The Annals of Probability
1982, Vol. 10, No. 2, 500–503

EDGEWORTH EXPANSIONS AND SMOOTHNESS

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We give a necessary and sufficient condition for the distribution function of $n^{-1/2}\sum_{i=1}^{n} X_i$, where the X_i are independently identically distributed with $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^{k+3} < \infty$, to possess an Edgeworth expansion to k terms. The condition is not practicable but clarifies the relation between the existence of an Edgeworth expansion and the smoothness of the distribution function of the sum.

1. Introduction. Let $X_1, \cdots, X_n$ be independently and identically distributed random variables with common distribution F and let F_n be the distribution function of $n^{-1/2}\sum_{i=1}^{n} X_i$. If $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^{k+2} < \infty$, for k a non-negative integer, then the kth order of Edgeworth expansion for F_n is

$$e_{k,n}(x) = \Phi(x) - \phi(x)\sum_{j=1}^{k} n^{-j/2}\sum_{r=j+2}^{3j} a_{jr}H_{r-1}(x), \tag{1}$$

where Φ is the distribution function of a standardised normal variate, $\phi(x) = \Phi'(x)$, $H_r(x)$ are the Hermite polynomials defined by

$$H_r(x)\phi(x) = (-1)^r\phi^{(r)}(x),$$

and the coefficients a_{jr} are expressible in terms of the moments of X_1 up to the $(k + 2)$th order.

It is well known (see, for example, Bhattacharya and Ranga Rao, 1976) that, (i) if $E|X_1|^3 < C$, then

$$\| F_n - e_{0,n} \| < Cn^{-1/2} \qquad \text{(Berry-Esseen bound)} \tag{2}$$

and (ii) if $E|X_1|^{k+3} < C$ and

$$\limsup_{|t|\to\infty} | Ee^{itX_1} | < 1 \qquad \text{(Cramér's condition)}$$

then

$$\| F_n - e_{k+1,n} \| < \varepsilon_n n^{-(k+1)/2},$$

where $C > \varepsilon_n \to 0$ as $n \to \infty$ and *a fortiori*

$$\| F_n - e_{k,n} \| < Cn^{-(k+1)/2}, \tag{3}$$

where here and in the sequel C is used generically as a constant depending only on F and k, and where we write

$$\| G \| = \sup_x | G(x)|,$$

for any function G: $R \to R$.

The Edgeworth expansion may be obtained formally by an asymptotic expansion of the characteristic function under the moment condition $E|X_1|^{k+3} < C$. Its validity, in the sense of (3), entails a degree of smoothness for F_n. Our theorem shows that a suitably defined degree of smoothness of F_n is both necessary and sufficient for (3) when $E|X_1|^{k+3} < C$. Although this result does not give a practical criterion, it does pinpoint the relation

Received December 1980.
[1] Prepared with the partial support of ONR contract N00014-80C-0163 and the Adolph and Mary Spragne Miller Foundation.
AMS 1970 subject classification. Primary 60F05.
Key words and phrases. Edgeworth expansions, central limit theorem.

between the Edgeworth expansion for F_n and smoothness of F_n which is implicit in Cramér's condition.

2. The result. For any $G: R \to R$ and $\sigma > 0$, define the first difference operator Δ_σ by

$$\Delta_\sigma G(x) = G(x + \sigma) - G(x)$$

and define the kth difference operator Δ_σ^k as the kth iterate of this, so that

$$\Delta_\sigma^k G(x) = \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} G(x + j\sigma). \tag{4}$$

The interpolating polynomial to $G(y)$ of degree k at the points $x, x + \sigma, \cdots, x + k\sigma$ is

$$P_{k,\sigma}(y; x, G) = G(x) + \sum_{j=1}^{k} \sigma^{-j}(j!)^{-1}\Delta_\sigma^j G(x) \prod_{i=1}^{j} (y - x - (i-1)\sigma)$$

(see for example Theorem 7.3.3 of Dahlquist and Björk, 1974). Moreover, if G has a bounded $(k + 1)$th derivative then for all y and x

$$| G(y) - P_{k,\sigma}(y; x, G)| \le C \| G^{(k+1)} \|(| y - x |^{k+1} + \sigma^{k+1}). \tag{5}$$

For convenience, write $\sigma_n = n^{-1/2}$.

THEOREM. *Suppose $E | X_1 |^{k+3} < C$. Then the following statements are equivalent:*

(a) *The Edgeworth expansion to k terms is valid in the sense of* (3).

(b) $\exists$ *C such that $\forall$ x, y, n*

$$| F_n(y) - P_{k,\sigma_n}(y; x, F_n)| < C(| y - x |^{k+1} + \sigma_n^{k+1}). \tag{6}$$

PROOF. From Theorem 9.10 of Bhattacharya and Ranga Rao (1976), if $| t | < B\sigma_n^{-1}$, where B is a constant depending only on k and F, then

$$| \hat{F}_n(t) - \hat{e}_{k,n} | < C\sigma_n^{k+1}(| t |^{k+3} + | t |^{3(k+1)})e^{-t^2/4}, \tag{7}$$

where for G a function of bounded variation we denote by $\hat{G}$ its Fourier-Stieltjes transform.

Let Y be a random variable, independent of $X_1, \cdots, X_n$ with distribution function K, such that (i) $EY = 0$, (ii) $E| Y|^{k+1} < C$ and (iii) $\hat{K}(t) = 0$ for $| t| > B$. Let $K_n(x) = K(x/\sigma_n)$. Then

$$F_n{*}K_n(x) = \int_{-\infty}^{\infty} F_n(x - \sigma_n y)\, dK(y)$$

has characteristic function $\hat{F}_n\hat{K}_n$. Now $(\hat{F}_n - \hat{e}_{k,n})\hat{K}_n$ vanishes for $| t | > Bn^{1/2}$, so from the Fourier inversion formula the density corresponding to $(F_n - e_{k,n}){*}K_n$ is

$$(2\pi)^{-1} \int_{-Bn^{1/2}}^{Bn^{1/2}} e^{-itx}(\hat{F}_n(t) - \hat{e}_{k,n}(t))\hat{K}_n(t)\, dt.$$

So

$$\|(F_n - e_{k,n}){*}K_n \| < (2\pi)^{-1} \int_{-Bn^{1/2}}^{Bn^{1/2}} | t |^{-1} | \hat{F}_n(t) - \hat{e}_{k,n}(t)| \, | \hat{K}_n(t)|\, dt < C\sigma_n^{k+1}, \tag{8}$$

from (7).

Let $\pi_{k,\sigma_n}(x, G) = EP_{k,\sigma_n}(x - \sigma_n Y; x, G)$. Then

$$F_n{*}K_n(x) - \pi_{k,\sigma_n}(x, F_n) = \int_{-\infty}^{\infty} [F_n(x - \sigma_n y) - P_{k,\sigma_n}(x - \sigma_n y; x, F_n)]\, dK(y).$$

From (6), the absolute value of the integrand is bounded by $C\sigma_n^{k+1}(1 + |y|^{k+1})$, so

(9) $$\| F_n * K_n - \pi_{k,\sigma_n}(\cdot, F_n)\| < C\sigma_n^{k+1},$$

since $E|Y|^{k+1} < C$. Also, since $e_{k,n}$ has a finite $(k+1)$th derivative, it follows in the same way from (5) that

(10) $$\| e_{k,n} * K_n - \pi_{k,\sigma_n}(\cdot, e_{k,n})\| < C\sigma_n^{k+1}.$$

Now (8), (9) and (10) imply that

(11) $$\| \pi_{k,\sigma_n}(\cdot, F_n) - \pi_{k,\sigma_n}(\cdot, e_{k,n})\| < C\sigma_n^{k+1}.$$

In order to complete the proof we need the following lemma.

LEMMA. *If* (6) *holds then*

(12) $$\| \Delta_{\sigma_n y}^{k+1} F_n \| < C\sigma_n^{k+1}(1 + |y|^{k+1}).$$

PROOF. From (4) and (6),

$$\Delta_{\sigma_n y}^{k+1} F_n(x) = \sum_{j=0}^{k+1} (-1)^{k+1-j}\binom{k+1}{j} F_n(x + j\sigma_n y)$$

$$= \sum_{j=0}^{k+1} (-1)^{k+1-j}\binom{k+1}{j}\left(F_n(x) + \sum_{s=1}^{k} \frac{\Delta_{\sigma_n}^s F_n(x)}{s!} \prod_{i=1}^{s} (jy - i + 1)\right)$$

$$+ \theta\sigma_n^{k+1}(1 + |y|^{k+1}),$$

where $|\theta| < C$. Now

$$\sum_{j=0}^{k+1} (-1)^{k+1-j}\binom{k+1}{j} j^{\ell} = 0,$$

for $\ell = 0, 1, \cdots, k$, so each term in the above sum vanishes and the lemma follows.

From the lemma it is clear that

(13) $$\| \Delta_{\sigma_n y}^{k+s} F_n \| < C\sigma_n^{k+1}(1 + |y|^{k+1}),$$

for $s = 1, 2, \cdots, k$. Taking the sth difference inside the norm in (11) gives

$$\| \Delta_{\sigma_n}^s (F_n - e_{k,n}) - \sum_{j=1}^{k} \frac{\Delta_{\sigma_n}^{j+s}(F_n - e_{k,n})}{j!} E \prod_{i=1}^{k} (Y - i + 1)\| < C\sigma_n^{k+1}.$$

For $s = k$, this, together with (13), implies that

$$\| \Delta_{\sigma_n}^k (F_n - e_{k,n})\| < C\sigma_n^{k+1}.$$

Repeating this for $s = k - 1, \cdots, 1$, we obtain (3).

Conversely, if (3) holds, then

(14) $$\| \Delta_{\sigma_n}^j (F_n - e_{k,n})\| < C\sigma_n^{k+1},$$

for $j = 0, 1, \cdots, k$. Now

$$|F_n(y) - P_{k,\sigma_n}(y; x, F_n)| \le |F_n(y) - e_{k,n}(y)| + |e_{k,n}(y) - P_{k,\sigma_n}(y; x, e_{k,n})|$$
$$+ |P_{k,\sigma_n}(y; x, e_{k,n}) - P_{k,\sigma_n}(y; x, F_n)|.$$

From (3), (5) and (14), each of these terms is bounded by $C(|y - x|^{k+1} + \sigma_n^{k+1})$, so (6) follows.

NOTES. (i) If $k = 0$, $\| \Delta_{\sigma_n} F_n \| < C\sigma_n$ implies the Berry-Esseen bound (2). This was used in a proof of the bound by Chen and Ho (1978), using Stein's method. We did, in fact, originally obtain our result using Stein's method.

(ii) If $k = 1$, it is sufficient to assume that (12) holds in order to obtain (3). To see this let K be symmetric about 0 and write

$$K_n * F_n(x) - F_n(x) = \int_{-\infty}^{\infty} [F_n(x - \sigma_n y) - F_n(x)]\, dK(y)$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} [F_n(x + \sigma_n y) + F_n(x - \sigma_n y) - 2F_n(x)]\, dK(y)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \Delta^2_{\sigma_n y} F_n(x - \sigma_n y)\, dK(y).$$

From (12), the integrand is bounded by $C\sigma_n^2(1 + y^2)$, so (3) follows since $EY^2 < C$.

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N.S.W. 2006
SYDNEY, AUSTRALIA