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STRONG LIMITING BOUNDS FOR MAXIMAL UNIFORM SPACINGS

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Let $U_1, U_2, \cdots$ be a sequence of independent uniformly distributed random variables on (0, 1) and M_n be the largest spacing induced by $U_1, \cdots, U_n$. We show that $P(M_n \geq (\log n + 2\log_2 n + \log_3 n + \cdots + \log_j n)/n \text{ i.o.}) = 1$, where $\log_j$ is the j times iterated logarithm, and $j \geq 4$. If $1 = N_1 < N_2 < \cdots < N_k < \cdots$ is the sequence of the successive times n where $M_n < M_{n-1}$, we derive strong limiting bounds for $\{N_k, k \geq 1\}$.

1. Introduction. Let $U_1, U_2, \cdots$ be a sequence of i.i.d. random variables uniformly distributed on (0, 1). If $U_0^{(n)} = 0 < U_1^{(n)} < \cdots < U_n^{(n)} < U_{n+1}^{(n)} = 1$ are the order statistics corresponding to $0, 1, U_1, \cdots, U_n$, then the maximal uniform spacing M_n is defined by

$$M_n = \max_{1 \leq i \leq n+1} S_i^{(n)},$$

where $S_i^{(n)} = U_i^{(n)} - U_{i-1}^{(n)}$ for $1 \leq i \leq n+1$. The $S_i^{(n)}$ are called spacings of order n.

Devroye [5] has shown that w.p.1,

$$(1) \qquad \limsup_{n\infty}(nM_n - \log n)/2\log_2 n = 1, \qquad \liminf_{n\infty}(nM_n - \log n)/\log_3 n = -1,$$

where $\log_j$ is the j times iterated logarithm.

The aim of this exposition is to make this result more precise by studying the sequence of the random times of decrease associated with $\{M_n, n \geq 1\}$ and defined in the following way:

$$N_1 = 1, \qquad N_k = \inf\{n > N_{k-1}; M_n < M_{N_{k-1}}\}, \qquad k = 2, 3, \cdots.$$

The definition of $N_1, N_2, \cdots$ corresponds to the fact that M_n remains constant when n varies between N_k and N_{k-1}, then decreases at the time $n = N_k$ when U_n takes its value in the spacing interval associated with M_n.

The main result about $N_1, N_2, \cdots$ is expressed in the following:

THEOREM 1. *When k tends to infinity, almost surely*

$$(2) \qquad N_k = \exp(\sqrt{2k} + \psi(k)), \quad \text{with} \quad |\psi(k)| \leq (\log k)(1 + o(1)),$$

and, for any $j \geq 4$, almost surely

$$(3) \quad \limsup_{n\infty}\left\{\left\{\left(\frac{N_{k+1} - N_k}{N_k}\right)\log N_k - 2\log_2 N_k - \log_3 N_k - \cdots - \log_{j-1} N_k\right\}/\log_j N_k\right\} = 1.$$

As a consequence of Theorem 1 and of (1), we will obtain that for $j \geq 4$,

$$(4) \qquad \limsup_{n\infty}\{nM_n - \log n - 2\log_2 n - \log_3 n - \cdots - \log_{j-1} n\}/\log_j n = 1 \quad \text{a.s.}$$

This makes precise the result obtained by Devroye [5], who showed that for $j \geq 4$,

$$(5) \qquad \limsup_{n\infty}\{nM_n - \log n - 2\log_2 n - \log_3 n - \cdots - \log_{j-1} n\}/\log_j n \leq 1 \quad \text{a.s.,}$$

and proves that the upper bound given in (5) is the best possible.

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2. Strong bounds for the times of decrease of the maximal spacing. It is clear that the sequence $M_1, M_2, \cdots$ is non-increasing, and that if i_n stands for the w.p.1 uniquely defined index such that $S_{i_n}^{(n)} = M_n$, $n = 1, 2, \cdots$, then for $k \geq 2$, N_k may be defined as the smallest value of m such that U_m falls into the interval $]U_{iN-1}^{(N)}, U_{iN}^{(N)}[$, where $N = N_{k-1}$.

If we let $m_k = M_{N_k}$, then it follows that the distribution of $N_k - N_{k-1}$, knowing the past anterior to N_{k-1}, depends only upon m_{k-1} and is given by

$$P(N_k - N_{k-1} \geq r \mid m_{k-1}) = (1 - m_{k-1})^{r-1}, \qquad r = 1, 2, \cdots.$$

The next step is given in a strong approximation lemma, analogous to [4]:

LEMMA 1. *On a possibly enlarged probability space, there exists an* i.i.d. *sequence* $\{Y_k, k \geq 1\}$ *of exponentially* $E(1)$ *distributed random variables, such that*

$$\text{(i)} \qquad N_k - N_{k-1} = \left[\frac{Y_{k-1}}{-\log(1 - m_{k-1})}\right] + 1, \qquad k = 2, 3, \cdots,$$

where $[u]$ *stands for the integer part of* u,

(ii) *for* $k = 2, 3, \cdots$, Y_{k-1} *is independent of* $m_1, \cdots, m_{k-1}$ *and of* N_{k-1}.

PROOF. Let us consider, in a more general setting, two r.v. G and Z, such that (i) $0 < Z < 1$ a.s., (ii) $P(G = r \mid Z = z) = z(1 - z)^{r-1}$, $r = 1, 2, \cdots$, or equivalently $P(G \geq r \mid Z = z) = (1 - z)^{r-1}$, $r = 1, 2, \cdots$, G taking integer values.

The lemma will be proved if we show that there exists an exponentially $E(1)$ distributed r.v. Y, independent of Z, such that $G = [Y/(-\log(1 - Z))] + 1$. The latter in turn follows from

LEMMA 2. *Let G and Z be two* r.v. *satisfying* (i) $0 < Z < 1$ a.s., (ii) $P(G \geq r \mid Z = z) = (1 - z)^{r-1}$, $r = 1, 2, \cdots$, *and let ζ be a uniformly distributed on* (0, 1) r.v., *independent of G and Z. If*

$$\text{(6)} \qquad Y = (G - 1)(-\log(1 - Z)) - \log(1 - \zeta Z),$$

then Y is exponentially $E(1)$ distributed, independent of Z, and such that

$$\text{(7)} \qquad G = [Y/(-\log(1 - Z))] + 1.$$

PROOF. First, we can see that $Y/(-\log(1 - Z)) + 1 = G + (\log(1 - \zeta Z))/(\log(1 - Z))$. Since $0 < 1 - Z < 1 - \zeta Z < 1$ a.s., $(\log(1 - \zeta Z))/(\log(1 - Z)) < 1$ a.s., and (7) follows.

Secondly, if Y is given by (6), then $U = 1 - e^{-Y} = 1 - (1 - Z)^{G-1} + \zeta((1 - Z)^{G-1} - (1 - Z)^G)$ has, given $G = r$ and $Z = z$, a uniform distribution on the interval $(1 - (1 - z)^{r-1}, 1 - (1 - z)^r)$; since $P(G = r \mid Z = z) = (1 - (1 - z)^r) - (1 - (1 - z)^{r-1})$, the distribution of U, given that $Z = z$, is uniform on (0, 1), and hence, U and $Y = -\log(1 - U)$ are independent of Z, Y being exponentially $E(1)$ distributed. Hence Lemmas 1 and 2 are proved.

We will now go back to the sequence $N_1, N_2, \cdots$, and evaluate its rate of increase to infinity. By definition, for $k = 2, 3, \cdots$, $N_k - N_{k-1} \geq 1$; hence $\liminf_{k\infty} N_k/k \geq 1$, and $\lim_{k\infty} N_k = +\infty$. Since $m_k = M_{N_k}$, it follows from Devroye's [6] results (1), that for an arbitrary $\varepsilon > 0$, there exists almost surely a k_ε such that if $k \geq k_\varepsilon$,

$$-\frac{(1 + \varepsilon)\log_3 N_k}{N_k} < m_k - \frac{\log N_k}{N_k} < \frac{(2 + \varepsilon)\log_2 N_k}{N_k}.$$

From this and Lemma 1, an easy deduction gives that, almost surely when $k \to \infty$,

$$\text{(8)} \qquad N_{k+1} - N_k \geq Y_k(N_k/\log N_k)(1 + O((\log_2 N_k)/\log N_k)).$$

By adding the inequalities of (8), we get that for $k \to \infty$,

$$\text{(9)} \qquad N_k \geq 1 + \textstyle\sum_{\iota=2}^{k-1} Y_\iota(N_\iota/\log N_\iota)(1 + o(1)).$$

We note that the Y_k are positive, and hence, by Kronecker's lemma, that

$$N_k = \left(\sum_{i=2}^{k-1} Y_i\right)/o(1), \tag{10}$$

which implies, by the law of large numbers, that $\lim_{k\infty} N_k/k = +\infty$. This enables one to iterate the reasoning to show in turn that $N_k \geq 1 + \sum_{i=2}^{k-1} Y_i(i/\log i)$ a.s. for k large enough. Using now Theorem 2.10.3 of [10] or Theorem 4.1.1 of [12] (see Jamison et al. [7]), it can be deduced that $\liminf_{k\infty} N_k/(k^2/\log k) \geq ½$ a.s.

By using this result again as in (9), (10), and by a straightforward feedback, we obtain:

LEMMA 3. *For an arbitrary* $r \geq 1$, $\lim_{k\infty} N_k/k^r = +\infty$ a.s.

Let us now consider the i.i.d. $E(1)$ sequence $Y_1, Y_2, \cdots$. By using any of the strong bounds given in [1], [3], or [11], it can be seen that for any $\varepsilon > 0$, there exists almost surely a k_ε, such that for $k \geq k_\varepsilon$,

$$1/(k(\log k)^{1+\varepsilon}) \leq Y_k \leq \log k + (1+\varepsilon)\log_2 k. \tag{11}$$

By (8), (11), and Lemma 3, we can deduce from this that for an arbitrary $r \geq 1$, $\lim_{k\infty}(N_{k+1} - N_k)/k^r = +\infty$ a.s. As a consequence, if we put

$$N_{k+1} - N_k = N_k(Y_k\rho_k/\log N_k), \qquad k = 2, 3, \cdots, \tag{12}$$

then $\lim_{k\infty}\rho_k = 1$, and by (11) and Lemma 3, $\lim_{k\infty} Y_k/\log N_k = 0$ a.s.

If we write (12) as $N_{k+1} = N_k(1 + Y_k\rho_k/\log N_k)$, the preceding result proves that if we put

$$\alpha_k = \log N_k, \quad \text{and} \quad \alpha_{k+1} = \alpha_k + Y_k\theta_k/\alpha_k, \qquad k = 2, 3, \cdots, \tag{13}$$

then $\lim_{k\infty}\theta_k = 1$, and $\lim_{k\infty} Y_k/\alpha_k = 0$ a.s. By taking squares of (13), we get

$$\alpha_{k+1}^2 - \alpha_k^2 = 2Y_k\theta_k + Y_k^2\theta_k^2/\alpha_k^2, \qquad k = 2, 3, \cdots. \tag{14}$$

It follows, by adding the inequalities in (14), and using the law of large numbers ($\lim_{n\infty} n^{-1}\sum_{k=1}^{n} Y_k = 1$ a.s.), that

$$\lim_{k\infty}\alpha_k^2/2k = 1 \quad \text{a.s.} \tag{15}$$

Likewise , we can deduce from (14), (15), as in the proof of Lemma 3, that

$$\lim_{n\infty}(\log n)^{-1}\sum_{k=2}^{n}\frac{Y_k^2\theta_k^2}{\alpha_k^2} = \frac{1}{2} \quad \text{a.s.} \tag{16}$$

Now (15) shows that $N_k = \exp(\sqrt{2k}(1 + o(1)))$, which does not enable one to get (2) without an evaluation of the rate of convergence of θ_k to 1 when $k \to \infty$. For this, getting back to Lemma 1, (i), and using (1), (11), (12) and (15), we can see that, as in (8),

$$\limsup_{k\infty}|\rho_k - 1|(\log N_k/2\log_2 N_k) = \limsup_{k\infty}|\rho_k - 1|(\sqrt{2k}/\log k) \leq 1 \quad \text{a.s.} \tag{17}$$

A close look at (12) and (13) shows that $(\theta_k - 1) - (\rho_k - 1) \sim Y_k/2\log N_k \leq (1 + o(1))(\log k)/(\sqrt{2k})$ a.s. when $k \to \infty$. Adding this to (17) and using the law of iterated logarithm for ΣY_k yields (2).

By (12), we get $(N_{k+1} - N_k)(\log N_k)/N_k = Y_k\rho_k$. Thus (3) is equivalent to

$$Y_k\rho_k \leq 2\log_2 N_k + \log_3 N_k + \cdots + (1+\varepsilon)\log_j N_k \quad \text{a.s. if} \quad k \to \infty \quad \text{when} \quad \varepsilon > 0, \quad \text{and}$$

$$Y_k\rho_k \geq 2\log_2 N_k + \log_2 N_k + \cdots + (1+\varepsilon)\log_j N_k \quad \text{i.o. w.p.1 when} \quad \varepsilon < 0.$$

To prove these assertions, we use the fact that, as in (11), $P(Y_k \geq \log k + \log_2 k + \cdots + (1+\varepsilon)\log_j k \text{ i.o.}) = 0$ when $\varepsilon > 0$, and 1 when $\varepsilon \leq 0$. If we note that, by (2), almost surely as $k \to \infty$,

$$\log N_k = \sqrt{2k} + O(\log k),$$

$$\log_2 N_k = \tfrac{1}{2}\log k + \tfrac{1}{2}\log 2 + O((\log k)/\sqrt{k}),$$

$$\log_3 N_k = \log_2 k - \log 2 + O(1/\log k),$$

$$\log_r N_k = \log_{r+1} k + O(1/\textstyle\prod_{i=2}^{r-2} \log_i k),\ r \geq 4,$$

we get easily:

$$\log k + \log_2 k + \cdots + (1+\varepsilon)\log_j k = 2\log N_k + \log_3 N_k + \cdots + (1+\varepsilon)\log_j N_k + O(1/\log_2 k).$$

Since by (17), $\rho_k = 1 + O((\log k)/\sqrt{k})$, it follows that

$$Y_k\rho_k \leq 2\log_2 N_k + \log_3 N_k + \cdots + (1+\varepsilon)\log_j N_k + O(1/\log_2 k)$$

a.s. if $k \to \infty$ when $\varepsilon > 0$, and

$$Y_k\rho_k \geq 2\log_2 N_k + \log_3 N_k + \cdots + (1+\varepsilon)\log_j N_k + O(1/\log_2 k) \text{ i.o. w.p.1 when } \varepsilon \leq 0.$$

Thus (3) is true and the proof of Theorem 1 is now complete.

Our next result is given in

THEOREM 2. *For any* $j \geq 4$,

$$\limsup_{n\infty}\{nM_n - \log n - 2\log_2 n - \log_3 n - \cdots - \log_{j-1} n\}/\log_j n = 1 \quad \text{a.s.} \tag{4}$$

PROOF. As noted before, Devroye [5] has proved (5), and we only need to show that, for an arbitrary $\varepsilon > 0$, the inequality

$$nM_n - \log n - 2\log_2 n - \log_3 n - \cdots - \log_{j-1} n - (1-\varepsilon)\log_j n \geq 0 \tag{18}$$

occurs infinitely often with probability 1.

In the proof, we use the fact that the values of n for which (18) occurs must include a subset of $\{N_j - 1, j \geq 1\}$. More precisely, suppose that (18) occurs, and let $k = k(n)$ be such that $N_k \leq n < N_{k+1}$. Since $M_m = M_n$ for any m such that $N_k \leq n \leq m < N_{k+1}$, we get for $0 < \varepsilon < 1$:

$$\begin{aligned} mM_m &- \log m - \cdots - (1-\varepsilon)\log_j m \\ &= \{nM_n - \log n - \cdots - (1-\varepsilon)\log_j n\} \\ &\quad + (m-n)M_n - \{\log m + \cdots + (1-\varepsilon)\log_j m - \log n - \cdots - (1-\varepsilon)\log_j n\} \\ &\geq (m-n)n^{-1}(\log n) - (j+1)(\log m - \log n) \\ &\geq (m-n)n^{-1}(\log n - (j+1)) \geq 0 \end{aligned}$$

for n large enough. Hence there exists a non-random n_0 such that if (18) occurs for $n \geq n_0$, then it also occurs for any $m : N_k \leq n \leq m < N_{k+1}$, and in particular for $m = N_{k+1} - 1$.

Let us now consider an arbitrary $n \geq 1$, and the corresponding integer $k = k(n)$ such that $N_k \leq n < N_{k+1}$. It can be seen that the distribution of $N_{k+1} - n$, knowing the past anterior to n, depends only upon M_n, and is given by

$$P(N_{k(n)+1} - n \geq r \mid M_n = m) = (1-m)^{r-1}, \qquad r = 1, 2, \cdots.$$

Hence, by Lemma 1, it follows that there exists an exponentially distributed random variable Z_n, independent of M_n and of $U_1, \cdots, U_n$, and such that

$$N_{k(n)+1} - n = \left[\frac{Z_n}{-\log(1-M_n)}\right] + 1. \tag{19}$$

We shall now consider the sequence $n_\ell = [\exp(\sqrt{2\ell})]$, $\ell = 1, 2, \cdots$ and put $T_\ell = Z_{n_\ell}$. Although $\{T_\ell, \ell \geq 1\}$ is a sequence of marginally exponentially $E(1)$ distributed random variables, it can be noted that they are not independent. In fact, if $M_{n_\ell} = M_{n_{\ell+1}}$, since then $N_{k(n_{\ell+1})} = N_{k(n_\ell)}$, $T_{\ell+1}$ is correlated with T_ℓ. On the other hand, if $M_{n_\ell} > M_{n_{\ell+1}}$, then clearly T_ℓ and $T_{\ell+1}$ are independent. Let us therefore put $\xi_\ell = I(M_{n_\ell} > M_{n_{\ell+1}})$, $\ell = 1, 2, \cdots$ and consider the random sequence defined by

$$\ell(1) = \min\{\ell \geq 1; M_{n_\ell} > M_{n_{\ell+1}}\} = \min\{\ell \geq 1; \xi_\ell = 1\},$$

$$(20) \qquad \ell(r) = \min\{\ell > \ell(r-1); M_{n_\ell} > M_{n_{\ell+1}}\}$$

$$= \min\{\ell > \ell(r-1); \xi_\ell = 1\}, \qquad r = 2, 3, \cdots.$$

It may be verified that $\{n_{\ell(r)+1}, r \geq 1\}$ is an increasing sequence of stopping times on $\{\sigma(U_1, \cdots, U_n), n \geq 1\}$. The preceding argument shows that

Lemma 4. *Let* $\{\ell(r), r \geq 1\}$ *be defined by* (20), *and put for* $r = 1, 2, \cdots, \omega_r = T_{\ell(r)+1}$, $\{T_\ell, \ell \geq 1\}$ *being defined in* (19)-(20), *then* $\{\omega_r, r \geq 1\}$ *is a sequence on independent exponentially* $E(1)$ *distributed random variables.*

Our next step is given in the following.

Lemma 5. *If for* $\ell = 1, 2, \cdots, n_\ell = [\exp(\sqrt{2\ell})]$ *and* $\{\ell(r), r \geq 1\}$ *is defined by* (20), *then*

$$(21) \qquad \lim_{r\infty} \ell(r)/r = \frac{e}{e-1} \quad \text{a.s.}$$

Proof. To prove (21), it is enough to prove that

$$(22) \qquad \lim_{N\infty} \frac{1}{N} \sum_{\ell=1}^{N} I(M_{n_\ell} > M_{n_{\ell+1}}) = \lim_{N\infty} \frac{1}{N} \sum_{\ell=1}^{N} \xi_\ell = 1 - e^{-1} \quad \text{a.s.}$$

It is easily seen that $P(\xi_\ell = 0 \mid M_{n_\ell} = m, M_{n_{\ell-1}}, \cdots, M_{n_1}) = (1-m)^{n_{\ell+1}-n_\ell}$. Next, $n_{\ell+1} - n_\ell \sim e^{\sqrt{2\ell}}(e^{\sqrt{2\ell+2}-\sqrt{2\ell}} - 1) \sim e^{\sqrt{2\ell}}/\sqrt{2\ell}$, and, by (1), $M_{n_\ell} \sim (\log n_\ell)/n_\ell \sim \sqrt{2\ell}/e^{\sqrt{2\ell}}$ a.s. as $\ell \to \infty$; hence $\lim_{\ell\infty} P(\xi_\ell = 0 \mid M_{n_\ell}, \cdots, M_{n_1}) = e^{-1}$ a.s., and, as a consequence, $\lim_{\ell\infty} E(\xi_\ell \mid \xi_{\ell-1}, \cdots, \xi_1) = 1 - e^{-1}$ a.s. Furthermore, $\lim_{\ell\infty} E(\xi_\ell) = 1 - e^{-1}$ and $\lim_{\ell\infty} D^2(\xi_\ell) = \lim_{\ell\infty} E(\xi_\ell)(1 - E(\xi_\ell)) = e^{-1} - e^{-2}$. This suffices for (22) and (21), since $\sum_{\ell=1}^{\infty} D^2(\xi_\ell)/\ell^2 < \infty$, which in turn implies that $(1/N)\sum_{\ell=1}^{N} (\xi_\ell - E(\xi_\ell \mid \xi_{\ell-1}, \cdots, \xi_1)) \to 0$ as $N \to \infty$ (see Loeve [9] page 387, Révész [10] page 137-138). The proof of Lemma 5 is now complete.

Lemma 6. *For any* $j \geq 4$ *and* $c > 0$,

$$(23) \qquad \lim_{N\infty} \frac{1}{N} \sum_{\ell=1}^{N} I(n_\ell M_{n_\ell} - \log n_\ell \geq -c \log_j n_\ell) = 1 \quad \text{a.s.}$$

Proof. Let $\eta_\ell = I(n_\ell M_{n_\ell} - \log n_\ell < -c \log_j n_\ell)$; (23) is equivalent to $\lim_{N\infty}(1/N)\sum_{\ell=1}^{N} \eta_\ell = 0$ a.s. For the proof, we will use the following evaluation given by Devroye [5], Lemma 3.2:

Lemma 7. *If* $a_n \to 0$ *and* $a_n \log n \to \infty$ *as* $n \to \infty$, *then,*

$$(24) \qquad P(nM_n/\log n - 1 < -a_n) \sim \exp(-n^{a_n}), \qquad n \to \infty.$$

If we put $a_n = (c \log_j n)/\log n$ in (24), we obtain that $P(nM_n < \log n - c \log_j n) \sim \exp(-\exp(a_n \log n)) = \exp(-(\log_{j-1} n)^c)$. Hence, $E(\eta_\ell) \sim \exp(-(\log_{j-2}\sqrt{2\ell})^c)$. This evaluation taken with $j = 4$, $c > 1$ yields $E(\eta_\ell) = o(1/(\log \ell)^2)$. It follows that $\sum_{\ell=1}^{\infty} \eta_\ell/\ell$, having a finite expectation, is finite a.s.; by Kronecker's lemma (see Stout [12] page 120-121), it implies that $\lim_{N\infty}(1/N)\sum_{\ell=1}^{N} \eta_\ell = 0$ a.s., given an easy proof of (23) in that case.

To get the result for $j \geq 5$, we must think that if we could treat $\eta_1, \eta_2, \cdots$ as independent r.v., since they are evidently bounded, the result would follow easily from $\lim_{N\infty}(1/N)\sum_{\ell=1}^{N}(\eta_\ell = E(\eta_\ell)) = 0$ a.s., the latter being true in that case (see Révész [10], page 59).

From this idea, we will prove (23) by classical techniques of the theory of laws of large numbers. First, it is easy to check that if $\zeta_N = (1/N)\sum_{\ell=1}^{N}\eta_\ell$, $\lim_{N\infty}\zeta_N = 0$ a.s. iff there exists an $a > 1$ such that $\lim_{n\infty}\zeta_{[a^n]} = 0$ a.s. (it follows from the positivity of the η_ℓ). This is in turn implied by

(25) $$\sum_{n=1}^{\infty} D^2(\zeta_{[a^n]}) < \infty.$$

Thus we have to evaluate $D^2(\zeta_N) = N^{-2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}(E(\eta_i\eta_\ell) - E(\eta_i)E(\eta_\ell))$. Let us now choose $c > 1$ and $j \geq 4$; by Lemma 7, we get

(26) $$E(\eta_\ell) = o((\log_{j-3}\ell)^{-2}), \qquad \ell \to \infty.$$

On an other hand, $|E(\eta_i\eta_\ell) - E(\eta_i)E(\eta_\ell)| = |E(\eta_i(\eta_\ell - E(\eta_\ell)))| \leq E(\eta_i)$. Hence, if $f_i = [i/(\log i)^{1+\varepsilon}]$, and $A_N = N^{-2}\sum_{i=4}^{N}\sum_{\ell=i-f_i}^{i+f_i}\operatorname{Cov}(\eta_i, \eta_\ell)$, it follows from (26) that $A_N = O((\log N)^{-1-\varepsilon}(\log_{j-3}N)^{-2})$ as $N \to \infty$. Consequently if $\varepsilon > 0$ and $a > 1$ then $\sum_{n=1}^{\infty} A_{[a^n]} < \infty$.

For (25), it suffices therefore to prove that $\sum_{n=1}^{\infty} B_{[a^n]} < \infty$, where $B_N = N^{-2}\sum_{i=4}^{N}\sum_{\ell=i+f_i+1}^{N}\operatorname{Cov}(\eta_i, \eta_\ell)$. This follows from the fact that $A_N + 2B_N \geq 0$, and hence, that it suffices for (25) to get an upper bound for B_N.

We will now evaluate $E(\eta_i\eta_\ell)$, when $\ell > i + f_i$. To do so, let $C_{m,n} = P(M_{n+m} < u, M_n < v)$, and consider the maximal spacing M'_m generated by $U_{n+1}, \cdots, U_{n+m}$. Clearly $P(M'_m < u) = P(M_m < u)$. Since $M_{m+n} \leq M'_m$, and because of the fact that M_n and M'_m are independent, we have therefore $C_{m,n} \leq P(M_m < u)P(M_n < v)$. This gives $B_N \leq N^{-2}\sum_{i=4}^{N}\sum_{\ell=i+f_i+1}^{N} E(\eta_i)\{P(n_\ell M_{n_\ell - n_i} < \log n_\ell - c\log_j n_\ell) - E(\eta_\ell)\}$.

Next, if $\ell > i + f_i$, then $n_i/n_\ell \leq \exp(-f_i(1+o(1))/\sqrt{2i}) = \exp(-\sqrt{i/2}(1+o(1)))/(\log i)^{1+\varepsilon} = c_i \to 0$ as $i \to \infty$. It follows that $\log n_\ell = \log(n_\ell - n_i) + O(c_i)$, and likewise, for any $r \geq 2$, that $\log_r n_\ell = \log_r(n_\ell - n_i) + o(c_i)$. By similar arguments, one can check that $n_i(\log n_\ell)/n_\ell \leq n_i(\log n_{i+f_i})/n_{i+f_i} = \exp(-f_i(1+o(1))/\sqrt{2i}) = c'_i \to 0$ as $i \to \infty$. Noting that $c_i = n_i/n_{i+f_i} = o(c'_i)$, it follows, by taking together the preceding evaluations, that $P(n_\ell M_{n_\ell - n_i} \leq \log n_\ell - c\log_j n_\ell) = P((n_\ell - n_i)M_{n_\ell - n_i} \leq \log(n_\ell - n_i) - c\log_j n_\ell + O(c'_i))$.

To conclude, we must now precise Devroye's bound (24) by evaluating an upper bound of $|P(nM_n/\log n - 1 < -a_n) - \exp(-n^{a_n})|$, with the assumption that $a_n \sim c(\log_j n)/\log n$.

Devroye's proof (see [5]) relies on the fact that $M_{n-1} = K_n$ is distributed as L'/T_n, where L' is the largest of n independent identically exponentially distributed random variables whose sum is T_n. It follows from the inequalities (see [5], (3.3)):

$$P(L'_n < (1-a-b)\log n) - P(T_n < n(1-b))$$
$$\leq P(nM_n/\log n < 1-a) \leq P(L'_n < (1-a+b)\log n) + P(T_n \geq n(1+b)),$$

where $a = a_n$ and $b = n^{-1/4}$. By [5], Lemma 3.1, $P(|T_n - n| \geq bn) \leq 2\exp(-\sqrt{n}/4)$. Thus, it remains to evaluate $P(L'_n < (1-a\pm b)\log n) = (1-n^{-(1-a\pm b)})^n = \exp(-n^{a\mp b} + O(n^{-1+2a\mp 2b})) = \exp(-n^a)(1 + O((n^{a-1/4})\log n))$. Finally, if $a = a_n \sim c(\log_j n)/\log n$, it follows that for any $\theta > 0$,

(27) $$P(nM_n/\log n - 1 < -a_n) = \exp(-n^{a_n})(1 + o(n^{\theta-1/4})).$$

Going back to B_N, we deduce from (27) the following upper bound:

$$B_N \leq N^{-2}\sum_{i=4}^{N}\sum_{\ell=i+f_i+1}^{N} E(\eta_i)$$
$$\cdot\{\exp(-c\log_j n_\ell + O(c'_i)) - \exp(-c\log_j n_\ell) + o(n_\ell^{\theta-1/4})\}$$
$$= N^{-2}\sum_{i=4}^{N}\sum_{\ell=i+f_i+1}^{N} E(\eta_i)E(\eta_\ell)(O(c'_i) + o(n_\ell^{\theta-1/4}))(1+o(1)).$$

By choosing $0 < \theta < 1/4$, a straightforward evaluation shows that $B_N = O(N^{-1})$. Hence $\sum_{n=1}^{\infty} B_{[a^n]} < \infty$. The proof of Lemma 6 is now complete.

Going back to the sequence $\{\ell(r),\ r \geq 1\}$ defined in (20), we extract from it a subsequence, by putting:

$$\lambda(1) = \min\{\ell(r),\ r \geq 1,\ n_{\ell(r)+1} M_{n_{\ell(i)+1}} - \log(n_{\ell(r)+1}) \geq -c \log_j (n_{\ell(r)+1})\},$$
$$\lambda(r) = \min\{\ell(i) > \lambda(r-1),\ n_{\ell(i)+1} M_{n_{\ell(i)+1}} - \log(n_{\ell(i)+1}) \geq -c \log_j (n_{\ell(i)+1})\}, \qquad r \geq 2, \tag{28}$$

LEMMA 8. *If* $\{\lambda(r), r \geq 1\}$ *is defined by* (28), *then*

$$\lim_{r\infty} \lambda(r)/r = \frac{e}{e-1} \quad \text{a.s.,} \tag{29}$$

and $\{n_{\lambda(r)+1}, r \geq 1\}$ *is an increasing sequence of stopping times on* $\{\sigma(U_1, \cdots, U_n)\}$.

PROOF. It is a direct consequence of (21) and of Lemma 6, (23).

LEMMA 9. *Let* $\{\lambda(r), r \geq 1\}$ *be defined by* (28), *and put for* $r = 1, 2, \cdots$ $\delta_r = T_{\lambda(r)+1}$, $\{T_\ell,\ \ell \geq 1\}$ *being defined in* (19)–(20), *then* $\{\delta_r,\ r \geq 1\}$ *is a sequence of independent exponentially* $E(1)$ *distributed random variables.*

PROOF. It follows easily from (28) and Lemma 4.

LEMMA 10. *For any* $j \geq 1$,

$$P(\delta_r \geq \log(\lambda(r)) + \log_2(\lambda(r)) + \cdots + \log_j(\lambda(r)) \text{i.o.}) = 1. \tag{30}$$

PROOF. By Lemma 4 and as in (11), we get easily that for any $j \geq 1$, $P(\delta_r \geq \log r + \log_2 r + \cdots + \log_{j+1} r \text{ i.o.}) = 1$. Let now $C = e/(e-1)$; it follows from Lemma 8 and the preceding result that $P(\delta_r \geq \log(\lambda(r)) - \log C + o(1) + \log_2(\lambda(r)) + \cdots + \log_{j+1}(\lambda(r))\text{i.o.}) = 1$. This proves (30).

We are now ready to derive the final step of the proof of Theorem 2. If $r \geq 1$ is arbitrary, put $n = n_{\lambda(r)+1}$, and $Z = Z_n = \delta_r$. From (28), we get:

$$nM_n \geq \log n - c \log_j n.$$

On the other hand, by Lemma 10 and (30), remembering that $n_\ell = [\exp(\sqrt{2\ell})]$,

$$Z_n \geq 2 \log_2 n + \log_3 n + \cdots + \log_j n \quad \text{i.o., w.p.1.}$$

Since by (19), $N_{k(n)+1} - 1 = n + (Z_n/M_n)(1 + O(M_n))$, if we put $N = N_{k(n)+1} - 1$, then:

$$NM_N = NM_n = nM_n + Z_n + Z_n O(M_n),$$

and hence

$$NM_N \geq \log n + 2 \log_2 n + \log_3 n + \cdots + \log_{j-1} n + (1 - c)\log_n n + o(1) \quad \text{i.o.,}$$

with probability one. Finally, since $\log n = \log N + O(1)$ a.s., $j \geq 4$ and $c > 0$ being arbitrary, it implies that (18) is true. Hence the proof of Theorem 2 is complete.

The result can be stated in an equivalent form:

COROLLARY. *For any* $j \geq 4$,

$$P(nM_n - \log n - 2 \log_2 n - \cdots - (1 + \varepsilon)\log_j n \geq 0 \quad \text{i.o.}) = 0 \quad \text{or} \quad 1, \tag{31}$$

according to whether $\varepsilon > 0$ *or* $\varepsilon \leq 0$.

PROOF. (31) can be deduced directly from (18) for $\varepsilon > 0$ and $\varepsilon < 0$. The case $\varepsilon = 0$ follows from the fact that $nM_n - \log n - 2\log_2 n - \cdots - (1 - \varepsilon)\log_j n \geq 0$ implies that $nM_n - \log n - 2 \log_2 n - \cdots - \log_{j-1} n \geq 0$ for $0 < \varepsilon < 1$.

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