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GENERALIZATION AND APPLICATION OF A RESULT OF C. C. HEYDE

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A result of C. C. Heyde, where it is shown that certain rates of convergence (expressed in terms of series conditions) in the CLT for i.i.d. random variables hold true if and only if corresponding moment conditions are fulfilled, is generalized to k-dimensional i.i.d. random vectors.

In his famous 1967 paper, C. C. Heyde has shown that certain moment conditions are equivalent to corresponding rates of convergence in the CLT for i.i.d. random variables. The rates of convergence being discussed there are expressed in terms of series conditions. In the proof of his result, C. C. Heyde employs subtle investigations on the behavior of characteristic functions of symmetric random variables and then reduces the general case to the case of symmetric random variables.

Following the lines of C. C. Heyde's proof, it apparently turns out to be a problem to generalize his result to i.i.d. random vectors. (For an ineffectual attempt see [1], bottom of page 315.)

As we show below, a relatively short proof (concerning the sufficiency of the moment conditions) can be given by using standard truncation methods and simple algebra on summation of series. (Essentially, this method has been used, e.g., by Heyde [6], proof of Theorem 1, page 904–906.) Since the above mentioned part of the proof is the crucial part in generalizing C. C. Heyde's theorem to i.i.d. random vectors, the result immediately follows.

Heyde's results complement those of Ibragimov [7]. Multidimensional extensions of Ibragimov's results have been obtained by Bikjalis [3], [4].

In the following we denote by Φ the k-dimensional standard normal distribution. Let, furthermore, $\mathscr{C}_k$ be the system of all convex Borel-sets in $\mathbb{R}^k$. Finally, vector- and matrix-norms are Euclidean norms.

As result of the present note we have the following.

Theorem. *Let* $X_1, X_2, \cdots$ *be a sequence of* i.i.d. *random vectors in* $\mathbb{R}^k$ *with* $EX_1 = 0$ *and* $\operatorname{Cov}(X_1) = I$. *Then for* $0 \le \delta < 1$,

$$E\|X_1\|^{2+\delta} < \infty, \quad \delta > 0, \qquad E\|X_1\|^2 \log(1 + \|X_1\|) < \infty, \quad \delta = 0, \tag{1}$$

if and only if

$$\textstyle\sum_{n\in N} n^{-1+\delta/2} \sup_{C\in\mathscr{C}_k} |P(n^{-1/2}\sum_{i=1}^n X_i \in C) - \Phi(C)| < \infty. \tag{2}$$

Proof. *Necessity of* (1). Follows immediately from the theorem in [5], page 12, by considering the systems of half-spaces

$$\mathscr{H}_k^{(j)} = \{\{y \in \mathbb{R}^k : y_j \le t\} : t \in \mathbb{R}\} \subset \mathscr{C}_k, \qquad j = 1, \cdots, k.$$

Sufficiency of (1). Let $\bar{X}_i$ denote X_i truncated at $rn^{1/2}$, i.e. $\bar{X}_i = X_i 1_{A_{n,r,i}^c}$, where

$$A_{n,r,i} = \{\|X_i\| \ge rn^{1/2}\}$$

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with $r \geq 1$ fulfilling

$$2kE\|X_1\|^2 1_{A_{1,r,1}} \leq \tfrac{1}{2}. \tag{3}$$

Define

$$\bar{\mu} = E\bar{X}_1 \quad \text{and} \quad \bar{\Sigma} = \operatorname{Cov}(\bar{X}_1). \tag{4}$$

Then by inequality (14.19), page 124, in [2],

$$\|\bar{\Sigma} - I\| \leq 2kE\|X_1\|^2 1_{A_{n,r,1}} \leq \tfrac{1}{2}. \tag{5}$$

Since (5) yields that $\bar{\Sigma}$ is positive definite, we have,

$$\begin{aligned}\sup_{C\in\mathscr{C}_k} &|P(n^{-1/2} \textstyle\sum_{i=1}^n X_i \in C) - \Phi(C)| \\ &\leq nP(A_{n,r,1}) + \sup_{C\in\mathscr{C}_k}[|\Phi(n^{1/2}\bar{\mu} + \bar{\Sigma}^{1/2}C) - \Phi(C)| \\ &\qquad + |P(n^{-1/2} \textstyle\sum_{i=1}^n \bar{X}_i \in n^{1/2}\bar{\mu} + \bar{\Sigma}^{1/2}C) - \Phi(C)|]\end{aligned}$$

In the following $d_k^{(1)}$, $d_k^{(2)}$, $d_k^{(3)}$ denote positive constants depending on k only. Furthermore, for notational convenience, we suppress the phrases "There exists a constant $d_k^{(i)}$ such that $\cdots$", $i = 1, 2, 3$.

From inequalities (14.73), page 133, and (14.86), page 135, in [2],

$$\sup_{C\in\mathscr{C}_k}|\Phi(n^{1/2}\bar{\mu} + \bar{\Sigma}^{1/2}C) - \Phi(C)| \leq d_k^{(1)} E\|X_1\|^2 1_{A_{n,r,1}}.$$

Furthermore, by the Berry-Esséen Theorem in k dimensions ([2], page 165, Corollary 17.2), for all $C \in \mathscr{C}_k$,

$$|P(n^{-1/2} \textstyle\sum_{i=1}^n \bar{X}_i \in n^{1/2}\bar{\mu} + \bar{\Sigma}^{1/2}C) - \Phi(C)| \leq d_k^{(2)} n^{-1/2} E\|\bar{\Sigma}^{-1/2}(\bar{X}_1 - \bar{\mu})\|^3$$

which implies (observe (5) and use (14.10), page 122 in [2])

$$\sup_{C\in\mathscr{C}_k}|P(n^{-1/2} \textstyle\sum_{i=1}^n \bar{X}_i \in n^{1/2}\bar{\mu} + \bar{\Sigma}^{1/2}C) - \Phi(C)| \leq d_k^{(3)} n^{-1/2} E\|X_1\|^3 1_{A_{n,r,1}^c}.$$

After these preliminaries we now prove the convergence of the series in (2). Let

$$B_{n,r} = \{rn^{1/2} \leq \|X_1\| < r(n+1)^{1/2}\}.$$

For $0 < \delta < 1$,

$$\begin{aligned}\textstyle\sum_{n\in N} n^{-1+\delta/2} E\|X_1\|^2 1_{A_{n,r,1}} &= \textstyle\sum_{n\in N} n^{-1+\delta/2} \sum_{j\geq n} E\|X_1\|^2 1_{B_{j,r}} \\ &= \textstyle\sum_{n\in N} E\|X_1\|^2 1_{B_{n,r}} \sum_{j=1}^n j^{-1+\delta/2} \leq 2\delta^{-1} \sum_{n\in N} n^{\delta/2} E\|X_1\|^2 1_{B_{n,r}} \\ &\leq 2\delta^{-1} E\|X_1\|^{2+\delta}.\end{aligned}$$

The same reasoning for $\delta = 0$ gives

$$\textstyle\sum_{n\in N} n^{-1} E\|X_1\|^2 1_{A_{n,r,1}} \leq 2(\log 2)^{-1} E\|X_1\|^2 \log(1 + \|X_1\|).$$

Furthermore, for $0 \leq \delta < 1$,

$$\begin{aligned}\textstyle\sum_{n\in N} n^{-(3-\delta)/2} E\|X_1\|^3 1_{A_{n,r,1}^c} &= \textstyle\sum_{n\in N} n^{-(3-\delta)/2} \sum_{j=1}^n E\|X_1\|^3 1_{B_{j-1,r}} \\ &= \textstyle\sum_{n\in N} E\|X_1\|^3 1_{B_{n-1,r}} \sum_{j\geq n} j^{-(3-\delta)/2} \leq 3(1-\delta)^{-1} r^{1-\delta} E\|X_1\|^{2+\delta}.\end{aligned}$$

Since for $\delta \geq 0$,

$$\textstyle\sum_{n\in N} n^{\delta/2} P(A_{n,r,1}) \leq E\|X_1\|^{2+\delta}$$

(observe $r \geq 1$) the proof of the theorem is completed.

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REFERENCES

[1] AHMAD, I. A. (1979). A note on rates of convergence in the multidimensional CLT for maximum partial sums. *J. Multivar. Anal.* **9** 314–321.

[2] BHATTACHARYA, R. N. and RAO, R. R. (1976). *Normal Approximation and Asymptotic Expansions.* Wiley, New York.

[3] BIKJALIS, A. (1971). On the central limit theorem in $\mathbb{R}^k$. I. *Litovsk. Mat. Sb.* **11** 27–58 (in Russian).

[4] BIKJALIS, A. (1972). On the central limit theorem in $\mathbb{R}^k$. II. *Litovsk. Mat. Sb.* **12** 53–84 (in Russian).

[5] HEYDE, C. C. (1967). On the influence of moments on the rate of convergence to the normal distribution. *Z. Wahrsch. verw. Gebiete* **8** 12–18.

[6] HEYDE, C. C. (1975). A nonuniform bound on convergence to normality. *Ann. Probability* **3** 903–907.

[7] IBRAGIMOV, I. A. (1966). On the accuracy of Gaussian approximation to the distribution functions of sums of independent variables. *Theor. Probability Appl.* **11** 559–579.

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