The Annals of Probability
1982, Vol. 10, No. 4, 1032–1040

MOMENT AND PROBABILITY BOUNDS WITH QUASI-SUPERADDITIVE STRUCTURE FOR THE MAXIMUM PARTIAL SUM[1]

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Let $X_1, \cdots, X_n$ be arbitrary random variables and put $S(i, j) = X_i + \cdots + X_j$ and $M(i, j) = \max\{|S(i, i)|, |S(i, i+1)|, \cdots, |S(i, j)|\}$ for $1 \le i \le j \le n$. Bounds for $E\{\exp tM(1, n)\}$, $E\, M^\gamma(1, n)$ and $P\{M(1, n) \ge t\}$ are established in terms of assumed bounds for $E\{\exp t|S(i, j)|\}$, $E|S(i, j)|^\gamma$ and $P\{|S(i, j)| \ge t\}$, respectively. The bounds explicitly involve a nonnegative function $g(i, j)$ assumed to be *quasi-superadditive* with index $Q\,(1 \le Q \le 2)$: $g(i, j) + g(j+1, k) \le Q\, g(i, k)$, all $1 \le i \le j < k \le n$. Results previously established for the case $Q = 1$ are improved and are extended to the case $1 < Q < 2$. When $g(i, j)$ is given by $\operatorname{Var} S(i, j)$, applications of the case $Q > 1$ include sequences $\{X_i\}$ exhibiting long-range dependence, in particular certain self-similar processes such as fractional Brownian motion.

1. Introduction.

Let $X_1, \cdots, X_n$ be arbitrary random variables. The only restrictions on the joint distribution of the X_k's will be those imposed by the assumed bounds on the quantities

$$(1.1) \qquad E\{\exp t|S(i, j)|\}, \qquad E|S(i, j)|^\gamma, \qquad P\{|S(i, j)| \ge t\}, \qquad 1 \le i \le j \le n,$$

where

$$S(i, j) = \textstyle\sum_{k=i}^{j} X_k, \qquad 1 \le i \le j \le n,$$

$\gamma \ge 1$ is a fixed real number, and t runs through an interval of the positive real line. Define

$$M(i, j) = \max\{|S(i, i)|, \quad |S(i, i+1)|, \cdots, |S(i, j)|\}, \qquad 1 \le i \le j \le n,$$

and for convenience set $S(i, j) = M(i, j) = 0$ for $j < i$. We will establish bounds for the quantities

$$E\{\exp tM(1, n)\}, \qquad E\, M^\gamma(1, n), \qquad P\{M(1, n) \ge t\}$$

in terms of the respective assumed bounds for the quantities in (1.1).

The bounds will relate to the variables $S(i, j)$ in specified ways through some function $g(i, j)$ satisfying

$$(1.2a) \qquad g(i, j) \ge 0, \quad \text{all} \quad 1 \le i \le j \le n,$$

$$(1.2b) \qquad g(i, j) \le g(i, j+1), \quad \text{all} \quad 1 \le i \le j \le n,$$

$$(1.2c) \qquad g(i, j) + g(j+1, k) \le Q\, g(i, k), \quad \text{all} \quad 1 \le i \le j < k \le n,$$

where $1 \le Q < 2$. We call the property (1.2c) *quasi-superadditivity with index* Q (or simply Q-*superadditivity*); the case $Q = 1$ corresponds to the usual notion of superadditivity.

Received July 1981; revised December 1981.

[1] Research supported by the Army, Navy and Air Force under Office of Naval Research Contract No. N00014-79-C-0801. Reproduction in whole or in part is permitted for any purpose of the United States Government.

AMS 1970 *subject classifications.* Primary, 60G99.

Key words and phrases. Fluctuation of sums; moment inequalities; probability inequalities; quasi-superadditivity.

REMARKS ON QUASI-SUPERADDITIVITY. (i) In many situations the function $g(i, j)$ is given by

$$g(i,j) = \operatorname{Var} S(i,j), \qquad 1 \le i \le j \le n, \tag{1.3}$$

in which case 1-superadditivity is equivalent to

$$\operatorname{Cov}\{S(i,j),\ \ S(j+1,k)\} \ge 0, \quad \text{all} \quad 1 \le i \le j < k \le n. \tag{1.4}$$

Thus, for this choice of $g(i, j)$, the notion of Q-superadditivity for $Q > 1$ is needed for any sequence $\{X_k\}$ for which any one of the inequalities in (1.4) is violated. Indeed, for $g(i, j)$ given by (1.3), it is easily seen that

$$\frac{1}{1+\rho^+} g(i,k) \le g(i,j) + g(j+1,k) \le \frac{1}{1-\rho^-} g(i,k), \qquad 1 \le i \le j < k \le n,$$

where ρ^+ and ρ^- are the maxima of the positive and negative parts of the quantities $\operatorname{Corr}\{S(i,j), S(j+1,k)\}$, respectively. In other words,

$$\rho^+ = \max_{1\le i\le j<k\le n}[\rho(i,j,k)]^+ \quad \text{and} \quad \rho^- = \max_{1\le i\le j<k\le n}[\rho(i,j,k)]^-,$$

where

$$\rho(i,j,k) = \operatorname{Corr}\{S(i,j), S(j+1,k)\}.$$

It is clear that in the case of independence, $\rho^+ = \rho^- = 0$.

(ii) For the case that $g(i, j)$ is of the form $g(i,j) = g(j - i + 1)$, (1.2c) becomes

$$g(i) + g(j-i) \le Q\, g(j), \quad \text{all} \quad 1 \le i \le j \le n. \tag{1.5}$$

In particular, for $g(n)$ of the form $g(n) = n^r$, (1.5) is satisfied with Q given by

$$\begin{aligned} Q_r &= 2^{1-r}, \qquad && 0 < r < 1, \\ &= 1, && r \ge 1. \end{aligned}$$

Note that for $0 < r < 1$ we have $1 < Q_r < 2$.

(iii) Combining the two previous remarks, we see that for $g(i, j)$ given by (1.3) the case of Q-superadditivity for $Q > 1$ corresponds, roughly speaking, to the property that $\operatorname{Var} S(1, n)$ increases slower than $O(n)$ as $n \to \infty$. Sequences $\{X_i\}$ exhibiting long-range dependence possess this type of dependence structure. See Mandelbrot and Taqqu (1979) and Taqqu (1979) for examples including fractional Brownian motion and other self-similar processes.

(iv) For $Q = 1$, as pointed out by Longnecker and Serfling (1977), there exist nonnegative constants $u_1, u_2, \cdots, u_n$ such that

$$g(1,n) = \textstyle\sum_{k=1}^{n} u_k \quad \text{and} \quad g(i,j) \le \sum_{k=i}^{j} u_k, \quad \text{all} \quad 1 \le i \le j \le n.$$

In this case the theorems below may be proved by reduction to the case that $g(i, j)$ is exactly of the form $g(i,j) = \sum_{k=i}^{j} u_k$.

(v) The superadditivity property (corresponding to $Q = 1$) has been considered in this context by Serfling (1970) and Móricz (1975), for example. However, the quasi-superadditivity notion is new with the present paper.

(vi) Note that in the case $Q = 1$, (1.2a) and (1.2c) imply (1.2b). □

Section 2 will present *exponential* estimates, that is, exponential bounds on $E\{\exp(tM(1,n))\}$ and $P\{M(1,n) \ge t\}$ in terms of similar bounds on $E\{\exp(t\,|S(1,n)|)\}$ and $P\{|S(1,n)| \ge t\}$, respectively. Section 3 will treat *power-type* estimates.

2. Exponential estimates. This section deals with bounds of exponential form.

THEOREM 2.1. *Suppose that there exist a constant $K \geq 1$, a function $g(i,j)$ satisfying (1.2) with $1 \leq Q < 2$ and a $t_0 \geq 0$ such that*

$$E\{\exp t|S(i,j)|\} \leq K \exp \phi(t)g(i,j), \quad \text{all} \quad t > t_0 \quad \text{and} \quad 1 \leq i \leq j \leq n, \tag{2.1}$$

where $\phi(t) > 0$ for $t > t_0$ and for each constant $C > 1$

$$\sup_{t>t_0}\phi(Ct)/\phi(t) = \chi(C) < \infty; \qquad \lim_{C\to 1+}\chi(C) = 1. \tag{2.2}$$

Then there exist constants $A \geq 1$ and $B \geq 1$, depending on Q and χ but not on n or $\{X_k\}$ or otherwise on g or ϕ, such that

$$E\{\exp t\,M(1,n)\} \leq AK \exp B\phi(t)g(1,n), \quad \text{all} \quad t > t_0. \tag{2.3}$$

Note that Condition (2.2) essentially means that $\phi(t)$ does not grow faster than a polynomial. In particular, for each $\gamma \geq 0$, the function $\phi(t) = t^\gamma$ satisfies (2.2). The special case $\phi(t) = t^2$ and $Q = 1$ is given by Móricz (1976b).

PROOF OF THEOREM 2.1 Let $\beta \in (Q-1, 1)$ be given, for example $\beta = Q/2$. Choose $q > 1$ such that

$$\chi(q) \leq \min\left\{\frac{1}{\beta}, \frac{1}{Q-\beta}\right\}, \tag{2.4}$$

which is possible due to (2.2), and set

$$A = 2^p \quad \text{and} \quad B = \chi(p), \tag{2.5}$$

where $1/p + 1/q = 1$.

The theorem holds trivially in the case $n = 1$. Assume the induction hypothesis that the result holds for all integers n satisfying $1 \leq n < N$. We will show that the result then follows also for $n = N$.

It is easily seen that, for any $m \in \{1, 2, \cdots, N\}$ and $t > 0$,

$$E\{\exp t\,M(1,N)\} \leq E\{\exp t\,M(1,m-1)\} + E\{\exp[t|S(1,m)| + t\,M(m+1,N)]\}. \tag{2.6}$$

For the β given above, let m be determined by

$$g(1,m-1) \leq \beta\, g(1,N) \leq g(1,m). \tag{2.7}$$

In the case $m = 1$, our convention is $g(1,0) = 0$. Thus also, by (1.2) and (2.7), we have

$$g(m+1,N) \leq (Q-\beta)g(1,N). \tag{2.8}$$

By Jensen's inequality, the induction hypothesis, and (2.7), we obtain

$$\begin{aligned} E\{\exp t\,M(1,m-1)\} &\leq (E\{\exp qt\,M(1,m-1)\})^{1/q} \\ &\leq (AK \exp B\phi(qt)g(1,m-1))^{1/q} \\ &\leq A^{1/q}K \exp[\beta B\phi(qt)g(1,N)/q]. \end{aligned}$$

Hence, using (2.2) and (2.4),

$$E\{\exp t\,M(1,m-1)\} \leq A^{1/q}K \exp B\phi(t)g(1,N). \tag{2.9}$$

Also, by Hölder's inequality, the induction hypothesis, (1.2), (2.1) and (2.8), we have

$$
\begin{aligned}
&E\{\exp[t|S(1,m)| + tM(m+1,N)]\}\\
&\qquad \le (E\{\exp pt|S(1,m)|\})^{1/p}(E\{\exp qt\,M(m+1,N)\})^{1/q}\\
&\qquad \le (K\exp\phi(pt)g(1,m))^{1/p}(AK\exp B\phi(qt)g(m+1,N))^{1/q}\\
&\qquad \le A^{1/q}K\exp B\,g(1,N)[\phi(pt)/pB + \phi(qt)(Q-\beta)/q].
\end{aligned}
$$

Taking into account (2.2), (2.4) and (2.5), we obtain

$$
\begin{aligned}
&E\{\exp[t|S(1,m)| + tM(m+1,N)]\}\\
&\qquad \le A^{1/q}K\exp B\phi(t)g(1,N)[\chi(p)/pB + \chi(q)(Q-\beta)/q]\\
&\qquad \le A^{1/q}K\exp B\phi(t)g(1,N).
\end{aligned}
\tag{2.10}
$$

Collecting (2.6), (2.9) and (2.10) together, we have

$$E\{\exp t\,M(1,n)\} \le 2A^{1/q}K\exp B\phi(t)g(1,N),$$

which is equivalent to the desired inequality (2.3) for $n = N$, owing to (2.5). This completes the induction argument and the proof of Theorem 2.1. □

THEOREM 2.2. *Suppose that there exist a constant* $K \ge 1$, *a function* $g(i,j)$ *satisfying* (1.2) *with* $1 \le Q < 2$, *and a* t_0, $0 < t_0 \le +\infty$, *such that*

$$P\{|S(i,j)| \ge t\} \le K\exp[-\phi(t)/g(i,j)], \quad \textit{all} \quad 0 < t < t_0 \tag{2.11}$$

and $1 \le i \le j \le n$, *where* $\phi(t) > 0$ *for* $0 < t < t_0$ *and for each constant* C, $0 < C < 1$,

$$\inf_{0<t<t_0}\phi(Ct)/\phi(t) = \chi(C) > 0; \qquad \lim_{C\to 1-}\chi(C) = 1. \tag{2.12}$$

Then there exist constants $A \ge 1$ *and* $B \ge 1$, *depending on* Q *and* χ *but not on* n *or* $\{X_k\}$ *or otherwise on* g *or* ϕ, *such that*

$$P\{M(1,n) \ge t\} \le AK\exp[-\phi(t)/Bg(1,n)], \quad \textit{all} \quad 0 < t < t_0. \tag{2.13}$$

REMARKS. (i) If $g(i,j) = 0$ for certain i and j, our convention is that the right-hand side of (2.11) is equal to 0 for all $t > 0$. That is, in this case $P\{|S(i,j)| = 0\} = 1$, which means that $S(i,j) = 0$ almost surely. Now, if $g(1,n) = 0$, then $g(1,j) = 0$ for all j, $1 \le j \le n$, *a fortiori* $M(1,n) = 0$ almost surely, and inequality (2.13) to be proved is trivially satisfied.

(ii) Setting $\phi^*(t) = 1/\phi(1/t)$, $\phi^*(t)$ is then defined for all $t > 1/t_0 = t_0^*$. It is easily checked that $\phi(t)$ satisfies (2.12) with a $\chi(C)$, $0 < C < 1$, if and only if $\phi^*(t)$ satisfies (2.2) with a $\chi^*(C) = 1/\chi(1/C)$, the latter function being defined for $C > 1$.

(iii) The case $Q = 1$ was treated (under somewhat more restrictive conditions on ϕ) by Móricz (1979). □

PROOF OF THEOREM 2.2 This time let $\beta \in (Q/2, 1)$. Choose q, $0 < q < 1$, in such a way that

$$\chi(q) \ge \frac{Q-\beta}{\beta}, \tag{2.14}$$

which is possible because of (2.12), and set

$$A = 3^{\beta/(1-\beta)} \quad \text{and} \quad B = 1/\beta\chi(p), \tag{2.15}$$

where $p + q = 1$.

The induction argument of the previous theorem is used. Thus, for N and m the same, we write

$$(2.16)\quad \begin{aligned} &P\{M(1, N) \ge t\} \\ &\le P\{M(1, m-1) \ge t\} + P\{|S(1, m)| + M(m+1, N) \ge t\} \\ &\le P\{M(1, m-1) \ge t\} + P\{|S(1, m)| \ge pt\} + P\{M(m+1, N) \ge qt\}. \end{aligned}$$

By the induction hypothesis and (2.11),

$$P\{M(1, N) \ge t\} \le AK \exp[-\phi(t)/Bg(1, m-1)] + K \exp[-\phi(pt)/g(1, m)] + AK \exp[-\phi(qt)/Bg(m+1, N)],$$

whence, using (1.2), (2.7) and (2.8) we obtain

$$P\{M(1, N) \ge t\} \le AK \exp[-\phi(t)/\beta Bg(1, N)] + K \exp[-\phi(pt)/g(1, N)] + AK \exp[-\phi(qt)/(Q-\beta)Bg(1, N)].$$

Now take (2.12), (2.14) and (2.15) into consideration:

$$(2.17)\qquad P\{M(1, N) \ge t\} \le 3AK \exp[-\phi(t)/\beta Bg(1, N)].$$

Here the right-hand side does not exceed $AK \exp[-\phi(t)/Bg(1, N)]$, provided that

$$\ln 3 - \phi(t)/\beta Bg(1, N) \le -\phi(t)/Bg(1, N),$$

which holds if

$$(2.18)\qquad \phi(t)/Bg(1, N) \ge (\beta \ln 3)/(1-\beta).$$

By (2.17), we obtain the desired (2.13) for $n = N$ under the assumption (2.18).

On the other hand, if (2.18) is not satisfied, then by (2.15) we have

$$A \exp[-\phi(t)/Bg(1, N)] > A \exp[-(\beta \ln 3)/(1-\beta)] = 1.$$

Consequently, (2.13) to be proved clearly holds, since in any case $P\{M(1, N) \ge t\} \le 1$. This completes the induction step and the proof of Theorem 2.2. □

REMARK. If, in addition to (2.12), we have

$$(2.19)\qquad \lim_{C \to 0+} \chi(C) = 0,$$

then Theorem 2.2 can be strengthened for the special case $Q = 1$. In fact, on account of (2.19), if $\beta \to 1-$ we can take $q \to 0+$ in (2.14), and consequently, $p \to 1-$ in (2.15). This means that in the conclusion (2.13) we can choose B as close to 1 as we wish. But we stress that $A \to +\infty$ as $\beta \to 1-$. □

The proof of Theorem 2.2 and this remark are modeled after that of Theorem 1 of Móricz (1979).

PROBLEM 1. In the special case $Q = 1$, can $B = 1$ in Theorem 2.2 under the additional assumption (2.18)? This assumption is satisfied, for example, in the case $\phi(t) = t^2$. □

3. Power-type estimates.

THEOREM 3.1. *Let $\alpha > 1$ and $\gamma \ge 1$ be given reals. Suppose that there exists a function $g(i, j)$ satisfying* (1.2) *with $1 \le Q < 2^{(\alpha-1)/\alpha}$, such that*

$$(3.1)\qquad E|S(i, j)|^\gamma \le g^\alpha(i, j), \quad \text{all} \quad 1 \le i \le j \le n.$$

Then there exists a constant A, depending on α, γ and Q but not on n, $\{X_k\}$ or otherwise on g, such that

$$E\,M^{\gamma}(1,n) \le A\,g^{\alpha}(1,n). \tag{3.2}$$

REMARK. The special case $Q = 1$ was proved by Móricz (1976a) and by Longnecker and Serfling (1977), independently of each other and with different determinations of the constant A. □

PROOF. The proof closely follows that of Theorem 1 of Móricz (1976a). Thus we only sketch it. Set $\beta = Q/2$ in (2.7) and (2.8), and estimate as follows:

$$M(1,N) \le |S(1,m)| + \{M^{\gamma}(1,m-1) + M^{\gamma}(m+1,N)\}^{1/\gamma}.$$

Via Minkowski's inequality, this yields

$$\{EM^{\gamma}(1,N)\}^{1/\gamma} \le \{E|S(1,m)|^{\gamma}\}^{1/\gamma} + \{EM^{\gamma}(1,m-1) + EM^{\gamma}(m+1,N)\}^{1/\gamma}. \tag{3.3}$$

Hence, using the induction hypothesis, (3.1), (2.7), and (2.8) with $\beta = Q/2$, we obtain

$$\begin{aligned}\{EM^{\gamma}(1,N)\}^{1/\gamma} &\le g^{\alpha/\gamma}(1,m) + \left\{\frac{Q^{\alpha}}{2^{\alpha-1}} A\,g^{\alpha}(1,N)\right\}^{1/\gamma}\\ &\le g^{\alpha/\gamma}(1,N)\left(1 + A^{1/\gamma}\frac{Q^{\alpha/\gamma}}{2^{(\alpha-1)/\gamma}}\right).\end{aligned}$$

For A large enough, this implies

$$\{EM^{\gamma}(1,N)\}^{1/\gamma} \le A^{1/\gamma} g^{\alpha/\gamma}(1,N),$$

which is equivalent to (3.2) for $n = N$. The smallest A satisfying the condition

$$1 + A^{1/\gamma}\frac{Q^{\alpha/\gamma}}{2^{(\alpha-1)/\gamma}} \le A^{1/\gamma}$$

is given by (observe that the assumption $Q < 2^{(\alpha-1)/\alpha}$ is essential)

$$A = \left(1 - \frac{Q^{\alpha/\gamma}}{2^{(\alpha-1)/\gamma}}\right)^{-\gamma}.$$

The proof of Theorem 3.1 is complete. □

THEOREM 3.2. *Let $\alpha > 1$ be a given real. Suppose that there exist a function $g(i,j)$ satisfying* (1.2) *with $1 \le Q < 2^{(\alpha-1)/\alpha}$, and a t_0, $0 < t_0 \le +\infty$, such that*

$$P\{|S(i,j)| \ge t\} \le g^{\alpha}(i,j)/\phi(t), \quad \textit{all} \quad 0 < t < t_0 \quad \textit{and} \quad 1 \le i \le j \le n, \tag{3.4}$$

where $\phi(t) > 0$ for $0 < t < t_0$ and (2.12) *is satisfied for each C, $0 < C < 1$. Then there exists a constant $A \ge 1$, depending on α, Q and χ but not on n, $\{X_k\}$ or otherwise on g or ϕ, such that*

$$P\{M(1,n) \ge t\} \le Ag^{\alpha}(1,n)/\phi(t), \quad \textit{all} \quad 0 < t < t_0. \tag{3.5}$$

For $\phi(t) = t^{\gamma}$ and $g(i,j) = \sum_{k=i}^{j} u_k$, $u_k \ge 0$. Theorem 3.2 was established by Billingsley (1968, page 94, Theorem 12.2).

PROOF OF THEOREM 3.2. Take $\beta = Q/2$ as in the proof of Theorem 3.1. For $p + q = 1, p > 0, q > 0$, choose q sufficiently close to 1 that

$$(Q/2)^{\alpha}[1 + 1/\chi(q)] < 1$$

(cf. (2.12)), and then choose A large enough to satisfy

$$(Q/2)^{\alpha}[1 + 1/\chi(q)] + 1/A\chi(p) \le 1. \tag{3.6}$$

Then apply the usual induction argument, starting with (2.16). By the induction hypothesis, (3.4), (2.7) and (2.8) with $\beta = Q/2$, we obtain

$$P\{M(1, N) \geq t\} \leq A(Q/2)^\alpha g^\alpha(1, N)/\phi(t) + g^\alpha(1, N)/\phi(pt) + A(Q/2)^\alpha g^\alpha(1, N)/\phi(qt)$$
$$= A\left[\left(\frac{Q}{2}\right)^\alpha + \left(\frac{Q}{2}\right)^\alpha \frac{\phi(t)}{\phi(qt)} + \frac{\phi(t)}{A\phi(pt)}\right] g^\alpha(1, N)/\phi(t).$$

On account of (2.12) and (3.6), we have immediately that

$$P\{M(1, N) \geq t\} \leq A\left[\left(\frac{Q}{2}\right)^\alpha \left(1 + \frac{1}{\chi(q)}\right) + \frac{1}{A\chi(p)}\right] g^\alpha(1, N)/\phi(t)$$
$$\leq A\, g^\alpha(1, N)/\phi(t),$$

which is the desired (3.5) for $n = N$. This completes the proof of Theorem 3.2. □

Finally, we treat the question of how to modify Theorem 3.1 in the case $\alpha = 1$. This will be done in a more general setting. To this effect, let $\{\lambda(n): n = 1, 2, \cdots\}$ be a given nondecreasing sequence of positive numbers. Set

$$\Lambda(1) = \lambda(1),$$
$$\Lambda(n) = \lambda\left(\left\lceil \frac{n}{2} \right\rceil\right) + Q^{1/\gamma}\Lambda\left(\left\lfloor \frac{n}{2} \right\rfloor\right), \qquad n \geq 2, \tag{3.7}$$

where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ denote the upper and lower integral parts, respectively. It is clear that $\{\Lambda(n): n = 1, 2, \cdots\}$ is also a nondecreasing sequence of positive numbers. Furthermore, a simple calculation gives that

$$\Lambda(n) = \sum_{k=0}^{\lfloor \log n \rfloor} Q^{k/\gamma} \lambda\left(\left\lceil \frac{n}{2^{k+1}} \right\rceil\right), \qquad \text{all } n \geq 1. \tag{3.8}$$

Here and in the sequel, the logarithms are to base 2.

THEOREM 3.3. *Let $\gamma \geq 1$ be a given real. Suppose that there exist a function $g(i, j)$ satisfying* (1.2) *with $Q \geq 1$ and a nondecreasing sequence $\{\lambda(n)\}$ of positive numbers such that*

$$E|S(i, j)|^\gamma \leq g(i, j)\lambda^\gamma(j - i + 1), \quad \textit{all} \quad 1 \leq i \leq j \leq n. \tag{3.9}$$

Then

$$EM^\gamma(1, n) \leq g(i, j)\Lambda^\gamma(n), \tag{3.10}$$

where $\Lambda(n)$ is defined by (3.7).

We state the special case $\lambda(n) \equiv 1$ in the form of a separate theorem, as follows (cf. (3.8)).

COROLLARY 3.1. *Let $\gamma \geq 1$ be a given real. Suppose that there exists a function $g(i, j)$ satisfying* (1.2) *with $Q \geq 1$ such that*

$$E|S(i, j)|^\gamma \leq g(i, j), \quad \textit{all} \quad 1 \leq i \leq j \leq n.$$

Then

$$EM^\gamma(1, n) \leq g(1, n) \left(\sum_{k=0}^{\lfloor \log n \rfloor} Q^{k/\gamma}\right)^\gamma. \tag{3.11}$$

REMARKS. (i) The right-hand side of (3.11) is of order of magnitude $g(1, n)(\log 2n)^\gamma$ as $n \to +\infty$ for $Q = 1$ and $g(1, n)\, n^{\log Q}$ for $Q > 1$.

(ii) Corollary 3.1 for $g(i,j) = \sum_{k=i}^{j} u_k$, $u_k \geq 0$, is given in Billingsley (1968), page 102.

(iii) The even more special case of Corollary 3.1 when $\gamma = 2$, $g(i,j) = \mathrm{Var}\{S(i,j)\}$, and the X_k's are mutually orthogonal, is the famous Rademacher–Menšov inequality (see, e.g., Doob, 1953, page 156.).

(iv) Corollary 3.1 for $Q = 1$ was essentially proved by Serfling (1970), while Theorem 3.3 also for $Q = 1$ (using a slightly different notation) by Móricz (1975).

Proof of Theorem 3.3. It runs along the same lines as the proof of Theorem 1 of Móricz (1975). Therefore, we present it here in a short form. We begin with (3.3). By the induction hypothesis and (3.9),

$$\{EM^{\gamma}(1, N)\}^{1/\gamma} \leq \lambda(m)g^{1/\gamma}(1, m) + \{\Lambda^{\gamma}(m-1)g(1, m-1) + \Lambda^{\gamma}(N-m)g(m+1, N)\}^{1/\gamma}.$$

Let us choose m at present to be $m = \lceil N/2 \rceil$. Then $m - 1 \leq \lfloor N/2 \rfloor$ and $N - m = \lfloor N/2 \rfloor$. Exploiting (1.2), we arrive at the inequality

$$\begin{aligned}\{EM^{\gamma}(1, N)\}^{1/\gamma} &\leq \lambda\left(\left\lceil \frac{N}{2} \right\rceil\right) g^{1/\gamma}(1, N) \\ &\quad + \left\{\Lambda^{\gamma}\left(\left\lfloor \frac{N}{2} \right\rfloor\right) g(1, m-1) + \Lambda^{\gamma}\left(\left\lfloor \frac{N}{2} \right\rfloor\right) g(m+1, N)\right\}^{1/\gamma} \\ &\leq \left\{\lambda\left(\left\lceil \frac{N}{2} \right\rceil\right) + Q^{1/\gamma}\Lambda\left(\left\lfloor \frac{N}{2} \right\rfloor\right)\right\} g^{1/\gamma}(1, N),\end{aligned}$$

which is equivalent to (3.10) for $n = N$, thanks to (3.7). This completes the induction step and the proof of Theorem 3.3. □

In closing, we mention an open problem concerning the probability inequality version of Theorem 3.3.

Problem 2. Let $\gamma \geq 0$ be a given real. Suppose that there exist a function $g(i, j)$ satisfying (1.2) with $Q \geq 1$ and a t_0, $0 < t_0 \leq +\infty$, such that

$$P\{|S(i,j)| \geq t\} \leq g(i,j)/t^{\gamma}, \quad \text{all} \quad 0 < t < t_0 \quad \text{and} \quad 1 \leq i \leq j \leq n.$$

How can one precisely estimate from above the probability $P\{M(1, n) \geq t\}$ for all $0 < t < t_0$ in terms of $g(i, j)$ and γ? The answer is not known even in the special case $g(i, j) = \sum_{k=1}^{j} u_k$, $u_k \geq 0$. □

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