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LIMIT THEOREMS AND INEQUALITIES FOR THE UNIFORM EMPIRICAL PROCESS INDEXED BY INTERVALS

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The uniform empirical process U_n is considered as a process indexed by intervals. Powerful new exponential bounds are established for the process indexed by both "points" and intervals. These bounds trivialize the proof of the Chibisov-O'Reilly theorem concerning the convergence of the process with respect to $\|\cdot/q\|$-metrics and are used to prove an interval analogue of the Chibisov-O'Reilly theorem. A strong limit theorem related to the well-known Hölder condition for Brownian bridge U is also proved. Connections with related work of Csáki, Eicker, Jaeschke, and Stute are mentioned. As an application we introduce a new statistic for testing uniformity which is the natural interval analogue of the classical Anderson-Darling statistic.

1. The main results. Let $\xi_1, \xi_2, \cdots$ be iid Uniform (0, 1) rv's having empirical df Γ_n and let

$$U_n(t) \equiv n^{1/2}[\Gamma_n(t) - t] \quad \text{for} \quad 0 \le t \le 1 \tag{1.1}$$

denote the empirical process. It is well known that

$$U_n \Rightarrow U \quad \text{as} \quad n \to \infty \tag{1.2}$$

where U denotes a Brownian bridge and $\Rightarrow$ denotes weak convergence.

An alternative form of this result that has certain technical advantages in applications is the so-called *Skorokhod* (1956) *construction*. Thus there exists a triangular array of row independent Uniform (0, 1) rv's $\xi_{n1}, \cdots, \xi_{nn}$, $n \ge 1$, and a Brownian bridge U, all defined on the same sample space, with the property that

$$\|U_n - U\| \to_{\text{a.s.}} 0 \quad \text{as } n \to \infty \text{ for the Skorokhod construction} \tag{1.3}$$

where U_n is the empirical process of $\xi_{n1}, \cdots, \xi_{nn}$ and where $\|\cdot\|$ denotes the sup norm.

From either (1.2) or (1.3) one can conclude the Mann-Wald result

$$T(U_n) \to_d T(U) \text{ for any functional } T \text{ that is } \|\cdot\|\text{-continuous a.s. } U \tag{1.4}$$

and for which $T(U_n)$, $n \ge 1$, and $T(U)$ are measurable. This is highly useful, but many useful functionals T are not $\|\cdot\|$-continuous. Many are, however, continuous in the $\|\cdot/q\|$ metric we now define. Let

$$\begin{aligned} &Q^* \equiv \{q : q > 0 \text{ on } (0, 1) \text{ is continuous, } \nearrow \text{ on } [0, \tfrac12] \text{ and symmetric about } t = \tfrac12\} \\ &Q \equiv \{q \in Q^* : t^{-1/2}q(t) \quad \text{is } \searrow \text{ on } [0, \tfrac12]\}. \end{aligned} \tag{1.5}$$

For any such q we define the $\|\cdot/q\|$ distance between functions f and g to be $\|(f-g)/q\|$.

For any interval $C = (s, t]$ with $0 \le s \le t \le 1$ we define $f(C)$ or $f(s, t]$ to be $f(t) - f(s)$ and we let $|C| \equiv t - s$. Let $\mathscr{C}$ denote the class of all such intervals. Let $\mathscr{C}(a, b) \equiv \{C \in \mathscr{C} : a \le |C| \le b\}$, and let $\mathscr{C}(a) \equiv \mathscr{C}(a, 1)$. For any subcollection $\mathscr{C}' \subset \mathscr{C}$ we let

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$\| h \|_{\mathscr{C}'} \equiv \sup\{ | h(C) | : C \in \mathscr{C}' \}$. Let $\| h \|_a^b \equiv \sup_{a \le t \le b} | h(t) |$. Let h^+ and h^- denote the positive and negative parts of h. Let I denote the identity function.

CLT results for points and intervals. The following theorem on *point convergence of weighted processes* is due to Chibisov (1964) and O'Reilly (1974). (Replacing (1.12) by (1.7) is found in Shorack (1979)).

THEOREM 1.1. *Let $q \in Q^*$. Then*

$$\|(U_n - U)/q\| \to_p 0 \quad \textit{as} \quad n \to \infty \textit{ for the Skorokhod construction} \tag{1.6}$$

if and only if

$$g(t) \equiv q^2(t)/[t \log\log(1/t)] \to \infty \quad \textit{as} \quad t \downarrow 0. \tag{1.7}$$

The next theorem on *interval convergence of weighted processes* is new.

THEOREM 1.2. *Let $q \in Q^*$. Then the Skorokhod construction* (1.3) *satisfies*

$$\sup_{\{C : \varepsilon n^{-1} \log n \le |C|\}} \frac{| U_n(C) - U(C) |}{q(|C|)} \to_p 0 \quad \text{as} \quad n \to \infty \tag{1.8}$$

for every $\varepsilon > 0$ if and only if

$$g(t) \equiv q^2(t)/[t \log(1/t)] \to \infty \quad \textit{as} \quad t \downarrow 0. \tag{1.9}$$

Bounding $|C|$ away from 0 in (1.8) is essential; if C shrinks down to just the single point ξ_1, say, then $U_n(C)/q(|C|) = n^{1/2}(n^{-1} - 0)/0 = \infty$ since all interesting q's have $q(0) = 0$.

REMARK ON (1.7). The Feller-Erdös-Kolmogorov-Petrovski criterion found in Ito and McKean (1965, page 33) implies that for $q \in Q$ we have

$$P(U(t) < q(t),\, t \downarrow 0) = \begin{cases} 0 \\ 1 \end{cases} \text{as} \quad \int_{0+} \frac{1}{t} \frac{q(t)}{t^{1/2}} \exp\left(-\frac{q^2(t)}{2t} \right) dt \begin{cases} = \infty \\ < \infty. \end{cases} \tag{1.10}$$

If the integral is finite (infinite), call such q *point upper class* (*point lower class*) for U. Now for $q \in Q^*$, Condition (1.7) is (as seen in the proof of Theorem 1.1) equivalent to

$$\varepsilon q \quad \text{is point upper-class for} \quad U \quad \text{for all} \quad \varepsilon > 0, \tag{1.11}$$

and to (the condition used by Chibisov and O'Reilly)

$$\int_{0+} t^{-1} \exp(-\lambda q^2(t)/t)\, dt < \infty \quad \text{for all} \quad \lambda > 0. \tag{1.12}$$

REMARK ON (1.9). The Chung, Erdös, and Sirao criterion found in Ito and McKean (1965, page 36) implies that for $q \in Q$

$$P(\sup_{|C| = \varepsilon} U(C) < q(\varepsilon),\, \varepsilon \downarrow 0) = \begin{cases} 0 \\ 1 \end{cases} \text{as} \quad \int_{0+} \frac{1}{t^2} \left[\frac{q(t)}{t^{1/2}} \right]^3 \exp\left(-\frac{q^2(t)}{2t} \right) dt \begin{cases} = \infty \\ < \infty. \end{cases} \tag{1.13}$$

If the integral is finite (infinite), call such q *interval upper-class* (*interval lower-class*) for U. We will show below that for $q \in Q^*$, (1.9) is equivalent to

$$\varepsilon q \text{ is interval upper-class for } U \quad \text{for all} \quad \varepsilon > 0, \tag{1.14}$$

and to

$$\int_{0+} t^{-2} \exp(-\lambda q^2(t)/t)\, dt \le \infty \quad \text{for all} \quad \lambda > 0. \tag{1.15}$$

Inequalities. We next present a powerful new inequality that trivializes the proof of (1.6) via (1.12); thus (1.23) is seen to be the "right inequality." Define h on $[0, \infty)$ and ψ on $[-1, \infty)$ by

$$h(\lambda) \equiv \lambda(\log \lambda - 1) + 1 \quad \text{and} \quad \psi(\lambda) \equiv 2h(1+\lambda)/\lambda^2 \tag{1.16}$$

so that

$$h(1+\lambda) = \int_0^\lambda \log(1+x)\,dx \quad \text{is} \uparrow \text{ for } \lambda > 0, \tag{1.17}$$

$$\psi \text{ is } \downarrow \text{ for } \lambda \geq 0 \text{ with } \psi(0) = 1, \tag{1.18}$$

$$\psi(\lambda) \sim (2\log\lambda)/\lambda \quad \text{as} \quad \lambda \to \infty \tag{1.19}$$

and, more crudely,

$$\psi(\lambda) \geq 1/(1 + \lambda/3) \tag{1.20}$$

$$\geq \begin{cases} 1-\delta & \text{for } 0 \leq \lambda \leq 3\delta, \quad (1.21) \\ 3\delta(1-\delta)/\lambda & \text{for } \lambda \geq 3\delta. \quad (1.22) \end{cases}$$

These functions arise in Bennett (1962) where exponential bounds on binomial rv's are obtained. See also Shorack (1980).

INEQUALITY 1.1. Let $q \in Q$, $0 \leq a \leq (1-\delta)b < b \leq \delta \leq \frac{1}{2}$ and $\lambda > 0$. Then

$$P(\| \mathbb{U}_n^{\pm}/q \|_a^b \geq \lambda) \leq \frac{3}{\delta}\int_a^b \frac{1}{t}\exp\left(-(1-\delta)^2\gamma^{\pm}\frac{\lambda^2}{2}\frac{q^2(t)}{t}\right)dt \tag{1.23}$$

where (recall (1.18) through (1.22) also)

$$\gamma^- \equiv 1 \geq \psi(\lambda) \quad \text{always works, and} \tag{1.24}$$

$$\gamma^+ \equiv \psi(\lambda q(a)/(a\sqrt{n})) \quad \text{always works} \tag{1.25}$$

$$\geq \psi(\lambda) \qquad \text{if} \quad a \geq q^2(1/n) \vee (1/n). \tag{1.26}$$

COROLLARY 1. *When* $q(t) = \sqrt{t}$ *we have*

$$P(\| \mathbb{U}_n^{\pm}/\sqrt{I} \|_a^b \geq \lambda) \leq \frac{3\log(b/a)}{\delta}\exp\left(-(1-\delta)^2\gamma^{\pm}\frac{\lambda^2}{2}\right) \tag{1.27}$$

where (1.24) *and* (1.25) *still hold* (*recall* (1.18) *through* (1.22) *also*).

The distinction between γ^- and γ^+ is due to the clean exponential bound on the short left tail of the Binomial (n, t) rv $\mathbb{U}_n(t)$ with $t \leq \frac{1}{2}$ and the necessarily messy exponential bound on the long, troublesome right tail.

If b is $\searrow 0$, it is an advantage to use a fixed small positive constant δ in (1.23) and (1.27) rather than $\delta = b$; the exponent doesn't matter either way, but the δ-term out front stays bounded.

For comparison, we note that Ito and McKean give a result for Brownian motion $\mathbb{B}$ that under the correspondence $(1+t)\mathbb{U}(t/(1+t)) \cong \mathbb{B}(t)$ yields, for $q \in Q$,

$$P(\| \mathbb{U}/q \|_0^b \geq \lambda) \leq 2\int_0^b \frac{\lambda q(t)/|t(1-t)|^{1/2}}{\sqrt{2\pi}t(1-t)}\exp\left(-\frac{\lambda^2 q^2(t)}{2t(1-t)}\right)dt. \tag{1.28}$$

We next present another new inequality that similarly leads to (1.8) via (1.15). Although we are no longer able to separate the positive and negative parts, the inequality (1.29) is seen to be the "right inequality" for intervals. (The constant γ^+ has been improved by a factor of 3^2 from the one in the original version of this paper after the authors saw Stute

(1982); our only byproduct, though, is that smaller constants would be possible in Theorem 3 had we chosen to evaluate the lim sups. At the same time, we have used our approach to modify Stute's inequalities; see (1.32) and (1.33) for a result to compare to his Lemma 2.5. His other key Lemma 2.4 should be compared to our inequality 3.2.)

INEQUALITY 1.2. Let $q \in Q$, $0 \le a \le (1-\delta)b < b \le \delta \le \frac{1}{2}$ and $\lambda > 0$. Then

$$P\left(\sup_{a \le |C| \le b} \frac{|U_n(C)|}{q(|C|)} \ge \lambda\right) \le \frac{24}{\delta^3} \int_a^b \frac{1}{t^2} \exp\left(-(1-\delta)^4 \gamma \frac{\lambda^2}{2} \frac{q^2(t)}{t}\right) dt \tag{1.29}$$

where (recall (1.18) through (1.22) also)

$$\gamma \equiv \psi\left(\frac{2^{1/2}\lambda q(a)}{\delta a \sqrt{n}}\right) \tag{1.30}$$

$$\ge \psi(2^{1/2}\lambda/\delta) \quad \text{if} \quad a \ge q^2(1/n) \vee (1/n). \tag{1.31}$$

COROLLARY 2. *When* $q(t) = \sqrt{t}$ *we have*

$$P\left(\sup_{a \le |C| \le b} \frac{|U_n(C)|}{|C|^{1/2}} \ge \lambda\right) \le \frac{24}{a\delta^3} \exp\left(-(1-\delta)^4 \gamma \frac{\lambda^2}{2}\right) \tag{1.32}$$

where (1.30) *holds* (*recall* (1.18) *through* (1.22)). *In particular* (*crudely*)

$$\gamma \ge \begin{cases} 1-\delta & \text{if } \lambda \le (3/2^{1/2})\delta^2\sqrt{an} \quad (1.33) \\ \dfrac{3\delta^2\sqrt{an}(1-\delta)}{2^{1/2}\lambda} & \text{if } \lambda \ge (3/2^{1/2})\delta^2\sqrt{an}. \quad (1.34) \end{cases}$$

Although we do not know an exact analogue of (1.28) in the case of intervals, by use of Lemma 1 of Csörgö and Révész (1979) we obtain the following inequality: for every $\delta > 0$ there exists a $M = M(\delta) > 0$ such that for $q \in Q$

$$P\left(\sup_{|C| \le b} \frac{|B(C)|}{q(|C|)} \ge \lambda\right) \le M \int_0^b t^{-2} \exp(-(1-\delta)^2 \lambda^2 q^2(t)/2t)\, dt \tag{1.35}$$

where B denotes standard Brownian motion.

Strong limit theorems for intervals. Now we turn to a theorem for the normalized process $\{U_n(C)/|C|^{1/2} : C \in C\}$. Cassels (1951) proved that

$$\limsup_{n\to\infty} \sup_{C \in \mathscr{C}} \left\{\frac{\pm U_n(C)}{(2 \log\log n)^{1/2}} - [|C|(1-|C|)]^{1/2}\right\} = 0 \quad \text{a.s.} \tag{1.36}$$

As noted by Wichura (1973, page 282) and Philipp (1977, page 325), Cassels's theorem is an immediate consequence of the functional law of the iterated logarithm for $\{U_n(t)/(2 \log \log n)^{1/2} : 0 \le t \le 1\}$. Our theorem will be stronger than (1.36) in the sense that the $|C|^{1/2}$ (or $[|C|(1-|C|)]^{1/2}$) is moved to the denominator of the first term; the resulting price is an increase in the magnitude of the sequence of constants from $(\log \log n)^{1/2}$ to $(\log n)^{1/2}$.

THEOREM 1.3.

$$\limsup_{n\to\infty} \sup_{C \in \mathscr{C}(n^{-1}\varepsilon \log n)} \frac{|U_n(C)|}{(|C| \log n)^{1/2}} < \infty \quad \text{a.s.} \quad \textit{for any} \quad \varepsilon > 0. \tag{1.37}$$

$$\limsup_{n\to\infty} \sup_{C \in \mathscr{C}(n^{-1}(\log n)^{\alpha})} \frac{|U_n(C)|}{|C|^{1/2}} \cdot \frac{\log \log n}{(\log n)^{1-\alpha/2}} < \infty \quad \text{a.s.} \tag{1.38}$$

for each $-\infty < \alpha < 1$.

Stute (1980) shows that if $a_n \to 0$, $na_n/\log(a_n^{-1}) \to \infty$, and $\log(a_n^{-1})/\log\log n \to \infty$ (e.g. $a_n = n^{-\lambda}$ with $0 < \lambda < 1$), then

$$\lim_{n\to\infty}\sup_{C\in\mathscr{C}(\underline{\tau}a_n,\bar{\tau}a_n)} \frac{|U_n(C)|}{\left(|C|\log\frac{1}{|C|}\right)^{1/2}} = 2^{1/2} \quad a.s. \tag{1.39}$$

for all $0 < \underline{\tau} < \bar{\tau} < \infty$. This should be compared with the known Hölder condition

$$\lim_{\varepsilon\searrow 0}\sup_{\{C:|C|=\varepsilon\}} \frac{|U(C)|}{\left(|C|\log\frac{1}{|C|}\right)^{1/2}} = 2^{1/2} \quad \text{a.s.} \tag{1.40}$$

However, Stute's methods do not seem to yield (1.37) since they do not allow intervals as short as $a_n = (\log n)/n$; note that $na_n/\log(a_n^{-1}) \to 1$ for this a_n.

Theorem 1.3 also deserves to be compared with the results of Csàki (1977) and Shorack (1980) for the normalized empirical process:

$$\limsup_{n\to\infty} \frac{\| U_n^-/(I(1-I))^{1/2}\|_0^{1/2}}{(\log\log n)^{1/2}} = 2 \quad \text{a.s.,} \tag{1.41}$$

$$\limsup_{n\to\infty} \frac{\| U_n^+/(I(1-I))^{1/2}\|_{n^{-1}\log\log n}^{1/2}}{(\log\log n)^{1/2}} = 2 \quad \text{a.s.,} \tag{1.42}$$

and

$$\limsup_{n\to\infty} \frac{\| U_n/(I(1-I))^{1/2}\|_{n^{-1}\log\log n}^{1/2}}{(\log\log n)^{1/2}} = 2 \quad \text{a.s.} \tag{1.43}$$

Interval versions of the results of Jaeschke (1979) and Eicker (1979) may also be possible; (1.27) and (1.32) may be useful in this connection.

2. An application. Watson's (1961) version of the Cramér-von Mises statistic for testing uniformity, T_n^2, may be written as

$$T_n^2 = \tfrac{1}{2}\int_0^1\int_0^1 [U_n(t) - U_n(s)]^2\, ds\, dt \tag{2.1}$$

$$= \int_0^1 [U_n(t) - \int_0^1 U_n\, dI]^2\, dt \tag{2.2}$$

$$\to_d \int_0^1 [U(t) - \int_0^1 U\, dI]^2\, dt = \tfrac{1}{2}\int_0^1\int_0^1 [U(t) - U(s)]^2\, ds\, dt; \tag{2.3}$$

see Durbin (1973, page 36). The process $U(t) - \int_0^1 U\, dI$ in (2.3) has constant variance so that weighting of the Anderson-Darling type seems inappropriate in (2.2) and (2.3). But Var$[U_n(t) - U_n(s)] = |t-s|(1-|t-s|)$, and hence weighting is appropriate in (2.1). We therefore define an "interval" version of the Anderson-Darling statistic by

$$S_n^2 \equiv \int_0^1\int_0^1 \frac{[U_n(t) - U_n(s)]^2}{|t-s|(1-|t-s|)}\, ds\, dt = n\int_0^1\int_0^1 \frac{[\Gamma_n(t) - \Gamma_n(s) - (t-s)]^2}{|t-s|(1-|t-s|)}\, ds\, dt. \tag{2.4}$$

THEOREM 2.1. *Under the null hypothesis of uniformity,*

$$S_n^2 \to_d S^2 \equiv \int_0^1\int_0^1 \frac{[U(t) - U(s)]^2}{|t-s|(1-|t-s|)}\, ds\, dt. \tag{2.5}$$

PROOF. Let $c_n \equiv n^{-1} \log n$, and break $[0, 1] \times [0, 1]$ into $D_{n0} \cup D_{n1}$ where

$$D_{n1} \equiv \{(s, t) : |t - s| \le c_n\}.$$

Let $q(t) \equiv [t(1-t)]^{1/4}$. Then it follows from Theorem 1.2 (and elementary calculations for the region D_{n1}) that, for the Skorokhod construction,

$$|S_n^2 - S^2| \le \int_{D_{n0}}\int \frac{|[U_n(t) - U_n(s)]^2 - [U(t) - U(s)]^2|}{|t-s|(1-|t-s|)}\, ds\, dt$$

$$+ \left(\int_{D_{n1}}\int \frac{[U_n(t) - U_n(s)]^2}{|t-s|(1-|t-s|)}\, ds\, dt + \int_{D_{n1}}\int \frac{[U(t) - U(s)]^2}{|t-s|(1-|t-s|)}\, ds\, dt\right)$$

$$\le \|(U_n - U)/q\|_{\mathscr{C}(c_n)} \cdot \{\|U_n/q\|_{\mathscr{C}(c_n, 1-c_n)} + \|U/q\|_{\mathscr{C}}\}$$

$$\cdot \int_0^1 \int_0^1 [|t-s|(1-|t-s|)]^{-1/2}\, ds\, dt + o_p(1)$$

$$= o_p(1) O_p(1) O(1) + o_p(1) = o_p(1). \quad \Box$$

The following computational formulas for the statistic S_n^2 are easily obtained by straightforward integration: Let $D_{ij} \equiv |\xi_i - \xi_j|$ for $1 \le i, j \le n$, and let $0 \le \xi_{n1} \le \cdots \le \xi_{nn} \le 1$ denote the order statistics corresponding to the first n ξ's. Then

$$S_n^2 = n + \frac{2}{n} \sum_{i=1}^n \sum_{j=1}^n \{D_{ij}\log(D_{ij}) + (1 - D_{ij})\log(1 - D_{ij})\}$$

$$(2.6) \qquad = n + \frac{4}{n} \sum_{i=1}^n \sum_{j=i+1}^n \{(\xi_{nj} - \xi_{ni})\log(\xi_{nj} - \xi_{ni})$$

$$+ (1 - (\xi_{nj} - \xi_{ni}))\log(1 - (\xi_{nj} - \xi_{ni}))\}.$$

The statistic S_n^2 is potentially suitable for testing uniformity, or goodness of fit, if its asymptotic null distribution, that of S^2 in (2.5), were known. The distribution is undoubtedly that of an infinite weighted sum of independent Chi squared random variables, but we do not know the weights.

3. Proof of the inequalities. Using the binomial exponential bounds due to Bennett (1962) and Hoeffding (1963), and the fact that $U_n(t)/(1-t)$ is a martingale in t (for each $n \ge 1$), Shorack (1980) improved results of James (1975) to give for $b \le \frac{1}{2}$ that

$$(3.1) \qquad P(\|U_n^-/(1-I)\|_0^b \ge \lambda/(1-b)) \le \exp(-\lambda^2/2b(1-b)),$$

$$(3.2) \qquad P(\|U_n^+/(1-I)\|_0^b \ge \lambda/(1-b)) \le \exp(-\lambda^2\psi(\lambda/bn^{1/2})/2b(1-b)),$$

$$(3.3) \qquad P(\|U_n/(1-I)\|_0^b \ge \lambda/(1-b)) \le 2\exp(-\lambda^2\psi(\lambda/bn^{1/2})/2b(1-b)).$$

PROOF OF INEQUALITY 1.1. Let

$$(a) \qquad A_n^{\pm} \equiv (\|U_n^{\pm}/q\|_a^b \ge \lambda).$$

Define

$$(b) \qquad \theta \equiv 1 - \delta$$

and integers $0 \le J \le K$ (note that $J \ge 2$ since θ is $\ge \frac{1}{2}$ and $b \le \frac{1}{2}$) by

$$(c) \qquad \theta^K < a \le \theta^{K-1} \quad \text{and} \quad \theta^J < b \le \theta^{J-1}.$$

(From here on, θ^i denotes θ^i for $J \le i < K$, but θ^K denotes a and θ^{J-1} denotes b.) Since q

is $\nearrow$ we have

$$\text{(d)}\qquad \begin{aligned} A_n^{\pm} &\subset \left(\max_{J\le i\le K} \sup_{\theta^i \le t\le \theta^{i-1}} \frac{U_n^{\pm}(t)}{q(t)} \ge \lambda\right) \\ &\subset \left(\max_{J\le i\le K} \sup_{\theta^i \le t\le \theta^{i-1}} \frac{U_n^{\pm}(t)}{q(\theta^i)} \ge \lambda\right) \end{aligned}$$

so that

$$\text{(e)}\qquad P(A_n^{\pm}) \le \sum_{i=J}^{K} P\left(\left\|\frac{U_n^{\pm}}{1-I}\right\|_0^{\theta^{i-1}} \ge \lambda q(\theta^i)\frac{(1-\theta^{i-1})}{(1-\theta^{i-1})}\right).$$

We first consider A_n^-. Now

$$\begin{aligned} P(A_n^-) &\le \sum_{i=J}^{K} \exp\left(-\frac{\lambda^2}{2}\frac{q^2(\theta^i)(1-\theta^{i-1})^2}{\theta^{i-1}(1-\theta^{i-1})}\right) \qquad \text{by (3.1)} \\ &\le \sum_{i=J}^{K} \exp\left(-\frac{\lambda^2}{2}\frac{q^2(\theta^i)}{\theta^{i-1}}\theta\right) \quad \text{by (b) and below (c)} \end{aligned}$$

$$\text{(f)}\qquad \begin{aligned} &\le \sum_{i=J+1}^{K-1} \frac{1}{1-\theta}\int_{\theta^i}^{\theta^{i-1}} \frac{1}{t}\exp\left(-\frac{\lambda^2}{2}\frac{q^2(t)}{t}\theta^2\right)dt \\ &\quad + \exp\left(-\frac{\lambda^2}{2}\frac{q^2(a)}{a}\theta^2\right) + \exp\left(-\frac{\lambda^2}{2}\frac{q^2(\theta b)}{\theta b}\theta^2\right) \end{aligned}$$

$$\text{(g)}\qquad \le \frac{3}{\delta}\int_a^b \frac{1}{t}\exp\left(-(1-\delta)^2\frac{\lambda^2}{2}\frac{q^2(t)}{t}\right)dt$$

by (b), (c) and the fact that $\int_{\theta b}^{b} t^{-1}\,dt/\delta \ge (\log(1/\theta))/\delta \ge 1$ follows from $a \le (1-\delta)b$ and $\delta \le ½$. This completes the proof in the "−" case.

We now consider A_n^+. From (e), (f) and (3.2) we have

$$\text{(h)}\qquad P(A_n^+) \le \sum_{i=J}^{K} \exp\left(-\frac{\lambda^2}{2}\frac{q^2(\theta^i)}{\theta^{i-1}}\theta\psi\left(\frac{\lambda q(\theta^i)(1-\theta^{i-1})}{\theta^{i-1}}\frac{1}{\sqrt{n}}\right)\right).$$

But $a = \theta^K \le \theta^i$ for $i \le K$ and $q(t)/t$ is $\searrow$, so

$$\text{(i)}\qquad \frac{q(\theta^i)(1-\theta^{i-1})}{\theta^{i-1}} \le \frac{q(\theta^i)}{\theta^i} \le \frac{q(a)}{a}.$$

Thus, as in case A^-, we have

$$\text{(j)}\qquad P(A_n^+) \le \frac{3}{\delta}\int_a^b \frac{1}{t}\exp\left(-\frac{\lambda^2}{2}\frac{q(t)}{t}\theta^2\psi\left(\frac{\lambda q(a)}{a\sqrt{n}}\right)\right)dt$$

completing the proof in the "+" case. We need only remark that for (1.26) we use

$$\text{(k)}\qquad \frac{q(a)}{a\sqrt{n}} = \frac{q(a)}{\sqrt{a}}\frac{1}{\sqrt{na}} \le \frac{q(1/n)}{\sqrt{1/n}}\frac{1}{\sqrt{na}} \le 1. \quad \square$$

Proof of Inequality 1.2. Let

$$\text{(a)}\qquad A_n \equiv \left[\sup_{a\le|C|\le b}\frac{|U_n(C)|}{q(|C|)} \ge \lambda\right].$$

Define (see Bolthausen, 1977)

$$\text{(b)}\qquad \theta = (1-\delta)$$

and integers $J \le K$ by (if $a = 0$, then $K = \infty$ and we consider $J \le i < K$)

$$\theta^K < a \le \theta^{K-1} \quad \text{and} \quad \theta^J < b \le \theta^{J-1} \tag{c}$$

(we again let θ^i denote θ^i for $J \le i < K$ while θ^K denotes a and θ^{J-1} denotes b); and let

$$\mathscr{C}_i \equiv \{C : \theta^i \le |C| \le \theta^{i-1}\} \quad \text{for} \quad J \le i \le K. \tag{d}$$

Since q is $\nearrow$ we have

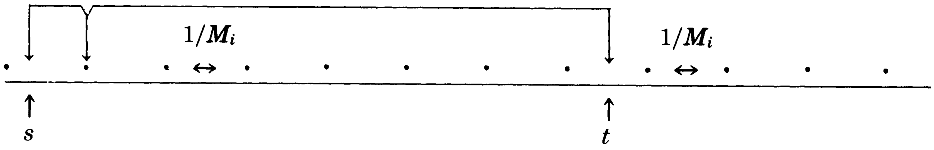
$$A_n \subset \left[\max_{J \le i \le K} \sup_{C \in \mathscr{C}_i} \frac{|\mathbb{U}_n(C)|}{q(\theta^i)} \ge \lambda\right]. \tag{e}$$

Now for any integer M_i

$$\max_{J \le i \le K} \sup_{C \in \mathscr{C}_i} \frac{|\mathbb{U}_n(C)|}{q(\theta^i)}$$

$$\le \max_{J \le i \le K} \max_{0 \le j \le M_i - 1} \sup_{0 \le t \le \theta^{i-1}} \left[\frac{\left|\mathbb{U}_n\left(\frac{j}{M_i}, \frac{j}{M_i} + t\right]\right|}{q(\theta^i)}\right] \tag{f}$$

$$+ \max_{J \le i \le K} \max_{0 \le j \le M_i - 1} \sup_{0 \le t \le 1/M_i} \left[\frac{\left|\mathbb{U}_n\left(\frac{j}{M_i} - t, \frac{j}{M_i}\right]\right|}{q(\theta^i)}\right]$$



but we will suppose that M_i is the smallest integer such that

$$1/M_i \le \delta^2 \theta^i, \tag{g}$$

and this entails for $J \le i \le K$ that

$$M_i < \frac{1}{\delta^2 \theta^i} + 1 < \frac{2}{\delta^2 \theta \theta^{i-1}} \le \frac{4}{\delta^2 \theta^{i-1}}. \tag{h}$$

(The use of M_i for the finer partition of (g) comes from Stute's version of inequality 3.1 below.) Thus, using the stationary increments of $\mathbb{U}_n$,

$$P(A_n) \le \sum_{i=J}^K \sum_{j=0}^{M_i - 1} P\left(\left\|\frac{\mathbb{U}_n}{1 - I}\right\|_0^{\theta^{i-1}} \ge \lambda q(\theta^i) \frac{1 - \theta^{i-1}}{1 - \theta^{i-1}} \frac{1}{1 + \delta}\right) \tag{i}$$

$$+ \sum_{i=J}^K \sum_{j=0}^{M_i - 1} P\left(\left\|\frac{\mathbb{U}_n}{1 - I}\right\|_0^{1/M_i} \ge \lambda q(\theta^i) \frac{1 - 1/M_i}{1 - 1/M_i} \frac{\delta}{1 + \delta}\right) \equiv B + D.$$

Now by (3.3) we have

$$B \le \sum_{i=J}^K M_i 2 \exp\left(-\frac{\lambda^2 q^2(\theta^i)(1 - \theta^{i-1})^2}{2\theta^{i-1}(1 - \theta^{i-1})} \frac{1}{(1 + \delta)^2} \psi\left(\frac{\lambda q(\theta^i)(1 - \theta^{i-1})}{\theta^{i-1}\sqrt{n}} \frac{1}{1 + \delta}\right)\right) \tag{j}$$

$$\le \sum_{i=J}^K \frac{4}{\delta^2 \theta^i} \exp\left(-\frac{\lambda^2}{2} \frac{q^2(\theta^i)}{\theta^{i-1}} (1 - \delta)^3 \psi\left(\frac{\lambda q(\theta^i)}{\theta^i \sqrt{n}}\right)\right) \quad \text{by (h)}$$

since $a = \theta^K \le \theta^i$ for $i \le K$ and $q(t)/t \searrow$ implies for $J \le i \le K$ that

$$\frac{q(\theta^i)}{\theta^{i-1}} \frac{(1 - \theta^{i-1})}{(1 + \delta)} \le \frac{q(\theta^i)}{\theta^i} \le \frac{q(a)}{a}. \tag{k}$$

Thus, using that ψ is $\searrow$ and $q^2(t)/t \searrow$,

$$B \le \frac{4}{\delta^2} \sum_{i=J+1}^{K-1} \frac{1}{1-\theta} \int_{\theta^i}^{\theta^{i-1}} \frac{1}{t^2} \exp\left(-\frac{\lambda^2}{2}\frac{q^2(t)}{t}(1-\delta)^4\gamma\right) dt$$

$$\text{(l)} \qquad + \frac{4}{\delta^2\theta^K} \exp\left(\frac{\lambda^2}{2}\frac{q^2(a)}{a}(1-\delta)^4\gamma\right) + \frac{4}{\delta^2 b\theta} \exp\left(-\frac{\lambda^2}{2}\frac{q^2(\theta^J)}{\theta b}(1-\delta)^4\gamma\right)$$

$$\le \frac{12}{\delta^3} \int_a^b \frac{1}{t^2}\, exp\left(-\frac{\lambda^2}{2}\frac{q^2(t)}{t}(1-\delta)^4\gamma\right) dt$$

since $\int_{b\theta}^b t^{-2}\,dt/\delta \ge (b - b\theta)/\theta b^2\delta \ge 1/b\theta$ and $\int_a^{a/\theta} t^{-2}\,dt/\delta = 1/a$. Likewise (note that $\theta^{i/2}$ means $a^{1/2}$ when $i = K$)

$$D \le \sum_{i=J}^K M_i P\left(\left\|\frac{U_n}{1-I}\right\|_0^{1/M_i} \ge \frac{\lambda q(\theta^i)(1-1/M_i)}{\theta^{i/2}(1+\delta)}\delta\theta^{i/2}\frac{1}{1-1/M_i}\right)$$

$$\le \sum_{i=J}^K M_i P\left(\left\|\frac{U_n}{1-I}\right\|_0^{1/M_i} \ge \frac{\lambda q(\theta^i)}{\theta^{i/2}}(1-\delta)^{3/2}\sqrt{\frac{1}{M_i}}\frac{1}{1-1/M_i}\right) \quad \text{by (9)}$$

$$\text{(m)} \qquad \le \sum_{i=J}^K 2M_i \exp\left(-\frac{\lambda^2 q^2(\theta^i)}{\theta^i}(1-\delta)^3\frac{1}{M_i}\frac{1}{2(1/M_i)(1-1/M_i)}\right.$$

$$\left.\cdot\psi\left(\frac{\lambda q(\theta^i)}{\theta^{i/2}}(1-\delta)^2\sqrt{\frac{1}{M_i}}\frac{1}{(1/M_i)\sqrt{n}}\right)\right)$$

$$\text{(n)} \qquad \le \frac{12}{\delta^3}\int_a^b \frac{1}{t^2}\exp\left(-\frac{\lambda^2}{2}\frac{q^2(t)}{t}(1-\delta)^4\gamma\right) dt$$

using in (m) that ψ is $\searrow$ and that (k) and (g) imply

$$\text{(o)} \qquad \frac{\lambda q(\theta^i)}{\theta^i}(1-\delta)^2\sqrt{M_i\theta^i}\frac{1}{\sqrt{n}} \le \frac{\lambda q(\theta^i)}{\theta^i\sqrt{n}}\frac{\sqrt{2}}{\delta} < \frac{\lambda q(a)}{a\sqrt{n}}\frac{\sqrt{2}}{\delta}. \quad \square$$

We now give an alternative to inequality (2.1) of Stute (1982).

INEQUALITY 3.1. *Let* $0 < a \le b < 1$ *and* $\lambda > 0$. *Then, with* $d = \sqrt{a/b}(1-b)/(1-a)$,

$$\text{(3.4)} \qquad P(\|U_n^{\pm}/\sqrt{I}\|_a^b \ge \lambda) \le \{\textit{any standard bound on: } P(|U_n^{\pm}(b)|/\sqrt{b} \ge \lambda d)\}.$$

PROOF. Now

$$P\left(\left\|\frac{U_n^{\pm}}{\sqrt{I}}\right\|_a^b \ge \lambda\right) \le P\left(\left\|\frac{U_n^{\pm}}{1-I}\right\|_a^b \ge \lambda\frac{\sqrt{a}}{1-a}\frac{(1-b)}{(1-b)}\right)$$

$$\text{(a)} \qquad \le P(\|U_n^{\pm}/(1-I)\|_a^b \ge (\lambda\sqrt{b})\, d/(1-b))$$

$$\text{(b)} \qquad \le \inf_{r>0}\exp(-r\lambda d\sqrt{b/n})E\exp(\pm rU_n(b)/\sqrt{n})$$

by the proof of (3.1) through (3.3) in Shorack (1980). Thus (3.1) through (3.3) give some examples for (3.4). All binomial exponential bounds we know are derived from (b). See (2.1) of Stute (1982) with $h(t) = t^{1/2}$. $\square$

Stute derived all his other inequalities from something to be compared with (3.4). If we repeat the proof of our inequality 1.2 with $q(t) = 1$, we obtain:

INEQUALITY 3.2. Let $0 < a \le \delta \le ½$. For $0 < \lambda < \delta^2\sqrt{an}$

$$\text{(3.5)} \qquad P(\sup_{t-s\le a}|U_n(s,t]| \ge \lambda\sqrt{a}) \le \frac{20}{a\delta^2}\exp(-(1-\delta)^4\lambda^2/2).$$

This is slightly different (λ need not be bounded away from zero by about $8/\delta^2$) than Stute's (1982) Lemma 2.4.

PROPOSITION 3.1. *For $q \in Q^*$, (1.9), (1.14), and (1.15) are equivalent.*

PROOF. Suppose (1.15) holds. Then for $0 < \lambda < 1$

$$\int_{\lambda t}^{t} \frac{1}{s^2} \exp\left(-\lambda \frac{q^2(s)}{s}\right) ds \geq \left(\int_{\lambda t}^{t} s^{-2}\, ds\right) \exp\left(-\frac{q^2(t)}{t}\right) \quad \text{since} \quad q(t) \nearrow$$

$$\text{(a)} \qquad = \left(\frac{1}{\lambda} - 1\right) \frac{1}{t} \exp\left(-\frac{q^2(t)}{t}\right)$$

$$= \left(\frac{1}{\lambda} - 1\right) \exp\left(-\left(\frac{q^2(t)}{t} - \log\left(\frac{1}{t}\right)\right)\right).$$

Since the left side converges to zero as $t \searrow 0$, we have $\alpha(t) \equiv t^{-1/2} q(t) \to \infty$ as $t \searrow 0$. In fact, from the last expression in (a)

$$\frac{q^2(t)}{t} - \log\left(\frac{1}{t}\right) = \left(\frac{q^2(t)}{t \log\left(\frac{1}{t}\right)} - 1\right) \log\left(\frac{1}{t}\right) \to \infty$$

as $t \searrow 0$, and hence

$$\text{(b)} \qquad g(t) \equiv \frac{q^2(t)}{t \log\left(\frac{1}{t}\right)} > 1 \quad \text{for} \quad t < b = b(q).$$

Since the preceding argument applies to q/M for any $M > 0$, it follows from (b) that

$$\text{(c)} \qquad g(t) = \frac{q^2(t)}{t \log\left(\frac{1}{t}\right)} > M^2 \quad \text{for} \quad t < b = b(q, M)$$

for every $M > 0$. Thus (1.15) implies (1.9).

By writing the integral in (1.15) as

$$\int_0^b t^{-2} \exp(-\lambda q^2(t)/t)\, dt = \int_0^b t^{-2+\lambda g(t)}\, dt$$

we see that (1.9) implies (1.15). Hence (1.9) and (1.15) are equivalent.

To show that (1.15) implies (1.14), note that from (b) we have, for any $\varepsilon > 0$ and all $t \leq b = b_\varepsilon$, $\alpha(t)^3 \leq \exp(\varepsilon^2 \alpha(t)^2/4)$. Thus, for $b \leq b_\varepsilon$ (note (1.13))

$$\int_{0+}^b t^{-2} \alpha(t)^3 \exp(-\varepsilon^2 \alpha(t)^2/2)\, dt \leq \int_{0+}^b t^{-2} \exp(-\varepsilon^2 \alpha(t)^2/4)\, dt < \infty,$$

and therefore εq is interval upper-class. Hence (1.15) implies (1.14). Now suppose (1.14) holds. Then $\alpha(t) \to \infty$ as $t \to 0$ (since q is interval upper-class), and hence for any $\varepsilon > 0$

$$\infty > \int_0^b t^{-2} (\varepsilon \alpha(t))^3 \exp(-\varepsilon^2 \alpha(t)^2/2)\, dt > \int_0^b t^{-2} \exp(-\varepsilon^2 \alpha(t)^2/2)\, dt$$

for b sufficiently small, and hence (1.14) implies (1.15). $\square$

4. Proof of theorems.

PROOF OF THEOREM 1.1. Suppose $g(t) \to \infty$ as $t \downarrow 0$. Then

(a) $$\hat{g}(t) \equiv \min\{\inf\{g(s): 0 < s \le t\}, \quad (\log\log 1/t)\} \nearrow \infty \quad \text{as} \quad t \downarrow 0.$$

Since $\hat{g} \le g$ it suffices to prove (1.6) for q redefined to be

(b) $$q(t) \equiv [t \log_2 1/t]^{1/2} \hat{g}(t)^{1/2} \quad \text{on} \quad [0, ½] \text{ with } q \text{ symmetric about } t = ½.$$

We now relabel $\hat{g}$ as g. Note moreover that (a) implies that $q(t)/t^{1/2}$ is now $\searrow$, so that

(c) $$q \in Q \quad \text{for the } q \text{ of (b).}$$

Now for $\theta \le ½$ we have

(d) $$\|(U_n - U)/q\|_0^{1/2} \le \|U_n/q\|_0^\theta + \|U/q\|_0^\theta + \|U_n - U\|/q(\theta).$$

We would like to use (1.23) directly on $\|U_n/q\|_0^\theta$; although this works on U_n^-, it fails on U_n^+ since $\gamma^+ = 0$ when $a = 0$. Thus we note that

(e) $$\|U_n/q\|_0^\theta \le \|U_n/q\|_0^{1/n} + \|U_n/q\|_{1/n}^{a_n} + \|U_n/q\|_{a_n}^\theta$$

when $a_n \equiv q^2(1/n)$. Now by (1.23), (1.26) and the fact that (1.7) implies (1.12) as in Shorack (1979), we can choose θ so small that

(f) $$P(\|U_n/q\|_{a_n}^\theta \ge \varepsilon) \le \varepsilon \quad \text{for} \quad n \ge \quad \text{some} \quad N'_\varepsilon.$$

Since $q(t)/t^{1/2}$ is $\searrow$, we have from (b) that

(g) $$\begin{aligned} P(\|U_n/q\|_{1/n}^{a_n} \ge \varepsilon) &\le P(\|U_n/I^{1/2}\|_{1/n}^{a_n} \ge \varepsilon(\log\log(1/a_n))^{1/2}(g(a_n))^{1/2}) \\ &\le P(\|U_n/I^{1/2}\|_{1/n}^{a_n} \ge (\log\log n)^{1/2}) \quad \text{for all large} \quad n \\ &\le 6\log(na_n)\exp\left(-\frac{1}{8}(\log\log n)\psi((\log\log n)^{1/2})\right) \quad \text{by (1.27)} \\ &\qquad \text{and (1.26) with } \delta = \frac{1}{2} \\ &\le 13\,(\log\log\log n)\exp\left(-\frac{1}{8}(\log\log n)\frac{(\log\log\log n)}{(\log\log n)^{1/2}}\right) \quad \text{by (1.19)} \\ &\le \varepsilon \quad \text{for} \quad n \ge \quad \text{some} \quad N''. \end{aligned}$$

We will now prove

(4.1) $$\|U_n/q\|_0^{1/n} \to_p 0 \quad \text{as} \quad n \to \infty.$$

To prove (4.1), let $A_n \equiv [\|U_n^+/q\|_0^{1/n} > \varepsilon]$ and $B_n \equiv [\|\Gamma_n/I\| > \lambda_n]$ with $\lambda_n \equiv (\log\log n)^{1/2}$. It is well-known that $P(B_n) = 1/\lambda_n \to 0$; see Shorack and Wellner (1978) and Wellner (1978) for related results. Since $t^{-1/2}q(t) \downarrow$ implies that $n^{1/2}\lambda_n\|I/q\|_0^{1/n} = 1/g^{1/2}(1/n) \to 0$ as $n \to \infty$, on B_n^c we have

(h) $$\begin{aligned} \|U_n^+/q\|_0^{1/n} &\le n^{1/2}(\|\Gamma_n/q\|_0^{1/n} + \|I/q\|_0^{1/n}) \\ &\le n^{1/2}(\lambda_n + 1)\|I/q\|_0^{1/n} \\ &\le 2/g^{1/2}(1/n) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $B_n^c \subset A_n^c$ for $n \ge N_\varepsilon$, $P(A_n^c) \ge P(B_n^c) \to 1$ as $n \to \infty$, and (4.1) holds. Combining (f), (g), and (4.1) into (e) shows

(i) $$P(\|U_n/q\|_0^\theta \ge \varepsilon) \le \varepsilon \quad \text{for} \quad n \ge \quad \text{some} \quad N_\varepsilon.$$

Combining (i), (1.28), and (1.3) into (d) gives (1.6).

Now for the necessity of (1.7). O'Reilly (1974) showed that (1.6) implies the equivalent conditions (1.11) and (1.12). We now show that (1.11) implies (1.7). Assume $\liminf_{t\downarrow 0} g(t) < \infty$. Then since (with $\varepsilon > 0$ chosen small enough) $\varepsilon q(t) = \varepsilon(t \log\log(1/t)g(t))^{1/2}$ is upper class, we have a sequence $t_i \downarrow 0$ on which

$$\limsup_{i\to\infty} \frac{U(t_i)}{[2t_i \log\log(1/t_i)]^{1/2}} < \frac{1}{2} \quad \text{a.s.}$$

As seen from the proof of Breiman (1968; pages 264–265), this is a contradiction (without loss we may assume that t_{i+1}/t_i does not exceed Breiman's q). Thus (1.7) holds. □

PROOF OF THEOREM 1.2. We first prove necessity: Suppose that (1.8) holds for every $\varepsilon > 0$. Choose and fix $\varepsilon > 0$. Then

$$\begin{aligned}
&\| U_n - U)/q \|_{\mathscr{C}(\varepsilon^{2}n^{-1}\log n)} \\
&\quad \geq \sup\{(U_n(C) - U(C))/q(|C|) : |C| = \varepsilon^2 n^{-1}\log n\} \\
\text{(a)}\quad &\quad \geq \sup\{(-n^{1/2}|C|^{1/2}|C|^{1/2} - U(C))/q(|C|) : |C| = \varepsilon^2 n^{-1}\log n\} \\
&\quad = \sup\{(-\varepsilon(1+o(1))[|C|\log(1/|C|)]^{1/2} - U(C))/q(|C|) : |C| = \varepsilon^2 n^{-1}\log n\} \\
&\quad \geq \sup\{(-2\varepsilon[|C|\log(1/|C|)]^{1/2} - U(C))/q(|C|) : |C| = \varepsilon^2 n^{-1}\log n\}
\end{aligned}$$

for n large; and hence for $n \geq N_\varepsilon$,

$$P(-U(C) < 2\varepsilon[|C|\log(1/|C|)]^{1/2} + \varepsilon q(|C|) \quad \text{for all } C \text{ with } \quad |C| = \varepsilon^2 n^{-1}\log n) > 1 - \varepsilon.$$

Thus $\varepsilon q_*(t) \equiv \varepsilon[2(t\log(1/t))^{1/2} + q(t)]$ is interval upper-class for every $\varepsilon > 0$, and hence, by the equivalence of (1.14) and (1.9)

$$\text{(b)}\qquad q_*(t)/[t\log(1/t)]^{1/2} = 2 + [q(t)/(t\log(1/t))^{1/2}] \to \infty$$

as $t \to 0$. But this clearly implies the same is true for q; i.e. εq is interval upper-class for every $\varepsilon > 0$.

To prove the sufficiency part of the theorem, replace g by

$$\text{(c)}\qquad \hat{g}(t) \equiv \min\{\inf\{g(s) : 0 < s \leq t\}, \quad \log(1/t)\} \nearrow \infty \quad \text{as} \quad t \downarrow 0$$

as in the proof of Theorem 1.1.

We will use (1.32) to handle intervals C with $\varepsilon n^{-1}\log n \leq |C| \leq a_n \equiv q^2(1/n)$; then (1.29) will handle intervals C with $a_n \leq |C| \leq \theta$ for fixed (small) θ.

Since $t^{-1/2}q(t)$ is $\searrow$, with $b_n \equiv \varepsilon n^{-1}\log n$ and M very large,

$$\begin{aligned}
\text{(d)}\quad & P(\| U_n/q \|_{\mathscr{C}(b_n, a_n)} \geq \varepsilon) \\
&\leq P(\sup_{\{C: b_n \leq |C| \leq a_n\}} |U_n(C)|/|C|^{1/2} \geq \varepsilon(\log(1/a_n))^{1/2}(g(a_n))^{1/2}) \\
\text{(e)}\quad &\leq P(\sup_{\{C: b_n \leq |C| \leq a_n\}} |U_n(C)|/|C|^{1/2} \geq M(\log n)^{1/2}) \quad \text{for all large} \quad n \\
&\leq \frac{200}{b_n} \exp\left(-\frac{M^2}{32}(\log n)\,\psi\left(\frac{3M(\log n)^{1/2}}{(\varepsilon \log n)^{1/2}}\right)\right) \quad \text{by (1.32), (1.30) with } \delta = \frac{1}{2} \\
&\leq \frac{200n}{\varepsilon \log n} \exp\left(-\frac{M^2}{32}(\log n)\psi\left(\frac{3M}{\varepsilon^{1/2}}\right)\right) \\
&\sim \frac{200n}{\varepsilon \log n} \exp\left(-\frac{M^2}{32}\frac{2\log M}{(3/\varepsilon^{1/2})M}(\log n)\right) \quad \text{for large} \quad M \\
\text{(f)}\quad &\leq \varepsilon \quad \text{for } M \text{ large enough, and } n \geq \text{some } N_M \text{ in (e).}
\end{aligned}$$

Also

$$\text{(g)}\qquad P(\| U/q \|_{\mathscr{C}(0,\theta)} \geq \varepsilon) \leq \varepsilon \quad \text{for } \theta \equiv \text{some } \theta_\varepsilon$$

by (1.35); and

(h) $$P(\| \mathbb{U}_n/q \|_{\mathscr{C}(a_n,\theta)} \geq \varepsilon) \leq \varepsilon \quad \text{for } n \geq \text{some } N_\varepsilon$$

by (1.29) and (1.31) with $\delta = \theta$ since $\gamma \geq \psi(2^{1/2}\varepsilon/\theta)$. Applying (f), (g), (h) and (1.3) (note that $\| \mathbb{U}_n - \mathbb{U} \|_{\mathscr{C}} \leq 2 \| \mathbb{U}_n - \mathbb{U} \|$) to

(i) $$\begin{aligned}\| (\mathbb{U}_n - \mathbb{U})/q \|_{\mathscr{C}(b_n,1-\theta)} \leq \| \mathbb{U}_n/q \|_{\mathscr{C}(b_n,a_n)} + \| \mathbb{U}_n/q \|_{\mathscr{C}(a_n,\theta)} \\ + \| \mathbb{U}/q \|_{\mathscr{C}(0,\theta)} + \| \mathbb{U}_n - \mathbb{U} |_{\mathscr{C}(\theta,1-\theta)}/q(\theta)\end{aligned}$$

shows that the term in (i) does $\to_p 0$. Finally, for $0 \leq s \leq t \leq 1$ with $t - s \geq 1 - \theta$ we have $q(t - s) \geq q(1 - t) \vee q(s)$; so that

$$\| \mathbb{U}_n/q \|_{\mathscr{C}(1-\theta,1)} \leq \sup\{ | \mathbb{U}_n(t) |/q(t) : t \geq 1 - \theta \} + \sup\{ | \mathbb{U}_n(s) |/q(s) : s \leq \theta \}$$

has (for $\theta \equiv \theta_\varepsilon$ sufficiently small), by the proof of Theorem 1.1,

(j) $$P(\| \mathbb{U}_n/q \|_{\mathscr{C}(1-\theta,1)} \geq \varepsilon) \leq \varepsilon \quad \text{for all } n \geq \text{some } N'_\varepsilon$$

and likewise

(k) $$P(\| \mathbb{U}/q \|_{\mathscr{C}(1-\theta,1)} \geq \varepsilon) \leq \varepsilon. \quad \square$$

Proof of Theorem 1.3. First note that for any fixed $0 < b \leq \frac{1}{2}$

(a) $$\sup_{\{C: |C| \geq b\}} \frac{| \mathbb{U}_n(C) |}{(|C| \log n)^{1/2}} \to_{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty$$

by Cassels' Theorem (1.36), and hence to prove (1.37) it suffices to show that (with $a_n = n^{-1}\varepsilon \log n$ and appropriate large M)

(b) $$\limsup_{n\to\infty} \sup_{\{C: a_n \leq |C| \leq b\}} \frac{| \mathbb{U}_n(C) |}{(|C| \log n)^{1/2}} \leq M \quad \text{a.s.}$$

To do this, let $a = a_n = n^{-1}\varepsilon \log n$ and $\lambda = M(\log n)^{1/2}$ in (1.32) and (1.30) to obtain

$$\begin{aligned} &P\left(\sup_{\{C: a_n \leq |C| \leq b\}} \frac{| \mathbb{U}_n(C) |}{|C|^{1/2}} \geq M(\log n)^{1/2} \right) \\ &\qquad \leq \frac{24n}{\varepsilon\delta^3 \log n} \exp\left(-\frac{(1-\delta)^4 M^2}{2} (\log n) \psi\left(\frac{2^{1/2} M}{\delta \varepsilon^{1/2}} \right) \right) \\ &\qquad \leq \text{constant} \cdot n^{-\tau}, \ \tau > 1 \end{aligned}$$

if M is sufficiently large by (1.19). Now (b) follows by Borel-Cantelli and (1.37) is proved.

To prove (1.38) we now specify $a = a_n \equiv n^{-1}(\log n)^\alpha$. By (a), it suffices to consider C's with $a_n \leq |C| \leq b$, $b > 0$ arbitrarily small. Then, with $a = a_n$ and $\lambda_n \equiv M(\log n)/[(na_n)^{1/2} \log \log n] = M(\log n)^{1-\alpha/2}/\log \log n$, (1.32) yields, for n sufficiently large,

$$\begin{aligned} &P\left(\sup_{\{C: a_n \leq |C| \leq b\}} \frac{| \mathbb{U}_n(C) |}{|C|^{1/2}} \geq \lambda_n \right) \geq \frac{24n}{\delta^3 (\log n)^\alpha} \exp\left(-(1-\delta)^4 \frac{\lambda_n^2}{2} \psi\left(\frac{2^{1/2}\lambda_n}{\delta (\log n)^{\alpha/2}} \right) \right) \\ &\sim \frac{24n}{\delta^3 (\log n)^\alpha} \exp\left(-(1-\delta)^4 \frac{M^2}{2} \frac{(\log n)^{2-\alpha}}{(\log \log n)^2} \frac{2^{1/2}\delta(1-\alpha)(\log \log n)}{M(\log \log n)^{1-\alpha}/(\log \log n)} \right) \quad \text{by (1.19)} \\ &\leq \text{constant} \cdot n^{-\tau}, \quad \tau > 1 \end{aligned}$$

for M sufficiently large. $\square$

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