The Annals of Probability
1982, Vol. 10, No. 4, 1047–1050

ON THE CENTRAL LIMIT THEOREM FOR STATIONARY MIXING RANDOM FIELDS

BY E. BOLTHAUSEN

Technische Universität Berlin

A simple proof of a central limit theorem for stationary random fields under mixing conditions is given, generalizing some results obtained by more complicated methods, e.g. Bernstein's method.

We consider a real valued stationary random field X_ρ, $\rho \in \mathbb{Z}^d$, i.e., the X_ρ are real random variables and the joint laws are shift invariant. We shall always assume that $EX_\rho^2 < \infty$. If $\rho_1, \rho_2 \in \mathbb{Z}^d$, let $d(\rho_1, \rho_2) = \max_{1 \le i \le d} |\rho_1(i) - \rho_2(i)|$, where $\rho(i)$, $1 \le i \le d$, are the components of ρ. If $\Lambda \subset \mathbb{Z}^d$ we write $|\Lambda|$ for the number of elements in Λ and $\partial\Lambda = \{\rho \in \Lambda$: there exists $\rho' \notin \Lambda$ with $d(\rho, \rho') = 1\}$. Let Λ_n be a fixed sequence of finite subsets of $\mathbb{Z}^d$, which increases to $\mathbb{Z}^d$ and satisfies

$$\lim_{n\to\infty} |\partial\Lambda_n|/|\Lambda_n| = 0. \tag{1}$$

Let $S_n = \sum_{\rho\in\Lambda_n}(X_\rho - \mu)$ where $\mu = EX_\rho$.

It is generally believed that the existence and positivity of $\sigma^2 = \sum_{\rho\in\mathbb{Z}^d} \text{cov}(X_0, X_\rho)$ has something to do with the validity of the central limit theorem, i.e., the asymptotic normality of $S_n/\sigma|\Lambda_n|^{1/2}$. Newman [6] proved that the existence and positivity of σ^2 together with the quite strong FKG conditions is sufficient for the central limit theorem. A central limit theorem is proved here under mixing conditions with the slowest possible decrease ensuring the existence of σ^2. In case $d = 1$, such theorems have been proved by Ibragimov and Linnik [3] and Gordin [2].

Actually, our Theorem as it is stated does not include these results. However, it is easy to prove a slightly more general result which includes the classical central limit theorems for $d = 1$. (See the remarks following the proof of our Theorem.) For $d > 1$, theorems in this direction have been obtained by Neaderhouser [5] and Nahapetian [4], but their conditions are stronger than those used here. Most of the proofs are based on Bernstein's method of dividing the sum into blocks and approximating then by independent variables. Gordin uses an approximation by martingales, but his method appears difficult to generalize to dimensions ≥ 2. Also Bernstein's method becomes a bit troublesome in higher dimensions for reasons I shall indicate below. We shall give here a very simple and direct proof which avoids any approximating techniques and which is based on an idea of Ch. Stein.

If $\Lambda \subset \mathbb{Z}^d$, let $\mathscr{A}_\Lambda$ be the σ-algebra generated by the X_ρ, $\rho \in \Lambda$. If $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$, let $d(\Lambda_1, \Lambda_2) = \inf\{d(\rho_1, \rho_2): \rho_1 \in \Lambda_1, \rho_2 \in \Lambda_2\}$. The mixing coefficients we use are defined as follows, if $n \in \mathbb{N}$, $k, \ell \in \mathbb{N} \cup \{\infty\}$

$$\alpha_{k,\ell}(n) = \sup\{|P(A_1 \cap A_2) - P(A_1)P(A_2)| : A_i \in \mathscr{A}_{\Lambda_i}, |\Lambda_1| \le k, |\Lambda_2| \le \ell, d(\Lambda_1, \Lambda_2) \ge n\}$$

$$\rho(n) = \sup\{|\text{cov}(Y_1, Y_2)| : Y_i \in L_2(\mathscr{A}_{\{\rho_i\}}), \quad \|Y_i\|_2 \le 1, \quad d(\rho_1, \rho_2) \ge n\}.$$

Theorems based on conditions on $\alpha_{\infty,\infty}$ are quite useless for applications to Gibbs fields as has been remarked by Dobrushin [1].

THEOREM. *If* $\sum_{m=1}^\infty m^{d-1}\alpha_{k\ell}(m) < \infty$ *for* $k + \ell \le 4$, $\alpha_{1\infty}(m) = o(m^{-d})$ *and if*

a) $$\sum_{m=1}^\infty m^{d-1}\rho(m) < \infty \quad or$$

b) *for some* $\delta > 0$ $\|X_\rho\|_{2+\delta} < \infty$ *and* $\sum_{m=1}^\infty m^{d-1}\alpha_{1,1}(m)^{\delta/(2+\delta)} < \infty.$

Received September 1981; revised April 1982.

AMS 1980 *subject classifications.* 60F05, 60G10, 60G60.

Key words and phrases. Central limit theorem, stationary random fields, mixing conditions.

Then $\sum_{\rho\in\mathbb{Z}^d}|\operatorname{cov}(X_0, X_\rho)| < \infty$ *and if* $\sigma^2 = \sum_\rho \operatorname{cov}(X_0, X_\rho) > 0$, *then the laws of* $S_n/\sigma|\Lambda_n|^{1/2}$ *converge to the standard normal one.*

PROOF. The convergence of $\sum_\rho|\operatorname{cov}(X_0, X_\rho)|$ follows from the following well-known lemma (see [3] Theorems 7.2.2 and 7.2.3).

LEMMA 1. *If the* $Z_\iota \in L_{2+\delta}(\mathscr{A}_{\{\rho_\iota\}})$ *for some* $\delta > 0$, *then*

$$|\operatorname{cov}(Z_1, Z_2)| \le c_\delta \alpha_{1,1}(d(\rho_1, \rho_2))^{\delta/(2+\delta)} \| Z_1 \|_{2+\delta} \| Z_2 \|_{2+\delta}.$$

We now apply the simple truncation technique which has also been used by Ibragimov and Linnik: If $N > 0$ let $f_N: \mathbb{R} \to \mathbb{R}$ be defined by $f_N(x) = (x \wedge N) \vee (-N)$ and $\tilde{f}_n = id - f_N$. $X_\rho = f_N(X_\rho) + \tilde{f}_N(X_\rho)$ and

$$E(\textstyle\sum_{\rho\in\Lambda_n}(\tilde{f}_n(X_\rho) - E\tilde{f}_n(X_\rho))/\sigma|\Lambda|^{1/2})^2 = \sum_{\rho,\tau\in\Lambda_n}\operatorname{cov}(\tilde{f}_N(X_\rho)\tilde{f}_N(X_\tau))/\sigma^2|\Lambda_n|$$

which converges to 0 as $N \to \infty$ uniformly in n. It therefore suffices to prove the theorem for bounded variables. For bounded variables we need only $\sum_{m=1}^\infty m^{d-1}\alpha_{k\ell}(m) < \infty$ and $\alpha_{1\infty}(m) = o(m^{-d})$. The proof is then based on the following lemma which is in the spirit of Stein's paper [7].

LEMMA 2. *Let* ν_n, $n \in \mathbb{N}$, *be a sequence of probability measures in* $\mathbb{R}$ *with*

a) $$\sup_n \int |x|^2 \nu_n(dx) < \infty,$$

b) $$\lim_{n\to\infty} \int (i\lambda - x)e^{i\lambda x}\nu_n(dx) = 0 \quad \textit{for all} \quad \lambda \in \mathbb{R}.$$

Then the ν_n *converge to the standard normal law.*

PROOF. a) implies tightness of the sequence $\{\nu_n\}$. If ν is any limit law, then

$$\int x^2\nu(dx) \le \sup_n \int x^2\nu_n(dx) \quad \text{and if} \quad \nu_{n_k} \to \nu,$$

then

$$\lim_{n\to\infty} \int x\nu_{n_k}(dx) = \int x\nu(dx).$$

Therefore

$$\int (i\lambda - x)e^{i\lambda x}\nu(dx) = 0.$$

It follows that ν is the standard normal law. So the lemma is proved.

We continue with the proof of the theorem for bounded variables: We may assume $EX_\rho = 0$. As $\alpha_{k,\ell}(m)$ is decreasing, we have for $k + \ell \le 4$ $\alpha_{k,\ell}(m) = o(m^{-d})$. We choose a sequence m_n, $n \in \mathbb{N}$, with $\alpha_{k,\ell}(m_n)|\Lambda_n|^{1/2} \to 0$ and $m_n^{-d}|\Lambda_n|^{1/2} \to \infty$ as $n \to \infty$. If $\alpha \in \mathbb{Z}^d$, let $S_{\alpha,n} = \sum_{\beta\in\Lambda_n, d(\alpha,\beta)\le m_n} X_\beta$ and $a_n = \sum_{\alpha\in\Lambda_n} E(X_\alpha S_{\alpha,n})$, $\bar{S}_n = a_n^{-1/2}S_n$, $\bar{S}_{\alpha,n} = a_n^{-1/2}S_{\alpha,n}$. From (1) and $\sum_{\beta\in\mathbb{Z}^d}|\operatorname{cov}(X_\alpha, X_\beta)| < \infty$ we obtain

$$a_n = \operatorname{var}(S_n)(1 + o(1)) = |\Lambda_n|\sigma^2(1 + o(1)). \tag{2}$$

It therefore suffices to prove that $\bar{S}_n$ is asymptotically standard normally distributed. As

$\sup_n E\bar{S}_n^2 < \infty$ it follows from Lemma 2 that it suffices to prove that

(3) $$\lim_{n\to\infty} E((i\lambda - \bar{S}_n)e^{i\lambda\bar{S}_n}) = 0 \quad \text{for any} \quad \lambda \in \mathbb{R}.$$

Write

(4) $$\begin{aligned}(i\lambda - \bar{S}_n)e^{i\lambda\bar{S}_n} &= i\lambda e^{i\lambda\bar{S}_n}(1 - a_n^{-1}\textstyle\sum_{\alpha\in\Lambda_n} X_\alpha S_{\alpha,n}) \\ &\quad - a_n^{-1/2}e^{i\lambda\bar{S}_n}\textstyle\sum_{\alpha\in\Lambda_n} X_\alpha[1 - e^{-i\lambda\bar{S}_{\alpha,n}} - i\lambda\bar{S}_{\alpha,n}] \\ &\quad - a_n^{-1/2}\textstyle\sum_{\alpha\in\Lambda_n} X_\alpha e^{i\lambda(\bar{S}_n - \bar{S}_{\alpha,n})} = A_1 - A_2 - A_3\end{aligned}$$

say.

Now $E(|A_1|^2) = \lambda^2 a_n^{-2}\sum_{\alpha,\alpha',\beta,\beta'\, d(\alpha,\beta)\le m,\, d(\alpha',\beta')\le m} \operatorname{cov}(X_\alpha X_\beta, X_{\alpha'}X_{\beta'})$, all summation indices belonging to Λ_n. If $d(\alpha, \alpha') = k \ge 3m$, we have

(5) $$|\operatorname{cov}(X_\alpha X_\beta, X_{\alpha'}X_{\beta'})| \le \alpha_{2,2}(k - 2m).$$

If $\min(d(\alpha, \alpha'), d(\alpha, \beta), d(\alpha, \beta')) = j$ we have

$$\begin{aligned}|\operatorname{cov}(X_\alpha X_\beta, X_{\alpha'}X_{\beta'})| &\le |E(X_\alpha X_{\alpha'}X_\beta X_{\beta'})| + |E(X_\alpha X_\beta)|\,|E(X_{\alpha'}, X_{\beta'})| \\ &\le c\alpha_{1,3}(j) \quad \text{for some constant} \quad c \ge 0.\end{aligned}$$

Therefore

$$\begin{aligned}E|A_1|^2 &\le \lambda^2 a_n^{-2}|\Lambda_n|(m^{2d}\textstyle\sum_{k=3m}^\infty k^{d-1}\alpha_{2,2}(k - 2m) + c'm^{2d}\sum_{j=0}^{3m} j^{d-1}\alpha_{1,3}(j)) \\ &= O(|\Lambda_n|^{-1}m^{2d}) = o(1). \\ E|A_2| &\le c a_n^{1/2}\sup_{\alpha\in\Lambda_n} E\bar{S}_{\alpha,n}^2 \le c' a_n^{-1/2}\textstyle\sum_{\beta,\beta',d(0,\beta)\le m,d(0,\beta')\le m} E(X_\beta X_{\beta'}) \\ &\le c'' a_n^{-1/2} m_n^d = o(1).\end{aligned}$$

$|EA_3| \le c a_n^{1/2}\alpha_{1,\infty}(m) = o(1)$. So (3) and therefore the theorem follow.

REMARK 1. The various types of mixing coefficients are a bit disturbing. If one prefers to use just one, one sees that e.g. $\|X\|_{2+\delta} < \infty$ and $\sum_{m=1}^\infty m^{d-1}\alpha_{2,\infty}^{\delta/(2+\delta)}(m)$ is sufficient for the theorem. It would be quite nice if one could replace $\alpha_{2,\infty}$ by $\alpha_{1,\infty}$; however, a proof eludes me.

REMARK 2. One of the advantages of the method is that one does not need conditions on $\alpha_{k,\ell}$ for arbitrary large k, ℓ. Nahapetian e.g. uses the condition

$$\textstyle\sum_{m=1}^\infty m^{d-1}[\sup_k(\varphi_{k,\infty}(m)/k)]^{1/2} < \infty$$

where $\varphi_{k,\infty}$ is the uniform strong mixing coefficient

$$\sup\{|P(A_2 \mid A_1) - P(A_2)| : A_i \in \mathscr{A}_{\Lambda_i}, \quad d(\Lambda_1, \Lambda_2) \ge m, \quad |\Lambda_1| \le k\}.$$

The need of such conditions seems to be inherent in Bernstein's method, as one has to estimate the dependence between blocks consisting of a large number of random variables.

REMARK 3. For $d = 1$ our theorem does not contain the central limit theorems usually stated for sequences (see [3]). These are based on the following mixing coefficients. Let:

$$\begin{aligned}&\mathscr{A}_m = \sigma(X_n : n \le m), \quad \mathscr{A}^m = \sigma(X_n : n \ge m), \quad m \in \mathbb{Z}, \quad \text{and} \\ &\tilde{\alpha}(n) = \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathscr{A}_0, \quad B \in \mathscr{A}^m\} \\ &\tilde{\rho}(n) = \sup\{|\operatorname{cov}(Z_1, Z_2)| : Z_1 \in L_2(\mathscr{A}_0), \quad Z_2 \in L_2(\mathscr{A}^n), \quad \|Z_i\|_2 \le 1\}.\end{aligned}$$

In fact, it seems to be impossible to estimate the $\alpha_{k,\ell}$ in terms of $\tilde{\alpha}$ or $\tilde{\rho}$.

However, our theorem can be slightly generalized: Define

$$\alpha'_{1,\infty}(n) = \sup\{|P(A \cap B \cap C) - P(A)P(B \cap C)| : A \in \sigma(X_0), \quad B \in \mathscr{F}_{-n}, \quad C \in \mathscr{F}^n\}$$

$$\alpha'_{k,\ell}(n) = \sup\{|P(A_1 \cap \cdots \cap A_k \cap A_{k+1} \cap \cdots \cap A_{k+\ell}) - P(A_1 \cap \cdots \cap A_k)P(A_{k+1} \cap \cdots \cap A_{k+\ell})| : A_i \in \sigma(X_{n_i}), d(\{n_1, \cdots, n_k\}, \{n_{k+1}, \cdots, n_{k+\ell}\}) \geq n\}.$$

Then it is easy to see that $\alpha'_{1,\infty}(n) \leq 3\tilde{\alpha}(n)$, $\alpha'_{k,\ell}(n) \leq 5\tilde{\alpha}(n)$ if $k + \ell \leq 4$. Furthermore, one easily checks that our theorem remains true if $\alpha_{1,\infty}$ and $\alpha_{k,\ell}$ are replaced by these coefficients.

So one obtains that, if

a) $\| X \|_{2+\delta} < \infty$, $\sum_{m=1}^{\infty} \tilde{\alpha}(m)^{\delta/(2+\delta)} < \infty$ or

b) $\sum_{m=1}^{\infty} \tilde{\rho}(m) < \infty$, then the conclusion of the theorem remains true. It is not difficult to give also a slight generalization in this direction of our theorems for $d > 1$, but as the notations become messy and there seems to be no use of this, I do not give it.

Acknowledgment. I thank the referee for suggesting a number of improvements of the original manuscript.

REFERENCES

[1] DOBRUSHIN, R. L. (1968). The description of the random field by its conditional distribution. *Theory Probab. Appl.* **13** 201–229.
[2] GORDIN, M. I. (1969). The central limit theorem for stationary sequences. *Soviet Math. Dokl.* **10** 1174–1176.
[3] IBRAGIMOV, I. A. and LINNIK, YU. V. (1971). *Independent and Stationary Random Variables.* Wolters-Noordhoff, Groningen.
[4] NAHAPETIAN, B. S. (1980). The central limit theorem for random fields with mixing conditions, in *Adv. in Probability* **6**, Multicomponent Systems, ed. R. L. Dobrushin and Ya. G. Sinai, 531–548.
[5] NEADERHOUSER, C. C. (1980). Convergence of block spins defined on random fields, *J. Statist. Phys.* **22** 673–684.
[6] NEWMAN, C. M. (1980). Normal fluctuations and the FKG inequalities; *Comm. Math. Phys.* **74** 119–128.
[7] STEIN, CH. (1973). A bound for the error in the normal approximation of a sum of dependent random variables, Proc. 6th Berkley Symp. Math. Stat. and Prob., **2** 583–602.

TECHNISCHE UNIVERSITÄT BERLIN
FACHBEREICH MATHEMATIK
D-1000 BERLIN 12