{-\epsilon}}{n_r^{1/\alpha}t^{1/\alpha}}\right\}$. Recalling from Theorem 1, Feller (1966, page 424), that $(e^{x^{-\alpha}})\, P(Y(1) \leq x) \to 0$ as $x \to 0$, one can find an $\epsilon_1 > 0$ such that

$$P\left\{Y_1(1) \leq \frac{n_{r+1}^{1/\alpha}}{n_r^{1/\alpha}} \frac{(\log n_r)^{-\epsilon}}{t^{1/\alpha}}\right\} \leq \exp(-r^{-\epsilon_1}).$$

Hence $\sum_{r=1}^{\infty} P(B_r) < \infty$. (10) is immediate by an appeal to the Borel Cantelli lemma and (12).

To establish (11), define,

$$U_r = \{X(n_{r+1}t) \geq n_r^{1/\alpha}(\log n_r)^{(1+\epsilon)/\alpha}\}, \qquad r \geq 2.$$

Then by Lemma 2, there exists an R_1 such that $P(U_r) \leq ctr^{-(1+\epsilon)}$ for all $r \geq R_1$. By the Borel Cantelli lemma we now get

(13) $$P(U_r \text{ i.o.}) = 0.$$

For all n in $n_r \leq n < n_{r+1}$, (13) implies that

$$\frac{X(nt)}{n^{1/\alpha}(\log n)^{(1+\epsilon)/\alpha}} \le \frac{X(n_{r+1}t)}{n_r^{1/\alpha}(\log n_r)^{(1+\epsilon)/\alpha}} \le 1 \text{ a.s.},$$

which in turn implies (11).

THEOREM. *The sequence* $Z_n(t) = (n^{-1/\alpha}X(nt))^{1/\log\log n}$, $t \in [0, 1]$, *is relatively compact with probability one and has K for the set of all its limit functions.*

PROOF. For each $n \ge 3$ and for each $w \in \Omega$, $Z_n(t, w)$ is nondecreasing in t. Hence all the sample functions of $Z_n(t)$ are located in D_1. By (5) and (11) the sequence $(Z_n(\cdot, w))$ is relatively compact over a set of probability one. Also, Lemma 3 establishes that the limit functions are bounded and they take values in $[1, e^{1/\alpha}]$.

For any $\epsilon > 0$, let

$$A_\epsilon = \{x \in D_1, e^{-\epsilon} \le x(t) \le e^{(1+\epsilon)/\alpha} \quad \text{for} \quad 0 < t \le 1\}$$

and

$$K_\epsilon = \{x \in A_\epsilon \cap D_2, \textstyle\prod_{j=1}^k x(t_j) \le e^{(1+\epsilon)/\alpha} \quad \text{whenever}$$

$$x(t_1) < x(t_2) \cdots < x(t_k), \qquad 0 < t_1 < t_2 \cdots < t_k \le 1, k = 1, 2 \cdots\}.$$

Notice that $A \subset A_\epsilon$, $K \subset K_\epsilon$ and $K_\epsilon \subset A_\epsilon$. By Lemma 3, our search for limit functions is restricted to the set A_ϵ. We first establish that no element of $A_\epsilon - K_\epsilon$ is a limit of (Z_n). This, along with the fact that the closure of $A_\epsilon - K_\epsilon$ is compact, implies that $P(Z_n \in A_\epsilon - K_\epsilon \text{ i.o.}) = 0$.

Let $q \ge 1$ be any arbitrary integer and let $0 < t_1 < t_2 \cdots < t_q \le 1$ be q arbitrary continuity points of a function $x \in A_\epsilon$. Then a necessary and sufficient condition for x to be a limit of the sequence (Z_n) is that

$$P\{\cap_{j=1}^q (Z_n(t_j) \in (x(t_j) - \delta, x(t_j) + \delta))\text{i.o.}\} = 1, \tag{14}$$

for every $\delta > 0$, for all $q \ge 1$ and for all arbitrary points $0 < t_1 < t_2 \cdots < t_q \le 1$, which are continuity points of x.

If $x \in A_\epsilon - K_\epsilon$, then one can choose either (a) continuity points $0 < t_1 < t_2 \cdots < t_q < 1$ of x such that $1 < x(t_1) < x(t_2) \cdots < x(t_q)$ and $\prod_{j=1}^q (x(t_j)) > e^{1/\alpha}$, or (b) a continuity point t such that $x(t) < 1$. In the case of (b), x fails to be a limit as a consequence of Lemma 3.

In the case (a), with no loss of generality, let $x(t_j) = e^{d_j}$, $j = 1, 2 \cdots q$. From the fact that $\prod_{j=1}^q x(t_j) > e^{1/\alpha}$ we have $\sum_{j=1}^q d_j > 1/\alpha$. Let $d_0 = 0$. We now claim that x fails to be a limit function of (Z_n) by establishing that for an $\epsilon_1 > 0$ with $\epsilon_1 < \min_{1\le i\le q}\left(\frac{d_i - d_{i-1}}{2}\right)$ and with $\sum_{j=1}^q d_j - \epsilon_1 q > 1/\alpha$

$$P\{\cap_{j=1}^q (Z_n(t_j) \in (e^{d_j-\epsilon_1}, e^{d_j+\epsilon_1})) \text{ i.o.}\} = 0. \tag{15}$$

i.e.
$$P\left\{\cap_{j=1}^q \left(\frac{\log X(nt_j) - \frac{1}{\alpha}\log n}{\log\log n} \in (d_j - \epsilon_1, d_j + \epsilon_1)\right)\text{i.o.}\right\} = 0.$$

Put
$$W_n(t) = \frac{\log X(nt) - \frac{1}{\alpha}\log n}{\log\log n}, \qquad t \in [0, 1], \qquad n \ge 3,$$

and notice that it satisfies the relation, $W_n(t) = a_{m,n}W_m(b_{m,n}t) + c_{m,n}$, $t \in [0, 1]$, where

$$a_{m,n} = \frac{\log\log m}{\log\log n}, \quad b_{m,n} = \frac{n}{m} \quad \text{and} \quad c_{m,n} = \frac{\log\frac{m}{n}}{\alpha\log\log n}.$$

If $m, n \to \infty$ in such a way that $m/n \to 1$ then observe that

$$a_{m,n} \to 1, \ b_{m,n} \to 1 \qquad \text{and} \qquad c_{m,n} \to 0.$$

Hence by Lemma 5.2 of Wichura (1974a), (15) is established once we prove that over the integer sequence $N_r = [\exp(r/\log r)]$, $r \geq 2$,

$$(16) \qquad P\{\cap_{j=1}^{q} (Z_{N_r}(t_j) \in (e^{d_j-\epsilon_1}, e^{d_j+\epsilon_1})) \text{i.o.}\} = 0.$$

Define $t_0 = 0$. Then

$$P\{\cap_{j=1}^{q} (Z_{N_r}(t_j) \in (e^{d_j-\epsilon_1}, e^{d_j+\epsilon_1}))\}$$

$$P\{\cap_{j=1}^{q}(X(N_r t_j) \in (N_r^{1/\alpha}(\log N_r)^{(d_j-\epsilon_1)}, N_r^{1/\alpha}(\log N_r))^{(d_j+\epsilon_1)})\}$$

$$\leq P\{\cap_{j=1}^{q}(X(N_r t_j) - X(N_r t_{j-1}) \geq \tfrac{1}{2} N_r^{1/\alpha}(\log N_r)^{(d_j-\epsilon_1)})\}.$$

By Lemma 2, one gets for all $r \geq R$ and for all $j = 1, 2 \cdots q$,

$$P(X(N_r t_j) - X(N_r t_{j-1}) \geq \tfrac{1}{2} N_r^{1/\alpha}(\log N_r)^{(d_j-\epsilon_1)}) \leq c_1(\log N_r)^{-(d_j-\epsilon_1)\alpha}.$$

Hence, for all $r \geq R$,

$$P\{\cap_{j=1}^{q} (Z_{N_r}(t_j) \in (e^{d_j-\epsilon_1}, d^{d_j+\epsilon_1}))\} \leq (c_2(\log N_r))^{-(\sum_{j=1}^{\prime} d_j - \mathscr{V}\epsilon_1)\alpha} \leq c_3 r^{-(1+\epsilon_2)},$$

for some $\epsilon_2 > 0$.

Now (16) follows by an appeal to the Borel Cantelli lemma. Below, we proceed to show that every $x \in K$ is a limit function of (Z_n).

For an $x \in K$, let $0 < t_1 < t_2 \cdots < t_q \leq 1$ be any q continuity points and let s denote the number of distinct members of the collection $\{1, x(t_1), x(t_2) \cdots x(t_q)\}$. When $s > 1$, let i_1 be the smallest j with $x(t_j) > 1$ and let i_2 be the smallest $j (j > i_1)$ for which $x(t_j) > x(t_{i_1})$ (if it exists) and so on. Define $x(t_j) = e^{\hat{d}_j}$, $j = 1,2, \cdots, \mathscr{V}$ and write $x(t_{i_k}) = e^{d_k}$, $0 < d_1 < d_2 \cdots < d_s \leq 1/\alpha$. Define the integer sequence $m_r = r^r$, $r = 1, 2 \cdots$ and set $t_0 = t_0(r) = m_{r-1} t_{\mathscr{V}}/m_r$. Then (14) is established once we prove that for $0 < \epsilon < d_1/2$,

$$(17) \qquad P\{\cap_{j=1}^{q} (Z_{m_r}(t_j) \in (e^{\hat{d}_j-\epsilon}, e^{\hat{d}_j+\epsilon})) \text{ i.o.}\} = 1.$$

$$P\{\cap_{j=1}^{q} (Z_{m_r}(t_j) \in (e^{\hat{d}_j-\epsilon}, e^{\hat{d}_j+\epsilon})) \text{ i.o.}\}$$

$$= P\{\cap_{j=1}^{q} (X(m_r t_j) \in (m_r^{1/\alpha}(\log m_r)^{(\hat{d}_j-\epsilon)}, m_r^{1/\alpha}(\log m_r)^{(\hat{d}_j+\epsilon)})) \text{ i.o.}\}.$$

For the ϵ fixed above, there exists an R such that for all $r \geq R$, the event

$$\{X_{m_r}(t_0) \leq m_r^{1/\alpha}(\log m_r)^{\epsilon/2},$$

$$\cap_{j=1}^{S} (X(m_r t_{i_j}) - X(m_r t_{i_j-1}) \in (m_r^{1/\alpha}(\log m_r)^{d_j-\epsilon_2}, m_r^{1/\alpha}(\log m_r)^{d_j+\epsilon/2})),$$

$$\cap_{\substack{j=1 \\ j\neq i_1, i_2 \cdots i_s}}^{q} (X(m_r t_j) - X(m_r t_{j-1}) \leq m_r^{1/\alpha}(\log m_r)^{\epsilon/2}\}$$

$$\subset \{\cap_{j=1}^{\mathscr{V}} (X(m_r t_j) \in (m_r^{1/\alpha}(\log m_r)^{\hat{d}_j-\epsilon}, m_r^{1/\alpha}(\log m_r)^{\hat{d}_j+\epsilon}))\}.$$

Hence (17) is established if we show that

$$(18) \qquad P\{X(m_r t_0) \geq m_r^{1/\alpha}(\log m_r)^{\epsilon/2} \text{ i.o.}\} = 0$$

and

$$(19) \qquad P\{\cap_{j=1}^{s} (X(m_r t_{i_j}) - X(m_r t_{i_j-1}) \in (m_r^{1/\alpha}(\log m_r)^{d_j-\epsilon/2}, m_r^{1/\alpha}(\log m_r)^{d_j+\epsilon/2})),$$

$$\cap_{j\neq i_1, i_2 \cdots i_s}^{\mathscr{V}} (X(m_r t_j) - X(m_r t_{j-1}) \leq m_r^{1/\alpha}(\log m_r)^{\epsilon/2}) \text{ i.o.}\} = 1.$$

By Lemma 2, there exists an R_1 such that for all $r \geq R_1$,

$$P\{X(m_r t_0) \geq m_r^{1/\alpha}(\log m_r)^{\epsilon/2}\}$$
$$= P(X(m_{r-1}t_q) \geq m_r^{1/\alpha}(\log m_r)^{\epsilon/2})$$
$$\leq C_1 m_{r-1}(m_r(\log m_r)^{\epsilon\alpha/2})^{-1} \leq C_2 r^{-(1+\epsilon_2)},$$

for some $\epsilon_2 > 0$. Consequently (18) follows by the Borel Cantelli lemma.

By (8), there exists an R_2 such that for all $r \geq R_2$ and for all j, $1 \leq j \leq q, j \neq i_1, i_2 \cdots i_s$

$$P(X(m_r t_j) - X(m_r t_{j-1}) \leq m_r^{1/\alpha}(\log m_r)^{\epsilon/2})$$
$$= 1 - P(X(m_r t_j) - X(m_r t_{j-1}) \geq m_r^{1/\alpha}(\log m_r)^{\epsilon/2}) \tag{20}$$
$$\geq 1 - P(Y_2(m_r t_j) - Y_2(m_r t_{j-1}) \geq m_r^{1/\alpha}(\log m_r)^{\epsilon/2}) \geq 1/2.$$

For $j = i_1, i_2 \cdots i_s$, one can find an R_3 such that for all $r \geq R_3$,

$$P\{X(m_r t_{i_j}) - X(m_r t_{i_{j-1}}) \in (m_r^{1/\alpha}(\log m_r)^{(d_j-\epsilon/2)}, m_r^{1/\alpha}(\log m_r)^{(d_j+\epsilon/2)})\} \tag{21}$$
$$\geq P\{X(m_r t_{i_j}) - X(m_r t_{i_{j-1}}) \geq m_r^{1/\alpha}(\log m_r)^{(d_j-\epsilon/2)}\}$$
$$- P\{X(m_r t_{i_j}) \geq m_r^{1/\alpha}(\log m_r)^{(d_j+\epsilon/2)}\}$$
$$\geq C_1(r \log r)^{-(d_j-\epsilon/2)\alpha} - C_2(r \log r)^{-(d_j+\epsilon/2)\alpha}$$
$$\geq C_3(r \log r)^{-(d_j-\epsilon/3)\alpha}.$$

Now (20) and (21) imply that for all $r \geq R = \max(R_2, R_3)$,

$$P\{\cap_{j=1}^{s} (X(m_r t_{i_j}) - X(m_r t_{i_{j-1}}) \in (m_r^{1/\alpha}(\log m_r)^{(d_j-\epsilon/2)}, m_r^{1/\alpha}(\log m_r)^{(d_j+\epsilon/2)}))$$
$$\cap_{\substack{j=1 \\ j\neq i_1, i_2 \cdots i_s}}^{q} (X(m_r t_j) - X(m_r t_{j-1}) \leq m_r^{1/\alpha}(\log m_r)^{\epsilon/2})\} \tag{22}$$
$$\geq C_4(r\log r)^{-\sum_{j=1}^{s}(d_j - \epsilon/3)\alpha} \geq C_5 r^{-(1-\epsilon_1)}$$

for some $\epsilon_1 > 0$. (22) along with the fact that $X(m_r t_j) - X(m_r t_{j-1}), j = 1, 2, \cdots q; r = 1, 2 \cdots$ are mutually independent, enable us to apply the Borel Cantelli lemma and claim (19).

REMARK 1. Here we introduce an important class of subordinators which come under the scope of our paper. Let $X(t)$ be a subordinator with

$$H(x) = -cx^{-\alpha}\{1 + \textstyle\sum_{i=1}^{k} a_i \cos \overline{\nu_i \log x} + \sum_{i=1}^{k} b_i \sin \overline{\nu_i \log x}\} \tag{23}$$

$x > 0$, where α, $0 < \alpha < 1$, $c > 0$, a_i, b_i and ν_i, $i = 1, 2 \cdots k$ are real constants such that $H(x)$ is non-decreasing, $xH'(x)$ is non-increasing and $xH(x) \in L(-1, 1)$. Then for any fixed $t > 0$, the random variable $X(t)$ is said to be a member of 'class P_r' for some integer r, $0 < r \leq 2k + 1$ (for a description of P_r see Zinger (1965)). This process $X(t)$ can well be called as a subordinator of class P_r. It is easy to see that (23) satisfies (1) and hence our result holds for the subordinators of class P_r.

REMARK 2. Let (X_n) be a sequence of independent random variables with a common positive stable distribution function of exponent α, $0 < \alpha < 1$, and let $S_n = \sum_{j=1}^{n} X_j$, $n \geq 1$. Then following the lines of proof of our theorem, one can show that the sequence

$$\{\xi_n(t) = (n^{-1/\alpha}S_{[nt]})^{1/\log\log n}\}, \qquad t \in [0, 1], \qquad n \geq 3,$$

is relatively compact with K as the set of all its limit functions. It is interesting to note that, with a different sequence (a_n) of normalising constants, M. J. Wichura (1974b) has established that the sequence $\left\{\eta_n(t) = \frac{S_{[nt]}}{a_n}\right\}$ is relatively compact with the limit set K_α. For a description of (a_n) and K_α see Wichura (1974b). Thus there are two functional laws of the interated logarithm when the summands are positive stable. The determination of the limit functions of (ξ_n) is based on $P(S_n \geq n^{1/\alpha}x)$ for large values of x and that of (η_n) is based on $P(S_n \leq n^{1/\alpha}x)$ for x near zero and the two limit functions are not comparable.

REMARK 3. Let (X_n) be a sequence of random variables described under Remark 2. Define $Y_n(t) = \sum_{k=1}^{[nt]} (1 - k/n)^P X_k$, $p \geq 0$, $t \in [0, 1]$ and $W_n(t) = (n^{-1/\alpha} Y_n(t))^{1/\log\log n}$, $t \in [0, 1]$ and $n \geq 3$. It is interesting to note that the sequence (W_n) is relatively compact and has the set K for the set of its limit functions. This can be established by proceeding with slight modifications in the proof of our theorem. The details are omitted.

Acknowledgement. The authors thank the referee for his valuable suggestions. In particular, the proof of Lemma 1 evolved out of a personal communication with Professor Bert E. Fristedt. The authors thank him for this help.

REFERENCES

[1] BREIMAN, L. (1968) *Probability.* Addison-Wesley.
[2] FELLER, W. (1966). *An introduction to probability theory and its applications,* Vol. 2, Wiley, New York.
[3] SKOROKHOD, A. V. (1956). Limit theorems for stochastic processes, *Theory Probability Appl.* **1** 261–290 (English translation).
[4] WICHURA, M. J. (1974a). On the functional form of the law of the iterated logarithm for the partial maxima of independent identically distributed random variables. *Ann. Probability* **2** 202–230.
[5] WICHURA, M. J. (1974b). Functional law of the iterated logarithm for partial sums of i.i.d. random variables in the domain of attraction of a completely asymmetric stable law. *Ann. Probability* **2** 1108–1138.
[6] ZINGER, A. A. (1965). On a class of limit distributions for sums of independent random variables. *Theory Probability Appl.* **10** 606–626 (English translation).

DEPARTMENT OF STATISTICS
UNIVERSITY OF MYSORE
MANASAGANGOTHRI,
MYSORE—570 006, INDIA.