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CONDITIONAL GENERALIZATIONS OF STRONG LAWS WHICH CONCLUDE THE PARTIAL SUMS CONVERGE ALMOST SURELY[1]

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Suppose that for every independent sequence of random variables satisfying some hypothesis condition H, it follows that the partial sums converge almost surely. Then it is shown that for every arbitrarily-dependent sequence of random variables, the partial sums converge almost surely on the event where the conditional distributions (given the past) satisfy precisely the same condition H. Thus many strong laws for independent sequences may be immediately generalized into conditional results for arbitrarily-dependent sequences.

1. Introduction. If every sequence of independent random variables having property A has property B almost surely, does every arbitrarily-dependent sequence of random variables have property B almost surely on the set where the conditional distributions have property A?

Not in general, but comparisons of the conditional Borel-Cantelli Lemmas, the conditional three-series theorem, and many martingale results with their independent counterparts suggest that the answer is affirmative in fairly general situations. The purpose of this note is to prove Theorem 1, which states, in part, that if "property B" is "the partial sums converge," then the answer is *always* affirmative, *regardless* of "property A." Thus many strong laws for independent sequences (even laws yet undiscovered) may be immediately generalized into conditional results for arbitrarily-dependent sequences.

2. Main Theorem. In this note, $\mathbb{Y} = (Y_1, Y_2, \cdots)$ is a sequence of random variables on a probability triple $(\Omega, \mathscr{A}, \mathscr{P})$, $S_n = Y_1 + Y_2 + \cdots + Y_n$, and $\mathscr{F}_n$ is the sigma field generated by $Y_1, \cdots, Y_n$. Let $\pi_n(\cdot, \cdot)$ be a regular conditional distribution for Y_n given $\mathscr{F}_{n-1}$, and $\Pi = (\pi_1, \pi_2, \cdots)$. Let $\mathscr{B}$ denote the Borel σ-field on $\mathbb{R}$, and $\mathscr{B}^\infty$ the product Borel σ-field on $\mathbb{R}^\infty$; let $\mathscr{P}(\mathbb{R})$ denote the space of probability measures on $(\mathbb{R}, \mathscr{B})$, and let $\mathscr{C} = \mathscr{P}(\mathbb{R}) \times \mathscr{P}(\mathbb{R}) \times \cdots$. (It might help the reader to think of $\mathbb{Y}$ as a random element of $\mathbb{R}^\infty$, and of Π as a random element of $\mathscr{C}$.) As a final convention, let $\mathscr{L}(X)$ denote the distribution of the random variable X.

Let $B \in \mathscr{B}^\infty$. With the above notation, the question this note addresses is: when is the following statement (S) true?

(S) If $A \subset \mathscr{C}$ is such that $(X_1, X_2, \cdots) \in B$ a.s. whenever $X_1, X_2, \cdots$ are independent and $(\mathscr{L}(X_1), \mathscr{L}(X_2), \cdots) \in A$, then for arbitrary $\mathbb{Y}$, $\mathbb{Y} \in B$ a.s. on the set where $\Pi \in A$.

A partial answer is given by the following.

THEOREM 1. (S) *holds in the following three cases*:
(i) $B = \{(r_1, r_2, \cdots) \in \mathbb{R}^\infty : \sum_1^n r_j \text{ converges}\}$;
(ii) $B = \liminf_{n\to\infty}\{(r_1, r_2, \cdots) : r_n \in A_n\}$; *and*
(iii) $B = \limsup_{n\to\infty}\{(r_1, r_2, \cdots) : r_n \in A_n\}$, *where* $A_n \in \mathscr{B}$, $(n = 1, 2, \cdots)$.

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3. Applications of Theorem 1. As a first "application" of Theorem 1, consider the following two well-known conditional results (both of which will be used in the proof of the Theorem): Levy's conditional form of the Borel-Cantelli Lemmas [4, page 249],

(1) For any sequence of random variables $Y_1, Y_2, \cdots$ taking only the values 0 and 1, $\sum_1^\infty Y_n$ is finite (infinite) almost surely where $\sum_1^\infty E(Y_n \mid \mathscr{F}_{n-1})$ is finite (infinite);

and the conditional three-series theorem [e.g., 5, page 66],

(2) For any sequence of random variables $Y_1, Y_2, \cdots$, the partial sums S_n converge almost surely on the event where the three series

$$\sum_1^\infty P(|Y_n| \geq c \mid \mathscr{F}_{n-1}), \sum_1^\infty E[Y_n I(|Y_n| \leq c) \mid \mathscr{F}_{n-1}], \quad \text{and}$$

$$\sum_1^\infty \{E[Y_n^2 I(|Y_n| \leq c) \mid \mathscr{F}_{n-1}] - E^2[Y_n I(|Y_n| \leq c) \mid \mathscr{F}_{n-1}]\} \quad \text{all converge.}$$

Both results (1) and (2) follow immediately from Theorem 1 and their classical counterparts for independent sequences. Similarly, in many martingale theorems the independent case is also the extremal one. As a second application of Theorem 1, for example, note that the following martingale results of Doob [2, page 320] and of Chow [1] follow immediately from (i) and the special case of independence:

(3) If $\{Y_n, \mathscr{F}_n, n \geq 1\}$ is a martingale difference sequence, then S_n converges a.s. where $\sum_1^\infty E[Y_n^2 \mid \mathscr{F}_{n-1}] < \infty$; and

(4) If $\{Y_n, \mathscr{F}_n, n \geq 1\}$ is a martingale difference sequence and $\{b_n, n \geq 1\}$ is a sequence of positive constants such that $\sum_1^\infty b_n < \infty$, then S_n converges almost surely where $\sum_1^\infty b_n^{1-p/2} E[|Y_n|^p \mid \mathscr{F}_{n-1}] < \infty$ for some $p > 2$.

Via Theorem 1 (i), one may deduce immediately a conditional generalization of practically *any* result for sequences of independent random variables in which the conclusion is "S_n converges almost surely." Although the above applications all have hypotheses involving conditional moments, virtually any hypothesis conditions will carry over. As one final example, consider the well-known fact [e.g., 5, page 102] that if $Y_1, Y_2, \cdots$ are independent and S_n converges in probability, then S_n converges almost surely. Theorem 1 allows the generalization of this fact given by Theorem 2 below.

DEFINITION. A sequence of probability measures $\mu_1, \mu_2, \cdots$ on $(\mathbb{R}, \mathscr{B})$ *sums in probability* if, for any independent sequence of random variables $X_1, X_2, \cdots$ with $\mathscr{L}(X_i) = \mu_i$, it follows that $X_1 + \cdots + X_n$ converges in probability.

THEOREM 2. *Let $Y_1, Y_2, \cdots$ be an arbitrarily-dependent sequence of random variables. Then $S_n(\omega)$ converges for almost all ω such that $\pi_1(\omega), \pi_2(\omega), \cdots$ sums in probability.*

4. Proof of Theorem 1. For fixed $B \in \mathscr{B}^\infty$, consider the statement

$$(S') \qquad P(\{\omega : P_{\Pi(\omega)}(B) = 1\} \cap \mathbb{Y} \notin B) = 0,$$

where $P_{\Pi(\omega)}$ is the product measure $\pi_1(\omega) \times \pi_2(\omega) \times \cdots$ on $(\mathbb{R}^\infty, \mathscr{B}^\infty)$.

Without loss of generality, assume $(\Omega, \mathscr{A}, P)$ is complete.

LEMMA 1. (S) $\Leftrightarrow$ (S′).

PROOF. "$\Rightarrow$" Let $A = \{\vec{\mu} \in C : P_{\vec{\mu}}(B) = 1\}$. Then $\{\omega : P_{\Pi(\omega)}(B) = 1\} = \{w : \Pi(\omega) \in A\}$, so $P(\{w : P_{\Pi(\omega)}(B) = 1\} \cap \mathbb{Y} \notin B) = P(\{\omega : \Pi(\omega) \in A\} \cap \mathbb{Y} \notin B) = 0$.

"$\Leftarrow$" Since $\{\omega : \Pi(\omega) \in A\} \subset \{\omega : P_{\Pi(\omega)}(B) = 1\} \in \mathscr{A}$, it follows (by completeness) that $P(\{\omega : \Pi(\omega) \in A\} \cap \mathbb{Y} \notin B) = P(\{\omega : P_{\Pi(\omega)}(B) = 1\} \cap \mathbb{Y} \notin B) = 0$. □

Proof of Theorem 1. For (i), Lemma 1 implies it is enough to show that (S′) holds for $B = \{(r_1, r_2, \cdots) \in \mathbb{R}^\infty : \sum_1^n r_j \text{ converges}\}$. Let $(\hat{\Omega}, \hat{\mathscr{A}}, \hat{P})$ be a copy of $(\Omega, \mathscr{A}, P)$, and

(enlarging this new space if necessary) for each $\omega \in \Omega$, let $Z_1(\omega), Z_2(\omega), \cdots$ be a sequence of independent random variables on $(\hat{\Omega}, \hat{\mathscr{A}}, \hat{P})$ with $\mathscr{L}(Z_n(\omega)) = \pi_n(\omega)$, (that is, $\hat{P}(Z_n(\omega) \in E) = \pi_n(\omega, E) = P(Y_n \in E \mid Y_1(\omega), \cdots, Y_{n-1}(\omega))$. Then

$$
\begin{aligned}
(5)\quad & P(\{\omega : P_{\Pi(\omega)}(B) = 1\} \cap \mathbb{Y} \notin B) \\
&= P(\{\omega : Z_1(\omega) + \cdots + Z_n(\omega) \text{ converges a.s. (in } \hat{\Omega}, \hat{\mathscr{A}}, \hat{P})\} \cap \mathbb{Y} \notin B) \\
&= P(\{\omega : \text{the three series } \textstyle\sum_1^\infty \hat{P}(|Z_n(\omega)| \geq 1), \sum_1^\infty \hat{E}(Z_n(\omega) \cdot I(|Z_n| \leq 1)), \\
&\qquad \text{and } \textstyle\sum_1^\infty \hat{\mathrm{V}}\mathrm{ar}(Z_n(\omega) \cdot I(|Z_n| \leq 1)) \text{ all converge}\} \cap \mathbb{Y} \notin B) \\
&= P(\{\text{event where the three series } \textstyle\sum_1^\infty P(|Y_n| \geq 1 \mid \mathscr{F}_{n-1}), \\
&\qquad \textstyle\sum_1^\infty E(Y_n \cdot I(|Y_n| \leq 1) \mid \mathscr{F}_{n-1}), \text{ and } \sum_1^\infty [E(Y_n^2 \cdot I(|Y_n| \leq 1) \mid \mathscr{F}_{n-1}) - \\
&\qquad E^2(Y_n I(|Y_n| \leq 1) \mid \mathscr{F}_{n-1})] \text{ all converge}\} \cap \mathbb{Y} \notin B) = 0,
\end{aligned}
$$

where the first equality in (5) follows by the definition of $Z_n(\omega)$, the second by Kolmogorov's Three-Series Theorem and independence of the $\{Z_i(\omega)\}$, the third by definition of $Z_n(\omega)$ and π_n, and the last by the conditional three series theorem (2). This completes the proof of (i).

For (ii) and (iii), application of the same technique using, in place of the three-series theorems, the classical and (Levy's) conditional form (1) of the Borel-Cantelli lemmas yields the desired conclusion. □

5. Remarks. The class of sets $B \in \mathscr{B}^\infty$ for which (S) and (S′)hold is not closed under complementation; a counterexample to the converse of the conditional three-series theorem due to Dvoretzky and to Gilat [3] demonstrates that (S) does not hold in general for $B =$ "S_n does not converge."

The following example shows that (S) does not hold for $B =$ "lim sup $S_n/a_n = 1$."

EXAMPLE. Let $Y_n = S_n - S_{n-1}$, where $\{S_n\}$ are i.i.d., $P(S_n = 0) = P(S_n = 1) = \frac{1}{2}$. There are only two possible conditional laws: $\pi^{(+)} = \delta_0/2 + \delta_1/2$ and $\pi^{(-)} = \delta_0/2 + \delta_{-1}/2$. Construct Z_n as in the proof of Theorem 1. Then the *unconditional* distribution of $\{Z_n\}_{n \geq 2}$ is i.i.d. with law $P(Z_n = 1) = P(Z_n = -1) = \frac{1}{4}$, $P(Z_n = 0) = \frac{1}{2}$. So lim sup $\sum_1^n Z_i/a_n = 1$ for $a_n = (n \log \log(n/2))^{1/2}$, while lim sup $\sum_1^n Y_i/a_n = 0$.

Whether (S) holds for "$S_n/n \to 0$" is not known to the author.

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