

GAUSSIAN MEASURES IN B_p^1

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For $p \geq 1$, conditions for a separable Gaussian process to have sample paths of finite p -variation are given in terms of the mean function and the covariance function. A process with paths of finite p -variation may or may not induce a tight measure on the nonseparable Banach space B_p . Consequences of tightness and conditions for tightness are given.

1. Introduction. For $p \geq 1$, we define B_p to be the class of real-valued functions f defined on $[0, 1]$ having finite p -variation,

$$(1.1) \quad V_p(f) = |f(0)|^p + \sup_{\pi} \sum_k |f(t_{k+1}) - f(t_k)|^p,$$

where the sup is taken over all finite partitions $\pi: 0 = t_0 < t_1 < \dots < t_n = 1$ of $[0, 1]$. It is easily seen that

$$(1.2) \quad \|f\|_p = (V_p(f))^{1/p}$$

defines a norm on B_p and $(B_p, \|\cdot\|_p)$ is a Banach space.

Consider a separable Gaussian process $X = \{X(t, \omega) : 0 \leq t \leq 1\}$ on a complete probability space (Ω, \mathcal{F}, P) . We are studying conditions for X to have sample paths $t \rightarrow X(t, \omega)$ of finite p -variation. Other authors have mainly considered the stationary-increment case. (See [13] and [15]). In [13] Kawada and Kono prove rather delicate results of the type Taylor [17] obtained for Brownian motion, but very stringent conditions must be imposed on the process to obtain such results. Our aim here is to obtain sharpest possible results in a very general setting (not even assuming the continuity of the covariance).

A zero-one law guarantees that the process has either paths in B_p with probability zero or with probability one. Since $B_p \subset B_q$ for $p < q$, it follows that there exists a number γ , $1 \leq \gamma \leq \infty$, such that $P\{X(\cdot, \omega) \in B_p\} = 0$ if $p < \gamma$, and $P\{X(\cdot, \omega) \in B_p\} = 1$ if $\gamma < p$.

Consider the mean function

$$(1.3) \quad m(t) = EX(t).$$

The process X has sample paths in B_p if and only if $m \in B_p$ and the centered process $X - m$ has paths in B_p . (See Theorem 2.3.) So in our study of conditions for Gaussian processes to have sample paths of finite p -variation we may focus on centered Gaussian processes. Assume that X is centered and define

$$(1.4) \quad \sigma(s, t) = E |X(s) - X(t)|.$$

A theorem by Fernique guarantees that if the sample paths have finite p -variation, then the expected p -variation will be finite, too. Since

$$E(\sup_{\pi} \sum_k |X(t_{k+1}) - X(t_k)|^p) \geq \sup_{\pi} E(\sum_k |X(t_{k+1}) - X(t_k)|^p) = c_p \sup_{\pi} \sum_k \sigma(t_k, t_{k+1})^p,$$

it follows that the condition

$$(1.5) \quad \sup \sum_k \sigma(t_k, t_{k+1})^p < \infty$$

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is *necessary* for X to have paths of finite p -variation. In the special case $p = 1$, this condition is also *sufficient*, because

$$E(\sup \sum |X(t_{k+1}) - X(t_k)|) = \sup \sum \sigma(t_k, t_{k+1}).$$

If $p > 1$, then condition (1.5) is *not* sufficient. Brownian motion, for example, satisfies condition (1.5) with $p = 2$, but does not have paths of finite quadratic variation. Under assumption (1.5) a.a. sample paths satisfy (see Theorem 3.1)

$$(1.6) \quad |X(s, \omega) - X(t, \omega)| \leq C(\omega)\sigma(s, t)[\log^*(\sigma(s, t))]^{1/2}$$

for all $s \leq t$, where $C(\omega)$ is some positive random variable and $\log^*(u) = \max\{1, |\log(u)|\}$. From (1.6) it immediately follows that if

$$(1.7) \quad \sup_{\pi} \sum_k \sigma(t_k, t_{k+1})^p [\log^*(\sigma(t_k, t_{k+1}))]^{p/2} < \infty$$

then the sample paths of X have finite p -variation almost surely.

In Theorem 3.2 we improve on this last result by showing that even the weaker assumption

$$(1.8) \quad \sup_{\pi} \sum_k \sigma(t_k, t_{k+1})^p [\log^*(\log^*(\sigma(t_k, t_{k+1})))]^{p/2} < \infty$$

is *sufficient* to guarantee that X has sample paths of finite p -variation. Intuitively, the reason for this is that for fixed t the sample paths almost surely satisfy

$$(1.9) \quad \lim_{\varepsilon \downarrow 0} \sup_{0 < \sigma(s, t) < \varepsilon, |s - t| < \varepsilon} |X(s, \omega) - X(t, \omega)| / \sigma(s, t) [\log |\log(\sigma(s, t))|]^{1/2} \leq \pi^{1/2}$$

(provided the quantity is defined). So for any fixed partition $\pi: 0 = t_0 < t_1 < \dots < t_n = 1$ and some large constant N , we have

$$|X(t_k, \omega) - X(t_{k+1}, \omega)|^p \leq N \sigma(t_k, t_{k+1})^p [\log^* \log^*(\sigma(t_k, t_{k+1}))]^{p/2}$$

for most indices k . For the remaining k we use the uniform bound (1.6) and show that the contribution from these k to the sum

$$\sum_{k=0}^{n-1} |X(t_k, \omega) - X(t_{k+1}, \omega)|^p$$

is small.

In Section 4 we give examples to show that although condition (1.8) is not necessary, it is the best possible condition of this type. For example, for $0 < \varepsilon < 1/2$ the condition

$$(1.10) \quad \sup_{\pi} \sum_k \sigma(t_k, t_{k+1})^p [\log^*(\log^*(\sigma(t_k, t_{k+1})))]^{p/2} < \infty,$$

is neither necessary nor sufficient for X to have sample paths of finite p -variation.

The second question investigated in this paper is motivated by the following well-known fact: If $X = \{X(t, \omega) : 0 \leq t \leq 1\}$ is a Gaussian process with paths in $C[0, 1]$ and L^2 -expansion

$$(1.11) \quad X(t) = \sum \varphi_j(t) \xi_j + m(t),$$

where the Gaussian variables $\{\xi_j\}$ are independent with mean 0 and variance 1, then for a.a. ω the partial sums

$$(1.12) \quad S_n(t, \omega) = \sum_{j=1}^n \varphi_j(t) \xi_j(\omega) + m(t)$$

converge uniformly in t to $X(t, \omega)$, i.e. $\{S_n\}$ converges to X in $C[0, 1]$. This fact is very closely related to the fact that the map $\omega \rightarrow X(\cdot, \omega)$ induces a tight probability measure in the Banach space $C[0, 1]$.

We pose the analogous question for Gaussian processes X with paths in B_p and L^2 -expansion (1.11): Do the random vectors S_n converge to X in B_p ? The reason that this is not obvious is that the Banach space B_p is *nonseparable*. Even the subset of continuous functions in B_p is nonseparable. For $p = 1$ this is well-known; for $p > 1$, consider the uncountable collection of continuous functions

$$(1.13) \quad f_\varepsilon(t) = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k/p} \sin(2^k \pi t), \quad 0 \leq t \leq 1,$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$, $\varepsilon_i = \pm 1$ (nonrandom). We show in Lemma 2.1 that $f_\varepsilon \in B_p$ and if $\varepsilon \neq \varepsilon'$, then $\|f_\varepsilon - f_{\varepsilon'}\|_p > 2$.

In [10] we proved that $\{S_n\}$ converges to X in B_p if and only if X induces a tight measure in B_p . Every Gaussian process with paths in B_1 induces a tight measure in B_1 . (See [10] or [11].) In this paper we give examples showing that a Gaussian process X with paths in B_p need not induce a tight measure in B_p if $p > 1$. However, if $p > \gamma = \gamma(X)$, then X does induce a tight measure in B_p . (Theorem 4.3).

Throughout, the letters c and c_i will denote positive constants. Their values are unimportant and may change from one context to another, even from line to line.

2. Preliminaries. The following lemma implies our assertions about the functions f_ε defined in (1.13).

LEMMA 2.1. *Let $(a_k) \in \ell^\infty$ be a real sequence. Then the function*

$$(2.1) \quad f(t) = \sum_{k=1}^{\infty} a_k 2^{-k/p} \sin(2^k \pi t), \quad t \in [0, 1]$$

is in B_p for $p > 1$. If f_ε is given by (1.13) and $\varepsilon \neq \varepsilon'$ are two ± 1 sequences, then $\|f_\varepsilon - f_{\varepsilon'}\|_p > 2$.

PROOF. We have for $0 \leq s < t \leq 1$

$$\begin{aligned} |f(t) - f(s)| &\leq \left| \sum'_k a_k 2^{-k/p} (\sin(2^k \pi t) - \sin(2^k \pi s)) \right| \\ &\quad + \left| \sum''_k a_k 2^{-k/p} (\sin(2^k \pi t) - \sin(2^k \pi s)) \right| \end{aligned}$$

where \sum' is the sum over $k \leq |\text{Log}(t - s)|$, and \sum'' is the sum over the remaining k 's; Log denotes the logarithm to base 2. If $|a_k| \leq c$, $k \geq 1$, we get the obvious estimate

$$\begin{aligned} |f(t) - f(s)| &\leq c\pi |t - s| \sum' 2^k 2^{-k/p} + 2c \sum'' 2^{-k/p} \\ &\leq c_1 |t - s|^{1/p} \end{aligned}$$

for some $c_1 > 0$ (independent of s, t). This estimate proves the first part. For the second part we first observe that for $j \geq 1$

$$(2.2) \quad \|\sin(2^j \pi t)\|_p > 2^{j/p}.$$

To see this, let $t_i = i2^{-j-1}$, $i = 0, 1, \dots, 2^{j+1}$. For this partition

$$(2.3) \quad |\sin(2^j \pi t_{i+1}) - \sin(2^j \pi t_i)|^p = 1$$

for each i , so (2.2) follows. If $\varepsilon \neq \varepsilon'$, then there exists j such that $\varepsilon_1 = \varepsilon'_1, \dots, \varepsilon_{j-1} = \varepsilon'_{j-1}, \varepsilon_j \neq \varepsilon'_j$. For this j we pick the partition of $[0, 1]$, as above, and we have

$$\begin{aligned} |(f_\varepsilon - f_{\varepsilon'})(t_{i+1}) - (f_\varepsilon - f_{\varepsilon'})(t_i)| &= |\varepsilon_j - \varepsilon'_j| 2^{-j/p} |\sin(2^j \pi t_{i+1}) - \sin(2^j \pi t_i)| \\ &= 2 \cdot 2^{-j/p} \end{aligned}$$

by using (2.3). This shows $\|f_\varepsilon - f_{\varepsilon'}\|_p > 2$.

We now turn to L^2 -expansions of Gaussian processes with paths in B_p . It is obvious from the definition that if a separable process has paths in B_p , then the sample paths must have right and left limits at every t , a.s. Therefore the following theorem, which is proved in [10], applies to Gaussian processes with paths in B_p . For a discussion of fixed and moving discontinuities of a process we refer to Doob [2]. As usual m denotes the mean function,

$$m(t) = EX(t), \quad 0 \leq t \leq 1.$$

THEOREM 2.2. *Let $\{X(t) : 0 \leq t \leq 1\}$ be a separable Gaussian process with almost all sample paths having right and left limits at every t . Then the following holds:*

(i) *Almost surely the paths of the process have no moving discontinuities. The set of fixed discontinuities is countable and coincides with the set of points of stochastic*

discontinuity of the process. In particular, if X is stochastically continuous, then X has continuous paths a.s. If X is not stochastically continuous, then with probability one, the sample paths are discontinuous at every point of stochastic discontinuity.

(ii) The $L^2(P)$ closure of the linear span of $\{X(t) - m(t) : 0 \leq t \leq 1\}$ is a separable Hilbert space with the $L^2(P)$ norm, and if $\{\xi_n\}$ is a complete orthonormal system (necessarily, ξ_n are independent, Gaussian with mean 0 and variance 1) in this Hilbert space, then

$$(2.4) \quad X(t) = \sum_n \varphi_n(t) \xi_n + m(t),$$

where

$$(2.5) \quad \varphi_n(t) = \int \xi_n X(t) dP.$$

For each fixed t , the series in (2.4) converges in $L^2(P)$ and a.s.

The following theorem contains a zero-one law. We give it here for later use.

THEOREM 2.3. *Let $\{X(t) : 0 \leq t \leq 1\}$ be a separable Gaussian process such that $P\{X(\cdot, \omega) \in B_p\} > 0$. Then this probability is 1, the mean function $m \in B_p$, $\varphi_n \in B_p$ for $n \geq 1$ (where φ_n is defined by (2.5)) and there exists $\varepsilon > 0$ such that $E[\exp(\varepsilon \|X(\cdot, \omega)\|_p^2)] < \infty$*

PROOF. The zero-one law and the fact that $m \in B_p$ and $\varphi_n \in B_p$ follow from Proposition 6.2 in [11], which loosely speaking states that if X is a separable Gaussian process with L^2 -expansion (2.4) and L is any linear function space then $P\{X(\cdot, \omega) \in L\} > 0$ implies that $P\{X(\cdot, \omega) \in L\} = 1$, $m \in L$ and $\varphi_n \in L$ for $n \geq 1$. The last assertion in Theorem 2.3 obviously follows from Fernique's result on measurable seminorms. However, since B_p is nonseparable and X may not induce a tight measure in B_p , we have to proceed with caution. Note that if $X(\cdot, \omega) \in B_p$ a.s. and $m \in B_p$, then the process $\{X(t, \omega) - m(t) : 0 \leq t \leq 1\}$ has paths with left and right limits and is therefore separable (since X is separable). Let S denote a separability set with $0 \in S$ and $1 \in S$. Let E be the linear space of all real-valued functions on S and \mathcal{E} the σ -algebra generated by the cylinder sets. Define the seminorm $N(f)$ on E by

$$N(f)^P = |f(0)|^P + \sup_n \sum_k |f(t_{k+1}) - f(t_k)|^P$$

where the sup is taken over all finite partitions $\pi : 0 = t_0 < t_1 < \dots < t_n = 1$ with $t_k \in S$, $k = 0, 1, \dots, n$. By Fernique's theorem ([5]) there exists $\varepsilon > 0$ such that

$$E\{\exp(\varepsilon N^2(X(\cdot, \omega) - m(\cdot)))\} < \infty.$$

Since $\|X(\cdot, \omega) - m(\cdot)\|_p = N(X(\cdot, \omega) - m(\cdot))$, a.s. we get

$$E \exp(\varepsilon \|X(\cdot, \omega) - m(\cdot)\|_p^2) < \infty.$$

The theorem now follows since $m \in B_p$.

The next lemma generalizes a lemma of Fernique [6].

We will write for a process X

$$(2.6) \quad \sigma_X(s, t) = E|X(s) - X(t)|.$$

LEMMA 2.4. *Let $\{X(t) : 0 \leq t \leq 1\}$ be a separable, centered Gaussian process. F is a non-decreasing function on $[0, 1]$ with $F(0) = 0$. If ψ is a nondecreasing continuous function on $[0, F(1)]$ such that for some $c_1 > 0$, $c_2 > 0$,*

$$(2.7) \quad \psi(0) = 0, u\psi(v) \leq c_1\psi(uv) \quad \text{for } 0 \leq v \leq F(1) \quad \text{and } 0 \leq u \leq 1,$$

$$(2.8) \quad \sigma_X(s, t) \leq c_2\psi(F(t) - F(s)), \quad s \leq t,$$

and

$$(2.9) \quad \int_0^\infty \psi(h e^{-x^2/2}) dx = 0(\psi(h)), \quad h \downarrow 0,$$

then there exists $c_3 > 0$ such that for $0 \leq a \leq b \leq F(1)$ and $x \geq 1$

$$(2.10) \quad P\{\sup_{s,t \in I(a,b)} |X(s) - X(t)| > c_3 \psi(b-a)x\} \leq \exp(-x^2/2),$$

where $I(a, b) = \{t : a \leq F(t) \leq b\}$.

PROOF. Let

$$\alpha = F(1), \quad R = \{F(t) : 0 \leq t \leq 1\}$$

and

$$V = [0, \alpha] - R.$$

We shall show that there exists a separable, centered, stochastically continuous Gaussian process $\{Y(\theta, \omega) : 0 \leq \theta \leq \alpha\}$ such that $P\{X(t, \omega) = Y(F(t), \omega) \text{ for all } t\} = 1$.

First observe that

$$(2.11) \quad V = \cup_{i=1}^\infty V_i,$$

where the V_i 's are disjoint open or half-open intervals. This is seen from the fact that if F is discontinuous at t_0 then either $F(t_0) < F(t_0+)$ or $F(t_0-) < F(t_0)$. If $F(t_0) < F(t_0+)$, then either the open interval $(F(t_0), F(t_0+))$ or the half-open interval $(F(t_0), F(t_0+)]$ is a maximal connected component of V —depending on whether $F(t_0+) \in R$ or $F(t_0+) \notin R$. Similarly, if $F(t_0-) < F(t_0)$, then either $(F(t_0-), F(t_0))$ or $[F(t_0-), F(t_0))$ is a maximal connected component of V .

We now construct the Gaussian process $Y(\theta)$, $\theta \in [0, \alpha]$ as follows: if $\theta \in R$, define

$$(2.12) \quad Y(\theta, \omega) = X(t, \omega)$$

where $F(t) = \theta$. This definition is unambiguous because if $H(\theta) = \{t : F(t) = \theta\}$ contains more than one point then it is a nonempty interval. By our assumption on the paths of X and (2.6) and (2.8) the process is a.s. constant on $H(\theta)$. Since such intervals are only countable in number, we can set aside a P -null set. Now let $\theta \in V$. If $\theta \in V_i$ and is an endpoint of V_i , say the right endpoint, then $\theta = F(t_0+)$ for some t_0 and we define $Y(\theta) = X(t_0+)$. (Pick any random variable representing the L^2 -limit of $X(s)$ as $s \downarrow t_0$, which exists by (2.8).) If $\theta \in V_i$ is the left endpoint of V_i , then $\theta = F(t_0-)$ for some t_0 and we define $Y(\theta) = X(t_0-)$. On the interior of V_i the paths of Y are defined linearly. The process Y we get is a centered, Gaussian process on $[0, \alpha]$ with $X(\cdot, \omega) = Y(F(\cdot), \omega)$ for a.a. ω . And Y is separable with separant $F(S) \cup S_1$, where S is a separant for X and S_1 is any countable dense subset of V .

It is clear from our construction that if $\theta_1, \theta_2 \in R$, then $\sigma_Y(\theta_1, \theta_2) \leq c_2 \psi(|\theta_1 - \theta_2|)$. Furthermore, if θ_1, θ_2 are in $V_i = (u_1, u_2]$, then

$$(2.13) \quad \begin{aligned} \sigma_Y(\theta_1, \theta_2) &= \frac{|\theta_1 - \theta_2|}{u_2 - u_1} \sigma_Y(u_1, u_2) \leq \frac{|\theta_1 - \theta_2|}{u_2 - u_1} c_2 \psi(u_2 - u_1) \\ &\leq c_1 c_2 \psi(|\theta_1 - \theta_2|) \end{aligned}$$

by Fatou, (2.7) and (2.8). It is easily seen that for any θ_1, θ_2 ,

$$(2.14) \quad \sigma_Y(\theta_1, \theta_2) \leq c_4 \psi(|\theta_1 - \theta_2|).$$

So Y is stochastically continuous. (Actually, Y has continuous paths a.s. But we don't need this fact in the proof of Lemma 2.4.)

To finish the proof of the lemma, let

$$(2.15) \quad Z((s_1, s_2), \omega) = Y(s_1, \omega) - Y(s_2, \omega), \quad (s_1, s_2) \in [0, \alpha]^2.$$

The Euclidean distance on $[0, \alpha]^2$ is denoted by d . As a consequence of (2.14) and (2.7) we get

$$(2.16) \quad \begin{aligned} \sigma_Z((s_1, s_2), (t_1, t_2)) &= E|Z(s_1, s_2) - Z(t_1, t_2)| \\ &\leq c_5 \psi(d((s_1, s_2), (t_1, t_2))/\sqrt{2}). \end{aligned}$$

We now apply a lemma of Fernique (Lemma 1.1 [9], page 138) to the Z -process for $a \leq s_1, s_2 \leq b$. We have by (2.14)

$$\sup_{a \leq s_1, s_2 \leq b} EZ^2(s_1, s_2) = \sup_{a \leq s_1, s_2 \leq b} E|Y(s_1) - Y(s_2)|^2 \leq c_6^2 \psi^2(b - a).$$

By Fernique's lemma we get for $x > 0$ ($k = 2$ here) and any integer $n > 1$,

$$(2.17) \quad \begin{aligned} P\{\sup_{a \leq s_1, s_2 \leq b} |Z(s_1, s_2)| > c_6 \psi(b - a)x + c_5 \sum_{p=1}^{\infty} \theta(p) \psi(n(p))^{-1}(b - a)\} \\ \leq n^4 \int_x^{\infty} e^{-t^2/2} dt + \sum_{p=1}^{\infty} n(p)^4 \int_{\theta(p)}^{\infty} e^{-t^2/2} dt, \end{aligned}$$

where $\theta(p) > 0$ and $n(p) = n^{(2^p)}$. If we pick $\theta(p) = 4x(\log n(p))^{1/2}$, then

$$(2.18) \quad \begin{aligned} \sum_p (\log n(p))^{1/2} \psi(n(p))^{-1}(b - a) &\leq c_7 \int_0^{\infty} \psi((b - a)e^{-u^2/2}) du \\ &\leq c_8 \psi((b - a)), \end{aligned}$$

where we used (2.9) at the last step. We also have for $x > 1, n \geq 2$

$$(2.19) \quad \begin{aligned} \sum_p n(p)^4 \int_{\theta(p)}^{\infty} e^{-u^2/2} du &\leq e^{-x^2/2} \sum_p n(p)^{4-6x^2} \\ &\leq c_9 e^{-x^2/2}. \end{aligned}$$

Using (2.18) and (2.19) in (2.17) we get c_{10} and c_{11} (independent of a, b) so that for $x \geq 1$

$$(2.20) \quad P\{\sup_{a \leq s_1, s_2 \leq b} |Z(s_1, s_2)| > c_{10} \psi(b - a)x\} \leq c_{11} e^{-x^2/2}.$$

This inequality readily implies (2.10) for some $c_3 > 0$ and the lemma is proved.

3. Conditions for paths to be in B_p . As we remarked in the previous section, if for some $p \geq 1$ we have $P\{X(\cdot, \omega) \in B_p\} = 1$, then almost all sample paths must have right and left limits at every t . By Theorem 2.2 such a process has no moving discontinuities and the set of its fixed discontinuities coincides with the set of stochastic discontinuities.

We now consider conditions in terms of the mean function and the covariance for the process to have sample paths in B_p , a.s. It follows from Theorem 2.3 that X has paths in B_p if and only if $m \in B_p$ and the centered process $X - m$ has paths in B_p . We now focus on centered Gaussian processes. Assume that $m = 0$ and write

$$(3.1) \quad \sigma(s, t) = E|X(s) - X(t)|$$

and

$$(3.2) \quad \begin{aligned} \log^*(s) &= \max(1, |\log(s)|), \quad s > 0 \\ \log_2^*(s) &= \log^*(\log^*(s)). \end{aligned}$$

For $p \geq 1$ we define

$$(3.3) \quad G(p; \sigma) = \sup_{\pi} \sum_k (\sigma(t_k, t_{k+1}))^p,$$

and

$$(3.4) \quad G_2(p; \sigma) = \sup_{\pi} \sum_k [\sigma(t_k, t_{k+1})(\log_2^*(\sigma(t_k, t_{k+1})))^{1/2}]^p,$$

where the sup is taken over all finite partitions $\pi: 0 = t_0 < t_1 < \dots < t_n = 1$ (and the summand is 0 if $\sigma(t_k, t_{k+1}) = 0$). Obviously, $G(p, \sigma) \leq G_2(p, \sigma)$. Let

$$(3.5) \quad \gamma = \inf\{p \geq 1 : G(p; \sigma) < \infty\}$$

(with the understanding that $\gamma = \infty$ if the set is empty). If $G(p; \sigma) < \infty$, then $G_2(p'; \sigma) < \infty$ for $p' > p$. So $G_2(p; \sigma)$ can be used in place of $G(p; \sigma)$ in the definition of γ .

THEOREM 3.1. *Let $\{X(t) : 0 \leq t \leq 1\}$ be a separable centered Gaussian process for which $G(p, \sigma) < \infty$ for some $p \geq 1$, then for almost every ω there exists $C(\omega)$ such that for $s \leq t$*

$$(3.6) \quad |X(s, \omega) - X(t, \omega)| \leq C(\omega) \sigma(s, t) |\log^*(\sigma(s, t))|^{1/2},$$

where the right side is 0 if $\sigma(s, t) = 0$.

PROOF. Since $G(p; \sigma) < \infty$, the function defined by

$$(3.7) \quad F(t) = \sup_{\pi} \sum_k E |X(t_{k+1}) - X(t_k)|^p,$$

where π ranges over all partitions $0 = t_0 < t_1 < \dots < t_n = t$, is a nondecreasing real-valued function (we have used the well-known property of Gaussian moments). For $s < t$

$$(3.8) \quad E |X(s) - X(t)|^p \leq F(t) - F(s).$$

The set D of discontinuities of F is countable and $X(t)$ is stochastically continuous at every $t \notin D$. Now consider each variable $X(t)$ a point in the Hilbert space $L^2(P)$. The function σ defines a pseudo metric on the set $\{X(t) : 0 \leq t \leq 1\}$ and by (3.8) we have (for some $c_2 > 0$)

$$(3.9) \quad \sigma(s, t) \leq c_2(F(t) - F(s))^{1/p}, \quad s < t.$$

For $\varepsilon > 0$ let $N(\varepsilon)$ denote the minimal number of balls of σ -radius $\leq \varepsilon$ that cover the set $\{X(t) : 0 \leq t \leq 1\}$. From (3.9) it is clear that there exists $c_3 > 0$ such that

$$(3.10) \quad N(\varepsilon) \leq c_3 \varepsilon^{-p}, \quad \varepsilon > 0.$$

Let Q denote the rational numbers in $[0, 1]$. Then the countable set of variables $\{X(t) : t \in D \cup Q\}$ is dense in $\{X(t) : 0 \leq t \leq 1\}$ in the σ -metric. Since X is separable with $D \cup Q$ as a separant, it follows from a theorem of Dudley [3] (see Theorem 5.1 in [9]) that for almost all ω , there exists $\delta(\omega) > 0$ such that $\sigma(s, t) \leq \delta(\omega)$ implies

$$(3.11) \quad |X(s, \omega) - X(t, \omega)| \leq c_4 \int_0^{\sigma(s,t)} (\log N(\varepsilon))^{1/2} d\varepsilon$$

where $c_4 > 0$ does not depend on ω . We now use the estimate in (3.10) to conclude that for $\sigma(s, t) \leq \delta(\omega)$

$$(3.12) \quad |X(s, \omega) - X(t, \omega)| \leq c_5 \sigma(s, t) |\log \sigma(s, t)|^{1/2}.$$

Since the paths of the process are bounded on $[0, 1]$, this clearly implies (3.6).

THEOREM 3.2. *Let $\{X(t) : 0 \leq t \leq 1\}$ be a separable, centered Gaussian process. If $G_2(p) < \infty$, then*

$$(3.13) \quad P\{X(\cdot, \omega) \in B_p\} = 1.$$

In particular, if $p > \gamma$, then (3.13) holds. Conversely, if (3.13) holds then $G(p) < \infty$, so if $p < \gamma$ then

$$(3.14) \quad P\{X(\cdot, \omega) \in B_p\} = 0.$$

PROOF. We have

$$(3.15) \quad \begin{aligned} E(V_p(X)) &\geq \sup_\pi \sum E |X(t_{k+1}) - X(t_k)|^p \\ &= c \sup_\pi \sum (\sigma(t_k, t_{k+1}))^p. \end{aligned}$$

If (3.13) holds, then by Theorem 2.3 we have $E(V_p(X)) < \infty$, so by (3.15) we have $G(p) < \infty$. We now prove the first assertion. In [11] we showed that $G(1) < \infty$ is both a necessary and a sufficient condition for X to have a.a. sample paths in B_1 . So we need only consider the case $p > 1$. The main part of the argument is contained in Theorem 3.1 and Lemma 2.4. After these two estimates are available, the rest of the argument is the same as given by Taylor [17]. See also [13]. We give the argument here for completeness. For $0 \leq t \leq 1$ define

$$(3.16) \quad F(t) = \sup_\pi \sum [\sigma(t_k, t_{k+1}) (\log_2^*(\sigma(t_k, t_{k+1})))^{1/2}]^p$$

where π ranges over all finite partitions $0 = t_0 < t_1 < \dots < t_n = t$. We also take

$$(3.17) \quad \psi(t) = t^{1/p} [\log_2^*(t)]^{-1/2}, \quad 0 \leq t \leq F(1).$$

Then it is easily checked that for $s \leq t$

$$(3.18) \quad \sigma(s, t) \leq c\psi(F(t) - F(s)).$$

As in Lemma 2.4, let

$$(3.19) \quad I(a, b) = \{t \in [0, 1] : a \leq F(t) \leq b\}.$$

We now define for $n \geq 1$

$$(3.20) \quad J_{n,j} = I(j2^{-n-1}, (j+2)2^{-n-1}), \quad j = 0, 1, \dots, [2^{n+1}F(1)],$$

where $[x]$ denotes the greatest integer less or equal to x . Let k and m be constants such that $2 + p/2 < m + 1 < k$, let c_3 be the constant in (2.10) and put

$$(3.21) \quad \Lambda_{n,j} = \{\omega : \sup_{s,t \in J_{n,j}} |X(s, \omega) - X(t, \omega)| \geq c_3 \sqrt{2k} 2^{-n/p}\},$$

$$(3.22) \quad Z_n(\omega) = \#\{j : \omega \in \Lambda_{n,j}\}.$$

By Lemma 2.4 we get

$$(3.23) \quad P(\Lambda_{n,j}) \leq (n \log(2))^{-k}.$$

Therefore,

$$(3.24) \quad E(Z_n) \leq 2^{n+1}F(1)(n \log(2))^{-k}.$$

By Markov's inequality, we get

$$(3.25) \quad \sum_n P\{Z_n > 2^n n^{-m}\} \leq \sum_n cn^{-(k-m)} < \infty.$$

By the Borel-Cantelli lemma, there exists for a.a. ω a number $c(\omega)$ such that

$$(3.26) \quad Z_n(\omega) \leq c(\omega)2^n n^{-m}, \quad n \geq 1.$$

Let $\pi: t_0 = 0 < t < \dots < t_N = 1$ be a partition of $[0, 1]$, and let

$$A(\omega) = \{i : |X(t_i, \omega) - X(t_{i-1}, \omega)|^p > 4(c_3 \sqrt{2k})^p (F(t_i) - F(t_{i-1}))\}$$

and

$$A_n = \{i : 2^{-n-2} \leq F(t_i) - F(t_{i-1}) < 2^{-n-1}\}.$$

Obviously

$$(3.27) \quad \sum_{i \notin A(\omega)} |X(t_i, \omega) - X(t_{i-1}, \omega)|^p \leq 4(c_3 \sqrt{2k})^p F(1).$$

Also,

$$(3.28) \quad \sum_{i \in A(\omega)} |X(t_i, \omega) - X(t_{i-1}, \omega)|^p = \sum_{n=1} \sum_{i \in A(\omega) \cap A_n} |X(t_i, \omega) - X(t_{i-1}, \omega)|^p.$$

Since $G_2(p) < \infty$ by assumption, we have $G(p) < \infty$. We can therefore use the estimate (3.6) of Theorem 3.1 to conclude that this last expression is dominated by

$$\sum_n \sum_{i \in A(\omega) \cap A_n} C(\omega) [\sigma(t_{i-1}, t_i) (\log^*(\sigma(t_{i-1}, t_i)))^{1/2}]^p.$$

By (3.18), this quantity has an upper bound

$$\sum_n \sum_{i \in A(\omega) \cap A_n} C(\omega) [\psi(2^{-n-1}) (\log^*(\psi(2^{-n-1})))^{1/2}]^p.$$

Finally, $\#(A(\omega) \cap A_n) \leq 4 Z_n(\omega)$, and using (3.26) we get the upper bound

$$\sum_n C'(\omega) 2^n n^{-m} 2^{-n-1} (n+1)^{p/2} < \infty.$$

Therefore by (3.27) and (3.28) we conclude that $X(\cdot, \omega) \in B_p$, a.s.

We will show in the next section that if $p > 1$, then $G_2(p) < \infty$ is the best possible sufficient condition for the process to have a.a. paths in B_p .

4. Conditions for tightness of the induced measure. If the process has paths in B_p a.s., does it induce a tight measure on the Borel subsets of B_p ? Recall that a probability measure μ on the Borel subsets of a Banach space B is *tight* if given $\varepsilon > 0$, there exists a compact subset K such that $\mu(K) > 1 - \varepsilon$; this is also equivalent to the existence of a separable Borel set A such that $\mu(A) = 1$. Suppose $X(\cdot, \omega) \in B_p$ a.s., and let \mathcal{A} denote the class of Borel subsets A of B_p such that $P\{X(\cdot, \omega) \in A\}$ is defined. Then \mathcal{A} is a σ -algebra. Since a.a. sample paths $t \rightarrow X(t, \omega)$ have left and right limits it follows that for any $g \in B_p$, the process $\{X(t) - g(t) : 0 \leq t \leq 1\}$ is separable, too. Let S denote a countable separant for this process. Then $V_p(X(\cdot, \omega) - g(\cdot))$ is almost surely attained if the partition points in (1.1) are restricted to S . Therefore, $V_p(X(\cdot, \omega) - g(\cdot))$ is a random variable. It follows that \mathcal{A} contains the σ -algebra generated by the balls $B(g, r) = \{f \in B_p : \|f - g\|_p < r\}$, $g \in B_p$, $r > 0$, which will be denoted by \mathcal{B}_0 . It is easily seen that \mathcal{B}_0 contains the compacts.

We first state a theorem (proved in [10]) which gives several equivalent conditions for the process with paths in B_p to induce a tight measure in B_p .

THEOREM 4.1. *Let $\{X(t) : 0 \leq t \leq 1\}$ be a separable, centered Gaussian process with paths in B_p , a.s. Then the following conditions are equivalent:*

$$(4.1) \quad \text{The process induces a tight measure on the Borel subsets of } B_p.$$

$$(4.2) \quad \text{If } \{\varphi_n\}, \{\xi_n\} \text{ are sequences as in Theorem 2.2(ii) and } S_n(t, \omega) \\ = \sum_{j=1}^n \varphi_j(t) \xi_j(\omega), \text{ then } P\{\lim_n \|S_n(\cdot, \omega) - X(\cdot, \omega)\|_p = 0\} = 1.$$

$$(4.3) \quad \text{For every } \varepsilon > 0, P\{\|X(\cdot, \omega)\|_p \leq \varepsilon\} > 0.$$

The next theorem deals with the case $p = 1$. It was proved in [10]; a different proof was given in [11].

THEOREM 4.2. *If the separable Gaussian process $\{X(t) : 0 \leq t \leq 1\}$ has paths in B_1 a.s., then it induces a tight measure in B_1 .*

For $p > 1$ the situation is quite different. The next theorem gives a sufficient condition for a B_p -path Gaussian process to induce a tight measure in B_p .

THEOREM 4.3. *Let $\{X(t) : 0 \leq t \leq 1\}$ be a separable centered Gaussian process. If for some $\varepsilon > 0$*

$$(4.4) \quad \sup_{\tau} \sum_k (\sigma(t_k, t_{k+1}))^p (\log^*(\sigma(t_k, t_{k+1})))^{p/2+\varepsilon} < \infty$$

(in particular, if $p > \gamma$, where γ is given by (3.5)) then the process has paths in B_p a.s. and induces a tight measure in B_p .

PROOF. Following Rudin [16], for $0 < \alpha < 1$ let $\text{lip } \alpha$ denote the set of bounded real-valued functions f on $[0, 1]$ such that

$$\lim_{\varepsilon \downarrow 0} \sup_{0 < |s-t| < \varepsilon} |f(s) - f(t)| / |s - t|^\alpha = 0.$$

It is well-known that the space $\text{lip } \alpha$ is a separable Banach space with norm

$$\|f\|_{\text{lip } \alpha} = |f(0)| + \sup_{s \neq t} |f(s) - f(t)| / |s - t|^\alpha.$$

If $p > 1$ and $\alpha = 1/p$, then clearly

$$\|f\|_p \leq \|f\|_{\text{lip } \alpha}, \quad f \in \text{lip } \alpha.$$

Therefore $\text{lip } 1/p$ is a separable subset of $(B_p, \|\cdot\|_p)$ when $p > 1$. Given $F: [0, 1] \rightarrow [0, 1]$, a nondecreasing function, the linear operator $\Phi_F: B_p \rightarrow B_p$ defined by

$$\Phi_F(f) = f \circ F$$

is continuous. Therefore $\Phi_F(\text{lip } 1/p)$ is a separable subset of B_p . This set consists of $g \in B_p$ for which $g(s) = g(t)$ if $F(s) = F(t)$ and

$$\lim_{\varepsilon \downarrow 0} \sup_{0 < |F(s) - F(t)| < \varepsilon} |g(s) - g(t)| / |F(s) - F(t)|^{1/p} = 0.$$

Let $\varepsilon > 0$ be as in (4.4), and define

$$(4.5) \quad F(t) = \sup_{\pi} \sum_k (\sigma(t_k, t_{k+1}))^p (\log^*(\sigma(t_k, t_{k+1})))^{p/2+\varepsilon}$$

where the sup ranges over partitions π of $[0, t]$. We will show that $X(\cdot, \omega)$ belongs a.s. to $\Phi_F(\text{lip } 1/p)$, where F is given by (4.5). This will prove the theorem.

By (4.4) we have $G(p) < \infty$. Therefore, by Theorem 3.1 we have a.s.

$$(4.6) \quad |X(s, \omega) - X(t, \omega)|^p \leq C^p(\omega) \sigma^p(s, t) |\log^*(\sigma(s, t))|^{p/2}.$$

If $s < t$ and $F(s) = F(t)$, then clearly $X(s, \omega) = X(t, \omega)$ a.s. Since X is separable, almost all paths satisfy the condition that $F(s) = F(t)$, $s \neq t$, implies $X(s, \omega) = X(t, \omega)$. By (4.6) we have

$$P\{X(\cdot, \omega) \in \Phi_F(\text{lip } 1/p)\} = 1.$$

We now proceed to discuss a class of examples which illustrate the sharpness of the conditions in Theorems 3.2 and 4.3.

In what follows, $p > 1$, m is a positive integer such that $m(1 - p^{-1}) > 1$ and $q = 4^m$. Let $(\varphi(n))$ be a real sequence satisfying

$$(4.7) \quad \sum_{n=1}^{\infty} |\varphi(n)| q^{-n/p} < \infty,$$

and define

$$(4.8) \quad f_n(t) = \varphi(n) q^{-n/p} \sin(q^n \pi t), \quad 0 \leq t \leq 1,$$

$$(4.9) \quad f(t) = \sum_{n=1}^{\infty} f_n(t), \quad 0 \leq t \leq 1.$$

We observe that the argument used to prove (2.2) shows

$$(4.10) \quad \|\sin(q^n \pi t)\|_p \geq q^{n/p},$$

therefore

$$(4.11) \quad \|f_n\|_p \geq |\varphi(n)|.$$

The following lemma will be useful.

LEMMA 4.4. *If f is defined by (4.9), then $f \in B_p$ if and only if $(\varphi(n)) \in \ell^\infty$. Furthermore, the series defining f converges in B_p if and only if $\varphi(n) \rightarrow 0$, as $n \rightarrow \infty$.*

PROOF. If $(\varphi(n)) \in \ell^\infty$, then $f \in B_p$ is shown by an argument similar to the one given

in the proof of Lemma 2.1. Now assume that $\sup_n |\varphi(n)| = \infty$. We can find $|\varphi(N)|$ arbitrarily large so

$$(4.12) \quad |\varphi(n)| \leq |\varphi(N)|, \quad n = 1, 2, \dots, N.$$

Consider the partition

$$(4.13) \quad t_k = \frac{k}{2} q^{-N}, \quad k = 0, 1, \dots, 2q^N.$$

For $n > N$, $f_n(t_k) = 0$, $0 \leq k \leq 2q^N$, so

$$(4.14) \quad |f(t_k) - f(t_{k-1})| \geq |f_N(t_{k+1}) - f_N(t_k)| - \sum_{n=1}^{N-1} |f_n(t_k) - f_n(t_{k-1})| \\ \geq |\varphi(N)| q^{-N/p} / 3.$$

Summing on $0 \leq k \leq 2q^N$, we conclude that

$$(4.15) \quad V_p(f) \geq 2 |\varphi(N)|^p / 3^p.$$

Since $|\varphi(N)|$ can be arbitrarily large, this proves the first assertion.

If the series defining f converges in B_p , then the n th term should tend to 0 as $n \rightarrow \infty$, so $\varphi(n) \rightarrow 0$ by (4.11). Now assume $\varphi(n) \rightarrow 0$. Then

$$|f_n(s) - f_n(t)| \leq |\varphi(n)| q^{-n/p} \min\{2, q^n \pi |s - t|\}.$$

Using this estimate and making the first part of the argument used in the proof of Lemma 2.1 we see that $\|\sum_{j=m}^n f_j\|_p \leq c \sup\{|\varphi(m)|, |\varphi(m+1)|, \dots, |\varphi(n)|\} \rightarrow 0$ as $m, n \rightarrow \infty$. This proves the lemma.

As an immediate corollary of this lemma we have:

PROPOSITION 4.5. *Let $\{\xi_n\}$ be independent Gaussian variables with mean 0 and variance 1, and let $(\varphi(n))$ be a monotone sequence satisfying (4.7) Then the process*

$$(4.16) \quad X(t, \omega) = \sum_{n=1}^{\infty} \varphi(n) q^{-n/p} \sin(q^n \pi t) \xi_n(\omega) \log^{-1/2}(n+1),$$

has paths in B_p a.s. if and only if $(\varphi(n)) \in \ell^\infty$. The defining series converges in B_p a.s. if and only if $\varphi(n) \rightarrow 0$ as $n \rightarrow \infty$.

PROOF. We only need to observe that by the Borel-Cantelli lemma $P\{|\xi_n| > \log^{1/2}(n+1) \text{ i.o.}\} = 1$ and $P\{|\xi_n| < 2 \log^{1/2}(n+1) \text{ eventually}\} = 1$. For the process defined by (4.16) we have

$$\sigma(s, t) = \left(\frac{2}{\pi}\right)^{1/2} \left\{ \sum_{n=1}^{\infty} \varphi(n)^2 q^{-2n/p} [\sin(q^n \pi t) - \sin(q^n \pi s)]^2 \log^{-1}(n+1) \right\}^{1/2}.$$

We write the infinite sum as $\Sigma' + \Sigma''$, where Σ' is the sum over $\{n: q^n \pi |s - t| < 2\}$ and Σ'' is the sum over the remaining n . Then using the estimates $|\sin(q^n \pi t) - \sin(q^n \pi s)| \leq q^n \pi |t - s|$ on Σ' and ≤ 2 on Σ'' , we get

$$\sigma(s, t) \leq \left(\frac{2}{\pi}\right)^{1/2} \left\{ \Sigma' \varphi(n)^2 q^{-2n/p} q^{2n} \pi^2 |t - s|^2 \log^{-1}(n+1) \right. \\ \left. + \Sigma'' \varphi(n)^2 q^{-2n/p} 4 \log^{-1}(n+1) \right\}^{1/2}.$$

Now assume that $(\varphi(n))$ is a slowly varying positive sequence. Then using standard facts about regularly varying summands (see [4]) we get

$$\sigma(s, t) \leq c |t - s|^{1/p} \varphi([\log^* |t - s|]) / (\log^* |t - s|)^{1/2},$$

where $c > 0$ is independent of s and t .

It follows that if $(\varphi(n))$ is slowly-varying in (4.16), then the Gaussian process defined by (4.16) has paths in B_p if and only if $G_2^*(p) < \infty$. This shows that the condition in Theorem

3.2 is the best condition of this type. If we take $\varphi(n) \equiv 1$, then the process has paths in B_p but does not induce a tight measure in B_p . This shows that $G_2(p) < \infty$ is not a sufficient condition for the process to induce a tight measure in B_p .

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