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AN APPLICATION OF TIME REVERSAL TO BROWNIAN LOCAL TIME

BY JOANNA B. MITRO

University of Cincinnati

By means of a time reversal argument based on Nagasawa's lemma, the equivalence of two representations for the Brownian local time is demonstrated.

1. Introduction. If we denote by $s(x, \cdot)$ the local time at x of the one-dimensional Brownian motion B, it is known that the process $x \to s(x, T)$ is a diffusion when T is a suitable terminal time. This remarkable fact was first proved by Ray [4] and Knight [3]; see also [6], [7], [8], and [9] for more recent reworkings. In [2] Knight gives the result when $T = T(-1) = \inf\{t: B(t) = -1\}$: for $-1 \le x \le 0$, $s(x, T(-1))$ is the diffusion with generator $(y(d^2/dy^2) + d/dy)$ (starting at the entrance boundary 0), then for $x \ge 0$ the generator changes to $y(d^2/dy^2)$, with the process absorbed at 0. Ray's analysis involved a change of variable and his description of the process takes the following form:

$$s(x, T(-1)) = (1 + x)^2\tau((1 + x)^{-1} - (1 + M)^{-1}), \quad 0 \le x \le M, \tag{1.1}$$

where M is a positive random variable satisfying $P^0(M > x) = (1 + x)^{-1}$ and τ is a diffusion equivalent to $(1/2)|B_4|^2$ and independent of M. Here $|B_4|$ is the Bessel process of dimension 4 (having generator $(1/2)(d^2/dy^2 + (3/y)d/dy)$) with $|B_4|(0) = 0$, and P^0 is the law of the Brownian motion starting at 0. For $-1 \le x \le 0$, $s(x, T(-1))$ is equivalent to $(1/2)\,|B_2|^2(x + 1)$, with $|B_2|(0) = 0$, and conditioned by $|B_2|(1) = [2\tau(1 - (1 + M)^{-1})]^{1/2}$ but otherwise independent of M and $|B_4|$. Here $|B_2|$ is the Bessel diffusion of dimension 2 (generator $(1/2)(d^2/dy^2 + (1/y)d/dy)$). For $x \ge M$ and $x \le -1$, $s(x,T(-1)) = 0$.

In fact the equivalence of Ray's and Knight's representations when $x \le 0$ is easily checked using the observation that $(1/2)|B_2|^2$ has generator $(y(d^2/dy^2) + d/dy)$. For $x \ge 0$ Knight derives Ray's representation from his own by means of an ingenious use of time reversal. He calls the argument "difficult", perhaps because it refers to an unconventional stationary process having Lebesgue measure as "initial distribution", and requires keeping track of some conditioning. In this note we prove the equivalence for $x \ge 0$ in another way. Our approach uses time reversal also; the main tool is Nagasawa's lemma. For the background of this technque, we refer the reader to [5] which contains a clear and self-contained exposition.

2. The time reversal. Acording to Knight's recipe, $s(x, T(-1))$ is equivalent to the process $(1/2)|B_0|^2$ (whose generator is $y(d^2/dy^2)$) with exponential initial distribution (parameter 1). In what follows we will let W stand for the process defined by (1.1), and Z for $(1/2)|B_0|^2$ with exponential initial distribution.

The first step toward equating W and Z is to establish the relationship between $(1/2)|B_4|^2$ and $(1/2)|B_0|^2$. From their generators, $(y(d^2/dy^2) + 2d/dy)$ and $y(d^2/dy^2)$ respectively, we compute their scale functions and speed measures and follow [5], Section 2. For $(1/2)|B_4|^2$, $s_4(y) = -1/y$, $m_4(dy) = y\,dy$; for $(1/2)|B_0|^2$, $s_0(y) = y$, $m_0(dy) = (1/y)\,dy$. The diffusion $(1/2)|B_4|^2$ has potential density $u(x, y) = \min(1/x, 1/y)$ and is self-dual relative to the measure $m_4(dy)$. By Nagasawa'a lemma, when $(1/2)|B_4|^2$ under P^0 is reversed from any cooptional time L, the resulting process is a diffusion with a transition function derived from that of $(1/2)|B_4|^2$ via an h-path transform. In fact $h(y) = u(0, y)$

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$= 1/y$, and a brief calculation shows that the new process has scale function $-1/s_4 = s_0$ and speed measure $s_4^2(y)m_4(dy) = m_0(dy)$. Thus the reversed process is equivalent to $(1/2)\cdot |B_0|^2$.

It remains to choose the cooptional time appropriately. Since we want the reversed process to have an exponential initial distribution, we take Y to be a non-negative random variable independent of $|B_4|$ with $P^0(Y > y) = e^{-y}$, and define

$$R(\omega) = L_{Y(\omega)}(\omega)$$

where $L_y = \sup\{x\colon (1/2)|B_4|^2(x) = y\}$. R is a cooptional time, and $(1/2)|B_4|^2(R)$ has an exponential distribution. The preceding shows that $(1/2)|B_4|^2(R-x)$ is equivalent to $Z(x)$.

We can compute the distribution of R. It was shown in [1] that for a Bessel process of dimension $d > 2$ starting at 0, the distribution of the last exit time from y has density

$$f_d(t) = y^{d-2}[2^{(d/2)-1}\Gamma((d/2)-1)t^{d/2}]^{-1}e^{-y^2/2t}.$$

From this formula it is easy to compute the distribution of L_y:

$$P^0(L_y \le t) = e^{-y/t},$$

and thus

$$P^0(R \le t) = \int_0^\infty e^{-y/t}e^{-y}\,dy = t/(t+1).$$

In fact R has the same distribution as the random variable M in (1.1), and will play the role of M in what follows.

3. The equivalence. Since $(1/2)|B_4|^2(R-x)$ $(0 \le x \le R)$ is equivalent to $Z(x)$, it remains to show that $(1/2)|B_4|^2(R-x)$ is equivalent to $W(x)$. We begin by observing that $(1+M)^{-1}$ is uniformly distributed on $(0, 1)$ and that $W(0) = (1/2)|B_4|^2(1-(1+M)^{-1})$ is indeed exponentially distributed ($(1/2)|B_4|^2(x)$ has density $g(y) = (y/x^2)e^{-y/x}$ under P^0). Thus the two processes have the same initial distribution, and it suffices to prove that for any initial point y_0 and $r > 0$ the processes $(1/2)|B_4|^2(r-x)$ with $(1/2)|B_4|^2(r) = y_0$ and $[(1+x)/2]|B_4|^2((1+x)^{-1}-(1+r)^{-1})$ with $(1/2)|B_4|^2(r/(1+r)) = y_0$ are equivalent. In fact since $|B_4|^2$ is the modulus of a four-dimensional Brownian motion, it suffices to check the equivalence with $(1/2)|B_4|^2$ replaced by the one-dimensional Brownian motion B. The familiar Gaussian density turns this into an easy exercise, which we leave to the reader to verify.

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DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF CINCINNATI
CINCINNATI, OHIO 45221