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A BOUND ON THE SIZE OF POINT CLUSTERS OF A RANDOM WALK WITH STATIONARY INCREMENTS

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Consider a random walk on $\mathbb{R}^d$ with stationary, possibly dependent increments. Let $N(V)$ count the number of visits to a bounded set V. We give bounds on the size of $N(t+V)$, uniformly in t, in terms of the behavior of N in a neighborhood of the origin.

1. Introduction. Let $(\xi_n)_{n\in\mathbb{Z}}$ be a stationary sequence of random vectors in the d-dimensional Euclidean space $(\mathbb{R}^d, \mathscr{B}^d)$. The process $(S_n)_{n\in\mathbb{Z}}$, determined by

$$S_0 := 0, \quad S_n = \xi_n + S_{n-1},\ n \in \mathbb{Z},$$

is called a *random walk with stationary increments.* This definition of S_n for all $n \in \mathbb{Z}$ is uncommon but will be useful in the present context. Define the point process N by

$$N(B) := \textstyle\sum_{n\in\mathbb{Z}} 1_B(S_n), \quad B \in \mathscr{B}^d.$$

We assume that the random walk is *transient*, i.e. N is finite on bounded Borel sets B.

For random walks on $\mathbb{R}^1$ with stationary, non-negative increments, Kaplan (1955) proved that $EN(t, t+h) \le EN(-h, h)$ for real t and $h > 0$. When the increments are independent, this inequality is a simple consequence of the Markov property (see Feller, 1970, VI.10) and in fact $N(t, t+h)$ is stochastically dominated by $N(-h, h)$. Below we shall see that this domination does not hold without independence.

Let us now consider random walks on $\mathbb{R}^d$. Assume V is a bounded Borel set with translate $V + t := \{s + t : s \in V\}$, and suppose $V_0 := \{s - t : s, t \in V\}$ is also a Borel set. We prove that if $f \ge 0$ is a function, growing not too slowly such that

$$n(f(n+1) - f(n)) \ge 0 \text{ is non-decreasing} \tag{1}$$

then

$$Ef(N(V)) \le Ef(N(V_0)). \tag{2}$$

The condition (1) is satisfied for e.g. $f(n) = n^\alpha$, $\alpha > 0$, or $f(n) = (\log n)_+$. If (2) were true for any non-decreasing f then $N(V)$ would be stochastically dominated by $N(V_0)$. However, we prove

$$P(N(V) \ge p) \le \gamma P(N(V_0) \ge p) \quad \text{where } \gamma = 2 - \frac{1}{p} \tag{3}$$

for $p = 1, \cdots$. An example will show that γ cannot be smaller without restricting V. The two results above will follow from the more general Theorem 1 below. Inequality (3) can also be proved directly using the method of Berbee (1979), Theorem 2.2.3.

Suppose $0 = f(0) \le f(1) \le \cdots$ is given. Let $c(n) := (1/n)\sum_{k=1}^n f(k)$ be a Cesaro average and let

$$h(n) := c(n) + \sup_{k\le n}(f(k) - c(k)).$$

We shall see that (1) implies that $f - c$ is non-decreasing and then $f \equiv h$. In Section 2 show

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THEOREM 1. $Ef(N(V)) \leq Eh(N(V_0))$.

This result and also (2), (3) and (5) can be improved slightly if $-V$ is a translate of V. In that case we may replace $N(V_0)$ by

$$\sup_{V' \ni 0} N(V') \tag{4}$$

where V' runs over the translates of V.

In Section 3 we pay special attention to random walks on the real line. We prove for an interval $V = (t, t+h)$

$$P(N(V) \geq p) \leq \gamma P(N(V_0) \geq p) \quad \text{where } \gamma = \frac{3}{2} - \frac{1}{2p} \tag{5}$$

for $p = 1, 2, \cdots$. An example shows that γ cannot be smaller.

Replacing V by $V + t$ in the inequalities does not change V_0. As a consequence an important application of our results concerns uniform integrability. Suppose that $EN(U) < \infty$ on a neighborhood U of the origin. Using the fact that the bounded set V is contained in a finite union of translates of U, it is proved easily from our inequalities that $N(V + t)$ is integrable, uniformly in t. This result is used in Berbee (1979) to obtain Blackwell's theorem for stationary processes. A related integrability problem is solved in Daley (1971) in connection with the global renewal theorem. A condition for finiteness of $EN(U)$ can be found in Lai (1977) in terms of strong mixing. In the limit theory of semi-Markov chains, very complicated integrability conditions are used (see Kesten, 1974).

2. Inequalities for General V. The proof of Theorem 1 is based on a combinatorial lemma. Let $A := (s_0, \cdots, s_n)$ be a finite sequence of points in $\mathbb{R}^k$. Define the *distant cluster* of $s \in A$ as the subsequence $A(s) \equiv A \cap (V + s)$ of points of A in $V + s$ (with the same multiplicities) and the *close cluster* as $A_0(s) \equiv A \cap (V_0 + s)$. Let $n(s)$ and $n_0(s)$ denote the number of points in the distant and close cluster of s; note that $s \in A_0(s)$ so $n_0(s) \geq 1$.

With f and h as in Theorem 1 we have the following comparison lemma for the sizes of distant and close clusters.

LEMMA 2. $\sum_s f(n(s)) \leq \sum_s h(n_0(s))$.

Here as in the proof below the sums are over the points in A with the right multiplicities.

PROOF. Obviously for $s \in A$

$$f(n(s)) \leq c(n_0(s)) + (f(n(s)) - c(n_0(s)))^+.$$

Observing that $n(s) \geq 1$ when $t \in A(s)$, define

$$h_1(s, t) := \frac{1}{n(s)} c(n_0(s)), \quad t \in A(s),$$

$$h_2(s, t) := \frac{1}{n(s)} (f(n(s)) - c(n_0(s)))^+, \quad t \in A(s),$$

$$h_1(s, t) = h_2(s, t) := 0, \quad \text{otherwise.}$$

Because $n(s) = \#A(s)$ we have, rewriting sums,

$$\sum_s f(n(s)) \leq \sum_s (\sum_t h_1(s, t) + \sum_r h_2(r, s))$$

and it suffices to prove that the term in brackets is at most $h(n_0(s))$. This term equals

$$c(n_0(s)) + \sum_{r: s \in A(r)} \frac{1}{n(r)} (f(n(r)) - c(n_0(r)))^+. \tag{6}$$

If $s \in A(r)$ then $V + r \subset V_0 + s$ so $n(r) \le n_0(s)$. Hence (6) is at most

$$c(n_0(s)) + \sum_{r:s\in A(r)} \sup_{n\le n_0(s)} \frac{1}{n} (f(n) - c(n_0(r)))^+.$$

The sum above is taken over $k := \#A \cap (-V + s)$ terms. If $s \in A(r)$ then $-V + s \subset V_0 + r$, so $k \le n_0(r)$. Because c is non-decreasing $(f(n) - c(j))^+$ is non-increasing in j. Hence (6) is at most

$$(7) \qquad c(n_0(s)) + k \sup_{n\le n_0(s)} \frac{1}{n} (f(n) - c(k))^+.$$

Since c is the Cesaro average of the monotomic sequence $\{f(n)\}$, the difference

$$\frac{k}{n} (f(n) - c(k)) - \frac{k-1}{n} (f(n) - c(k-1)) = \frac{1}{n} (f(n) - f(k))$$

is non-negative for $k \le n$ and non-positive for $k \ge n$. So the expression (7) is maximal for $k = n$. Therefore (7) and so also (6) is at most $h(n_0(s))$. □

REMARK 3. If $-V$ is a translate of V we can strengthen Lemma 2 by replacing $n_0(s)$ there by

$$(8) \qquad n_0'(s) := \sup_{V'\ni s} \#A \cap V'$$

where V' runs over all translates $V + t$ of V: in proving this assertion we use the facts that $n(r) \le n_0'(s)$ and $k \le n_0'(r)$ if $s \in A(r)$, and follow the arguments as above with the obvious changes.

Theorem 1 follows from Lemma 2 using the ergodic theorem as follows.

PROOF OF THEOREM 1. Take $A := (S_0, \cdots, S_n)$ and define

$$\bar{N}(B) := \sum_{k=0}^{n} 1_B(S_k).$$

By Lemma 2

$$(9) \qquad \sum_{k=0}^{n} f(\bar{N}(S_k + V)) \le \sum_{k=0}^{n} h(\bar{N}(S_k + V_0)).$$

Choose some large constant m and define for $-\infty < k < \infty$ a stationary sequence

$$N_k^{(m)} := N(S_k + V) \quad \text{if } S_{j+k} \notin S_k + V \quad \text{for all} \quad |j| \ge m,$$
$$:= 0 \qquad \text{otherwise.}$$

With these definitions

$$N_k^{(m)} \le \bar{N}(S_k + V) \quad \text{for } m \le k \le n - m,$$
$$\le 2m - 1 \qquad \text{for all } k,$$

and hence

$$\sum_{k=0}^{n} f(N_k^{(m)}) - 2mf(2m - 1) \le \sum_{k=0}^{n} f(\bar{N}(S_k + V)).$$

By (9) the right hand side is dominated by

$$\sum_{k=0}^{n} h(\bar{N}(S_k + V_0)) \le \sum_{k=0}^{n} h(N(S_k + V_0)),$$

where to deduce the last inequality we have used the facts that h is non-decreasing and $\bar{N} \le N$. Hence

$$\sum_{k=0}^{n} f(N_k^{(m)}) - 2mf(2m - 1) \le \sum_{k=0}^{n} h(N(S_k + V_0)).$$

Divide by $n + 1$, let $n \to \infty$ and apply the ergodic theorem. After taking expectations we obtain

$$Ef(N_0^{(m)}) \leq Eh(N(V_0)).$$

Let $m \to \infty$. By the monotone convergence theorem this implies the assertion. □

To get (2) from (1) we apply Theorem 1 and the following remark.

REMARK 4. Obviously $h \equiv f$ if and only if $f(n) - c(n)$ is non-decreasing. This property holds under (1). To see this observe that f can be expressed as $f \equiv \sum_1^\infty a_p f_p$ where $a_1 := f(1)$ and

$$(n-1)(f(n) - f(n-1)) = a_2 + \cdots + a_n, \quad n \geq 2,$$

species the other a_p. They are non-negative by (1). Here f_p is defined by

$$f_p(n) := \textstyle\sum_p^n \frac{1}{k-1} \quad n \geq p > 1$$

$$:= 1 \qquad n \geq p = 1$$

$$:= 0 \qquad \text{else.}$$

That $f - c$ is non-decreasing is checked easily for $f \equiv f_p$, and hence also holds for $f \equiv \sum_1^\infty a_p f_p$.

Inequality (3) follows from Theorem 1 by using $f \equiv 1_{[p,\infty)}$ and observing that for $n \geq p$

$$h(n) = 1 - \frac{p-1}{n} + \frac{p-1}{p} \leq \gamma = 2 - \frac{1}{p}. \tag{10}$$

The constant in (3) cannot be smaller because of the following example for $d = 1$.

EXAMPLE 5. Fix some $m \geq 1$. We construct a sequence $\bar{A}$ of reals $x_1 < y_1 < \cdots < x_m < y_m < z$ and a set V such that $y_i \in x_i + V$, $z \in y_i + V$ and $(x_i + V_0) \cap \bar{A} = \{x_i\}$.

Suppose this is done. Let $A = (s_0, \cdots, s_n)$ consist of $(p-1)$-tuplets at $x_1, \cdots, x_m$ and p-tuplets at $y_1, \cdots, y_m, z$. Then, counting with the right multiplicities

$$\# \{s \in A : n(s) \geq p\} = m(p-1) + mp$$
$$\# \{s \in A : n_0(s) \geq p\} = mp + p.$$

If m is large the ratio γ_m of these numbers is close to $2 - 1/p$.

To construct the probabilistic example, let $\omega := (\omega_k)_{k \in \mathbb{Z}}$ have period $n + 1$ such that $\omega_i = s_i - s_{i-1}$, $1 \leq i \leq n$, and ω_0 is some very large number. Let each element of $\Omega := \{T^i\omega, 0 \leq i \leq n\}$ have equal probability. The identitity ξ on Ω is stationary and the ratio of the probabilities in (3) is γ_m as above.

To construct $\bar{A}$ let $2 < p_1 < p_2 < \cdots$ be primes. Take $z := 0$ and

$$y_i := -p_1 * \cdots * p_{m+i}$$

$$x_i := y_i - p_1 * \cdots * p_i, \; 1 \leq i \leq m,$$

and let $V := \{p_1 * \cdots * p_i : 1 \leq i \leq 2m\}$. The only property of $\bar{A}$ that is not obvious is $(x_i + V_0) \cap \bar{A} = \{x_i\}$. Let us call products of more than m primes long and the other products short. Each $v \in V_0$ is uniquely represented as difference of two elements in V. Let v' be obtained by replacing in this difference the short products by 0. Also $(x_i)' := y_i$.

Suppose $x_j \in x_i + V_0$. It is easily proved that for the long products in $x_j - x_i = v \in V_0$ we have $y_j - y_i = v'$ and then we should have $v = v'$. So $y_j - y_i = x_j - x_i$ and $i = j$. Similar considerations disprove y_j or $0 \in x_i + V_0$. Hence $(x_i + V_0) \cap \bar{A} = \{x_i\}$.

3. Inequalities for intervals. Let $d = 1$ and assume $V = (t, t + h)$. Let $A := (s_0, \cdots, s_n)$ and take $n(s) := \# A \cap (V + s)$ as before but define $n_0(s)$ by (8). Because $-V$

is a translate of V, Lemma 2 holds. We get (5) from Lemma 6 as in the proof of Theorem 1. Counting $s \in A$ with its multiplicity, we have

LEMMA 6. $\# \{s \in A : n(s) \geq p\} \leq (3/2 - 1/2p) \# \{s \in A : n_0(s) \geq p\}$.

PROOF. Let $f \equiv 1_{[p,\infty)}$. Then $h(n) \leq 3/2 - 1/2p$ for $n \leq 2p$ by (10). Hence if $n_0(s) \leq 2p$ for all $s \in A$, then the assertion follows from Lemma 2.

Let $\gamma(A) := \#/\#_0$ be the ratio of the numbers at the left and right in the assertion. If $\gamma(A) \leq 1$, nothing has to be proved. Otherwise there may exist an interval $I = (x, x + h)$ with more than $2p$ points of A. We will remove one of these points to get A' and will show $\gamma(A) \leq \gamma(A')$. Continuing this procedure, we would come in finitely many steps to A'' with no such intervals I. For such A'' we already obtained the assertion and so $\gamma(A) \leq \gamma(A'')$ $\leq 3/2 - 1/2p$ would complete the proof.

So consider A and I as above and remove $\bar{s} \in A \cap I$ from A such that both in $(x, \bar{s}]$ and $[\bar{s}, x + h)$ at least p points of A are left. One checks easily that then $\# A' \cap V' \geq p$ if $\# A \cap V' \geq p$ for any translate V' of V. Hence in $\gamma(A) := \#/\#_0$ the removal of $\bar{s}$ causes the denominator (numerator) to decrease by (at most) 1. Because $\gamma(A) \geq 1$ we may conclude $\gamma(A') \geq \gamma(A)$. □

EXAMPLE 7. The constant γ in (5) cannot be smaller than $3/2 - 1/2p$. To see this let $0 < \varepsilon_0 < \cdots < \varepsilon_m < 1$. Let A contain p-tuplets at $5k$ and $5k + \varepsilon_k$ and $(p - 1)$-tuplets at $5k + \varepsilon_k + 1$, $0 \leq k \leq m$. With $V := (5, 6)$ the ratio γ_m of

$$\# \{s \in A : n(s) \geq p\} = (3p - 1)m$$
$$\# \{s \in A : n_0(s) \geq p\} = 2p(m + 1)$$

is close to $3/2 - 1/2p$ for large m. Here we may take $n_0(s) := \# A \cap (V_0 + s)$. Just as in example 5 we can construct a probability space where the ratio of the probabilities in (5) is γ_m.

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