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HIGH DENSITY LIMIT THEOREMS FOR INFINITE SYSTEMS OF UNSCALED BRANCHING BROWNIAN MOTIONS[1]

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The fluctuations of an infinite system of unscaled branching Brownian motions in R^d are shown to converge weakly under a spatial central limit normalization when the initial density of particles tends to infinity. The limit is a generalized Gaussian process M which can be written as $M = M^I + M^{II}$, where M^I is the fluctuation limit of a Poisson system of Brownian motions obtained by Martin-Löf, and M^{II} arises from the spatial central limit normalization of the "demographic variation process" of the system. In the critical case M^I and M^{II} are independent and M^{II} coincides with the generalized Ornstein-Uhlenbeck process found by Dawson and by Holley and Stroock as the renormalization limit of an infinite system of critical branching Brownian motions when $d \geq 3$. Generalized Langevin equations for M, M^I and M^{II} are given.

1. Introduction. Let $N \equiv \{N_t,\ t \geq 0\}$ denote a random field of particles in R^d ($d \geq 1$) such that at time $t = 0$ it is a homogeneous Poisson field, and each initial particle generates a branching Brownian motion where the particle lifetime distribution is exponential and the branching law has finite second moment. N_t is the point process determined by the locations of the particles in R^d at time t.

In recent investigations (e.g. [4], [5], [6], [10], [12]) on asymptotic behaviors of such systems, scalings have been used where the initial density of particles tends to ∞ and the mean particle lifetime goes to 0 (and possibly other scalings). Of particular interest are limits in the critical case, i.e. when the mean of the branching law is 1. In these models the branching structure originated by each initial particle disappears in the limit due to the time scaling which makes the mean particle lifetime tend to 0. In this paper we study the asymptotic behavior of N when the initial density tends to ∞ but all other parameters of the process remain unchanged. Since the branching structure is preserved, this model portrays an infinite noninteracting branching particle system where the particles undergo many more scatterings than fissions.

We show that the random field N^T, where the initial Poisson field has intensity $T > 0$, obeys a law of large numbers and a functional central limit theorem when $T \to \infty$. The weak limit $M = \{M_t,\ t \geq 0\}$ of the fluctuation process is a generalized Gauss-Markov process which has continuous paths and satisfies the generalized Langevin equation

$$\partial M/\partial t = \tfrac{1}{2}\Delta M + \alpha M + \mathscr{W}, \quad t \geq 0,$$

where M_0 is the standard Gaussian white noise on R^d, α is the Malthusian parameter of the branching process and $\mathscr{W}$ is a certain space-time noise.

M has the decomposition $M = M^I + M^{II}$, where M^I is the high-density fluctuation limit of a Poisson system of Brownian motions obtained by Martin-Löf [15], and M^{II} is the fluctuation limit of the "demographic variation process" of the system, which traces the excess or defect of Brownian particles due to reproduction and deaths in the population

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with respect to the "basic population process" where there are no reproduction and deaths. M^I and M^{II} are continuous generalized Gauss-Markov processes satisfying a system of generalized Langevin equations, and in the critical case ($\alpha = 0$) they become independent.

In dimensions $d \geq 3$ and the critical case M^{II} coincides with the generalized Ornstein-Uhlenbeck process found by Dawson [4], [5] and by Holley and Stroock [12] as the renormalization limit of an infinite system of critical branching Brownian motions. This coincidence is interesting in view of the fact that the same limit arises from two qualitatively different scalings: renormalization with scaled time and central limit normalization with unscaled time.

Section 2 contains the limit theorems, the descriptions of the processes M, M^I and M^{II} and the generalized Langevin equations they satisfy. The proofs form Section 3. We refer to [1] for the theory of branching processes and to [8] and [14] for the theory of generalized random fields.

This work owes a great deal to ideas of Martin-Löf [15], and Holley and Stroock [12, 13].

2. Results. The special *branching random field of particles* $N^T \equiv \{N_t^T, t \geq 0\}$ in R^d ($d \geq 1$) considered in this paper is described as follows: the initial field N_0^T is a homogeneous Poisson field on R^d with intensity $T > 0$, each initial particle generates an independent branching Brownian motion in R^d such that the particle lifetime distribution is exponential with parameter V, and the branching law $\{p_n\}_{n=0}^{\infty}$ has mean m_1 and finite second factorial moment m_2. The Malthusian parameter is then $\alpha = V(m_1 - 1)$. The case $m_1 = 1$ (or $\alpha = 0$) is called *critical.* $N_t^T(A)$ represents the number of particles of N^T in the Borel set $A \subset R^d$ at time t. The existence of such a process on a probability space $(\Omega, \mathscr{A}, P)$ is established e.g. in [6] and [12]. Our objective is to study the behavior of N^T as $T \to \infty$.

Since the fluctuation limit of N^T will be a generalized random field, we regard N^T as taking values in $\mathscr{S}'(R^d)$, the Schwartz space of tempered distributions on R^d; this can be done because N_t^T is a random tempered Radon measure on R^d. Let $\mathscr{S}(R^d)$ denote the Schwartz space of (rapidly decreasing infinitely differentiable) test functions on R^d whose topological dual is $\mathscr{S}'(R^d)$, and $\langle , \rangle$ the canonical bilinear form on $\mathscr{S}'(R^d) \times \mathscr{S}(R^d)$. When $\mu \in \mathscr{S}'(R^d)$ is a (signed) measure, then $\langle \mu, \phi \rangle = \int \phi \, d\mu$, $\phi \in \mathscr{S}(R^d)$. We recall that a nuclear space topology on $\mathscr{S}(R^d)$ is defined by a sequence of norms $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \cdots \leq \|\cdot\|_p \leq \cdots$, and $\|\cdot\|_{-p}$ denotes the operator norm on the dual space of the $\|\cdot\|_p$-completion of $\mathscr{S}(R^d)$. We will use the norms

$$\|\phi\|_p = \max_{0 \leq |k| \leq p} \sup_x \prod_{j=1}^d (1 + |x_j|)^p |D^k \phi(x)|, \quad x = (x_1, \cdots, x_d),$$

where $k = (k_1, \cdots, k_d)$, $|k| = k_1 + \cdots + k_d$, and $D^k = \partial^k / \partial x_1^{k_1} \cdots \partial x_d^{k_d}$ (see [8, 9]).

The process N^T has mean $E\langle N_t^T, \phi \rangle = Te^{\alpha t} \int \phi(x) \, dx$ and $\operatorname{Var}\langle N_t^T, \phi \rangle$ is of order T (see Corollary 1, Section 2).

The first result is a law of large numbers for the system.

THEOREM 1. *For each $t \geq 0$ and $\phi \in \mathscr{S}(R^d)$,*

$$T^{-1}\langle N_t^T, \phi \rangle \to e^{\alpha t} \int \phi(x) \, dx \quad \textit{in mean square as } T \to \infty.$$

Now we consider the fluctuation process $M^T \equiv \{M_t^T, t \geq 0\}$ defined by

$$M_t^T = T^{-1/2}(N_t^T - Te^{\alpha t}\lambda), \quad t \geq 0,$$

where λ is the Lebesgue measure on R^d.

In the following we denote by $\Rightarrow$ weak convergence of probability measures and W is the standard Gaussian white noise on R^d (i.e. the $\mathscr{S}'(R^d)$-valued random variable whose characteristic functional is $\exp\{-\frac{1}{2} \int \phi^2(x) \, dx\}$).

The fluctuation M^T has the following asymptotic behavior.

THEOREM 2. *There exists an $\mathscr{S}'(R^d)$-valued centered Gaussian process $M \equiv \{M_t, t \geq 0\}$ with covariance functional*

$$\mathrm{Cov}(\langle M_s, \phi\rangle, \langle M_t, \psi\rangle) = e^{\alpha t}\int\int \phi(x)\psi(y)e^{-\|x-y\|^2/2(t-s)}(2\pi(t-s))^{-d/2}\,dx\,dy$$

$$+ m_2 V e^{\alpha t}\int\int \phi(x)\psi(y)\int_0^s e^{\alpha r-\|x-y\|^2/2(t-s+2r)}(2\pi(t-s+2r))^{-d/2}\,dr\,dx\,dy,$$

$$\phi, \psi \in \mathscr{S}(R^d), \quad s \leq t,$$

and $M^T \Rightarrow M$ as $T \to \infty$.

The process M is Markovian, has a norm-continuous version (i.e. for each $\tau \in (0, \infty)$ there is an integer $p > 0$ such that M_t is $\|\cdot\|_{-p}$-continuous on $[0, \tau]$ almost surely), and viewed as a space-time process it satisfies the generalized Langevin equation

$$\partial M/\partial t = \tfrac{1}{2}\Delta M + \alpha M + \mathscr{W}, \quad t \geq 0, \quad M_0 = W,$$

where $\mathscr{W}$ is a generalized space-time centered Gaussian noise with covariance functional

$$\mathrm{Cov}(\langle \mathscr{W}_s, \phi\rangle, \langle \mathscr{W}_t, \psi\rangle) = \delta(s-t)e^{\alpha t}\left\{\int \nabla\phi(x)\cdot\nabla\psi(x)\,dx + (m_2V-\alpha)\int\phi(x)\psi(x)\,dx\right\},$$

$$\phi, \psi \in \mathscr{S}(R^d)$$

($\cdot$ denotes inner product).

From the branching random field N^T we now form another process $N^{I,T} \equiv \{N_t^{I,T}, t \geq 0\}$ as follows: when a particle in N^T splits into $n > 1$ particles we leave only one of them and remove the other $n - 1$, and when a particle in N^T dies without leaving descendants ($n = 0$) we resuscitate it and let it live forever without reproducing doing its Brownian motion. $N^{I,T}$ is then a Poisson system of independent Brownian motions; such systems have been investigated by Martin-Löf [15]. In the present case $N^{I,T}$ is a stationary process, $N_t^{I,T}$ being distributed as the initial Poisson field N_0^T for all t. We call $N^{I,T}$ the *basic population process* of the system.

We also define the process $N^{II,T} \equiv \{N_t^{II,T}, t \geq 0\}$ by

$$N_t^{II,T} = N_t^T - N_t^{I,T}, \quad t \geq 0,$$

and call it the *demographic variation process* of the system because it traces the excess or defect of Brownian particles due to reproduction and deaths in the population with respect to the basic population $N^{I,T}$.

Clearly M^T can be written as

$$M^T = M^{I,T} + M^{II,T},$$

where

$$M_t^{I,T} = T^{-1/2}(N_t^{I,T} - T\lambda)$$

and

$$M_t^{II,T} = T^{-1/2}(N_t^{II,T} - (e^{\alpha t} - 1)T\lambda).$$

The following theorem concerns this decomposition.

THEOREM 3. *The process M of Theorem 2 has the decomposition*

$$M = M^I + M^{II},$$

where M^I and M^{II} are generalized centered Gaussian fields satisfying the system of generalized Langevin equations

$$\partial M^I/\partial t = \tfrac{1}{2}\Delta M^I + \mathscr{W}^I$$

$$\partial M^{II}/\partial t = \tfrac{1}{2}\Delta M^{II} + \alpha(M^I + M^{II}) + \mathscr{W}^{II}, \quad t \geq 0,$$

$$M_0^I = W, \; M_0^{II} = 0,$$

where $\mathscr{W}^I$ and $\mathscr{W}^{II}$ are independent generalized space-time Gaussian noises with covariances

$$\mathrm{Cov}(\langle \mathscr{W}_s^I, \phi\rangle, \langle \mathscr{W}_t^I, \psi\rangle) = \delta(s-t)\int \nabla\phi(x)\cdot\nabla\psi(x)\,dx$$

and

$$\mathrm{Cov}(\langle \mathscr{W}_s^{II}, \phi\rangle, \langle \mathscr{W}_t^{II}, \psi\rangle) = \delta(s-t)\left\{(e^{\alpha t}-1)\int \nabla\phi(x)\cdot\nabla\psi(x)\,dx + e^{\alpha t}(m_2 V - \alpha)\int \phi(x)\psi(x)\,dx\right\}.$$

M^I and M^{II} are Markovian, have norm-continuous versions, and their covariances are

$$\mathrm{Cov}(\langle M_s^I, \phi\rangle, \langle M_t^I, \psi\rangle) = \iint \phi(x)\psi(y)e^{-\|x-y\|^2/2(t-s)}(2\pi(t-s))^{-d/2}\,dx\,dy, \quad s \leq t,$$

$$\begin{aligned}\mathrm{Cov}(\langle M_s^{II}, \phi\rangle, \langle M_t^{II}, \psi\rangle) &= e^{\alpha t}(1-e^{-\alpha s})\iint \phi(x)\psi(y)e^{-\|x-y\|^2/2(t-s)}(2\pi(t-s))^{-d/2}\,dx\,dy \\ &\quad + (m_2 V e^{\alpha t} - \alpha(1+e^{\alpha(t-s)}))\iint \phi(x)\psi(y)\int_0^s e^{\alpha r - \|x-y\|^2/2(t-s+2r)} \\ &\quad \cdot (2\pi(t-s+2r))^{-d/2}\,dr\,dx\,dy, \quad s \leq t,\end{aligned}$$

and

$$\mathrm{Cov}(\langle M_s^I, \phi\rangle, \langle M_t^{II}, \psi\rangle)$$

$$= \begin{cases} (e^{\alpha(t-s)}-1)\iint \phi(x)\psi(y)e^{-\|x-y\|^2/2(t-s)}(2\pi(t-s))^{-d/2}\,dx\,dy \\ \quad + \alpha e^{\alpha(t-s)}\iint \phi(x)\,\psi(y)\int_0^s e^{\alpha r-\|x-y\|^2/2(t-s+2r)}(2\pi(t-s+2r))^{-d/2}\,dr\,dx\,dy, & s \leq t, \\ \alpha\iint \phi(x)\psi(y)\int_0^t e^{\alpha r-\|x-y\|^2/2(s-t+2r)}(2\pi(s-t+2r))^{-d/2}\,dr\,dx\,dy, & s \geq t. \end{cases}$$

In particular M^I and M^{II} are independent if and only if $\alpha = 0$ (the critical case).
Finally, $M^{I,T} \Rightarrow M^I$ and $M^{II,T} \Rightarrow M^{II}$ as $T \to \infty$.

REMARKS.

1) The laws of large numbers for $N^{I,T}$ and $N^{II,T}$ are

$$T^{-1}\langle N_t^{I,T}, \phi\rangle \to \int \phi(x)\,dx$$

and

$$T^{-1}\langle N_t^{II,T}, \phi\rangle \to (e^{\alpha t}-1)\int \phi(x)\,dx \quad \text{as} \quad T \to \infty$$

in the mean square.

2) The limit $M_t^{I,T} \Rightarrow M_t^I$ for each t and the properties of M^I were proved by Martin-Löf [15].

3) In the critical case ($\alpha = 0$) and in dimensions $d \geq 3$ the process M^{II} coincides with the renormalization limit of the infinite system of critical branching Brownian motions. In the renormalization scaling the initial field is also Poisson with intensity T, but there is a space-time scaling of particle motion under which the mean particle lifetime V^{-1} is replaced by $(VT^{2/d})^{-1}$ and the normalization is $T^{-1/2-1/d}(N^T - T\lambda)$. This limit was obtained by Holley and Stroock [12] (using binary branching) and independently by Dawson [4], [5] (with particles doing a symmetric stable process). In the terminology of [12] the limit process M^{II} with $\alpha = 0$ is the generalized Ornstein-Uhlenbeck process with characteristics $\frac{1}{2}\Delta$ and $(m_2 V)^{1/2}I$ starting from 0. In the present paper this process arises from the spatial central limit normalization of the demographic variation process $N^{II,T}$ without space-time scaling of particle motion and it holds for $d \geq 1$.

4) In the critical case ($\alpha = 0$) and dimensions $d \geq 3$ the process M has an invariant random field M_∞ which is the generalized centered Gaussian field with covariance kernel

$$\delta(x-y) + m_2 V\Gamma(d/2 - 1)(4\pi)^{-1}\|x-y\|^{-d+2}, \quad x, y \in R^d,$$

i.e. the same as for the generalized Ornstein-Uhlenbeck process above plus an independent standard Gaussian white noise.

5) M^I and in the critical case M^{II} are both self-similar. Indeed, given any constant $k > 0$ the distribution of M^I is invariant under the transformation $\langle k^{-d/2}M^I_{k^2t}, \phi(k^{-1}\cdot)\rangle$, and when $\alpha = 0$ the distribution of M^{II} is invariant under the transformation $\langle k^{-d/2-1}M^{II}_{k^2t}\ \phi(k^{-1}\cdot)\rangle$. Hence M is never self-similar but in the critical case it is a sum of two independent self-similar processes with different similarity transformations.

6) Remarks 3 and 5 lead to the following observations. In the critical case and $d \geq 3$ it can be shown that the renormalization limit of $N^{II,T}$ is the generalized Ornstein-Uhlenbeck process, and since that of $N^{I,T}$ is 0, then the renormalization limit of N^T is also the generalized Ornstein-Uhlenbeck process, thus agreeing with the results of Dawson, and Holley and Stroock. The process $N^{II,T}$ has the generalized Ornstein-Uhlenbeck process as a limit under two different scalings: renormalization, and spatial central limit normalization with unscaled motion.

7) Here we have restricted our attention to Brownian particles in order to bring out the main features of the model in a relatively simple case. By a more elaborate analysis, but essentially the same approach, similar results can be obtained for infinite systems of branching random motions with particles performing other Markov processes, or systems where for each initial particle the corresponding branching motion process converges under a scaling to a branching diffusion, as in [11] for example.

3. Proofs. Due to deaths in the population, N^T will contain some terminated (Brownian) trajectories $\omega : [0, \infty) \to R^d$. It will be convenient to consider all trajectories as being infinite. In order to achieve this we add a point † to the state space and denote $\tilde{R}^d = R^d \cup \{\dagger\}$; $\{\dagger\}$ is the cemetery where dead particles live, and every trajectory ω in N^T will belong to the space $\mathscr{F}$ of right-continuous functions $x : [0, \infty) \to \tilde{R}^d$ with the property that $x(t) = \dagger$ implies $x(s) = \dagger$ for all $s > t$. Test functions $\phi \in \mathscr{S}(R^d)$ are extended to $\tilde{R}^d$ by $\phi(\dagger) = 0$. This is a device to take care of dead particles in order to make the following definitions precise.

We will consider N^T as an $\mathscr{S}'(R^d)$-valued stochastic process and also as an $\mathscr{S}'(R^d \times R^+)$- valued random field. Let $\{\omega_i\}_{i=1}^{\infty}$ be the set of all trajectories in N^T. Given

$\phi \in \mathscr{S}(R^d)$, let

(1) $$\langle N_t^T, \phi \rangle = \sum_{i=1}^{\infty} \phi(\omega_i(t)), \quad t \geq 0.$$

Then $\{N_t^T, t \geq 0\}$ is a random element of the Skorohod space $D([0, \infty), \mathscr{S}'(R^d))$. Given $\phi \in \mathscr{S}(R^d \times R^+)$, let

(2) $$\tilde{\phi}(x) = \int_0^\infty \phi(x(t), t)\, dt, \quad x \in \mathscr{F},$$

and define $\tilde{N}^T$ on $\mathscr{S}(R^d \times R^+)$ by

(3) $$\langle \tilde{N}^T, \phi \rangle = \sum_{i=1}^{\infty} \tilde{\phi}(\omega_i) = \int_0^\infty \langle N_t^T, \phi(\cdot, t) \rangle\, dt.$$

Thus $\tilde{N}^T$ is N^T viewed as a linear functional on $\mathscr{S}(R^d \times R^+)$, and we will verify later that it is an $\mathscr{S}'(R^d \times R^+)$-valued random field. The norms defining the topology on $\mathscr{S}(R^d)$ were given in Section 2. For $\mathscr{S}(R^d \times R^+)$ we use the norms

$$\|\phi\|_p = \max_{0 \leq |k| \leq p} \sup_x \sup_{t \geq 0} \prod_{j=1}^{d} (1 + |x_j|)^p (1 + t)^p e^{\alpha t} |D^k \phi(x, t)|, \quad p = 0, 1, \cdots,$$

if $\alpha > 0$, and without the $e^{\alpha t}$ if $\alpha \leq 0$, where D^k involves differentiation with respect to x and t; the induced topology is nuclear (see [8, 9]).

Before proving the theorems, we need to establish some basic results about N^T.

Since $N^T = \sum_{i=1}^{\infty} N^{x_i}$, where $\{x_i\}_{i=1}^{\infty}$ are the points of the initial field N_0^T and $N^x \equiv \{N_t^x, t \geq 0\}$ denotes a branching Brownian motion as described in Section 2 with a single initial particle starting from $x \in R^d$, and since N_0^T is Poisson with intensity measure $T\,dx$, it can be seen from (1) that the random vector $(\langle N_{t_1}^T, \phi_1 \rangle, \cdots, \langle N_{t_n}^T, \phi_n \rangle)$, $t_1 < t_2 < \cdots < t_n$, has characteristic function

(4) $$E \exp\{i \textstyle\sum_{j=1}^{n} u_j \langle N_{t_j}^T, \phi_j \rangle\} = \exp\left\{T \displaystyle\int [E \exp\{i \textstyle\sum_{j=1}^{n} u_j \langle N_{t_j}^x, \phi_j \rangle\} - 1]\, dx\right\},$$
$$u_1, \cdots, u_n \in R, \qquad \phi_1, \cdots, \phi_n \in \mathscr{S}(R^d).$$

Similarly, from (3), $\tilde{N}^T$ has characteristic functional

(5) $$E \exp\{i \langle \tilde{N}^T, \phi \rangle\} = \exp\left\{T \int [E \exp\{i \langle \tilde{N}^x, \phi \rangle\} - 1]\, dx\right\}, \quad \phi \in \mathscr{S}(R^d \times R^+),$$

where $\tilde{N}^x$ is N^x as an $\mathscr{S}'(R^d \times R^+)$-random field.

From (4) with $n = 1, 2$ follows that

(6) $$E\langle N_t^T, \phi \rangle = T \int E\langle N_t^x, \phi \rangle\, dx, \quad \phi \in \mathscr{S}(R^d),$$

and

(7) $$\mathrm{Cov}(\langle N_s^T, \phi \rangle, \langle N_t^T, \psi \rangle) = T \int E\langle N_s^x, \phi \rangle \langle N_t^x, \psi \rangle\, dx, \quad \phi, \psi \in \mathscr{S}(R^d).$$

The process $\{N_t^T\}$ is Markovian with infinitesimal generator $\mathscr{L}$ given by

(8) $$\mathscr{L}f(\langle \mu, \phi \rangle) = f'(\langle \mu, \phi \rangle)\langle \mu, \tfrac{1}{2}\Delta\phi \rangle + \tfrac{1}{2} f''(\langle \mu, \phi \rangle)\langle \mu, |\nabla \phi|^2 \rangle$$
$$+ V \int \mu(dx) \textstyle\sum_{n=0}^{\infty} p_n [f(\langle \mu, \phi \rangle + (n-1)\phi(x)) - f(\langle \mu, \phi \rangle)],$$

where $f \in C_b^2(R)$ (functions with bounded continuous derivatives of up to second order), μ is a point measure on R^d and $\phi \in \mathscr{S}(R^d)$, and

(9) $$f(\langle N_t^T, \phi \rangle) - \int_0^t \mathscr{L}f(\langle N_s^T, \phi \rangle)\, ds, \quad t \geq 0$$

is a right-continuous martingale with respect to the σ-fields generated by $\{\langle N_s^T, \phi\rangle, s \le t, \phi \in \mathscr{S}(R^d)\}$, $t \ge 0$. This is shown as in [12].

We define the semigroup

$$\mathscr{T}_t^\alpha = e^{\alpha t}\mathscr{T}_t, \quad t \ge 0, \tag{10}$$

where $\mathscr{T}^0 \equiv \mathscr{T}$ is the Brownian semigroup

$$\mathscr{T}_t\phi(x) = \int \phi(y)e^{-\|y-x\|^2/2t}(2\pi t)^{-d/2}\,dy, \quad t \ge 0.$$

$\mathscr{T}^\alpha$ has infinitesimal generator

$$\mathscr{A}^\alpha = \tfrac{1}{2}\Delta + \alpha, \quad t \ge 0, \tag{11}$$

and

$$\mathscr{T}_t^\alpha\phi - \phi = \int_0^t \mathscr{T}_s^\alpha\mathscr{A}^\alpha\phi\,ds = \int_0^t \mathscr{A}^\alpha\mathscr{T}_s^\alpha\phi\,ds. \tag{12}$$

The following lemma contains basic calculations.

LEMMA. *For $\phi, \psi \in C_b^2(R^d) \cup \mathscr{S}(R^d)$,*

$$\int E\langle N_t^x, \phi\rangle\,dx = e^{\alpha t}\int \phi(x)\,dx, \quad t \ge 0, \tag{13}$$

and

$$\int E\langle N_s^x, \phi\rangle\langle N_t^x, \psi\rangle\,dx = C(s, \phi; t, \psi), \tag{14}$$

where

$$\begin{aligned} C(s, \phi; t, \psi) &= e^{\alpha t}\left\{\int \phi(x)\mathscr{T}_{t-s}\psi(x)\,dx + m_2V\int\int_0^s e^{\alpha r}\mathscr{T}_{2r}\phi(x)\,dr\mathscr{T}_{t-s}\psi(x)\,dx\right\} \\ &= e^{\alpha s}\left\{\int \phi(x)\mathscr{T}_{t-s}^\alpha\psi(x)\,dx \right. \\ &\qquad \left. + m_2V\int\int_0^s e^{-\alpha r}\mathscr{T}_{2r}^\alpha\phi(x)\,dr\mathscr{T}_{t-s}^\alpha\psi(x)\,dx\right\}, \quad 0 \le s \le t, \end{aligned} \tag{15}$$

if these expressions are finite.

PROOF. By a renewal argument,

$$\begin{aligned} E\langle N_t^x, \phi\rangle &= e^{-Vt}\int \phi(y)e^{-\|y-x\|^2/2t}(2\pi t)^{-d/2}\,dy \\ &\quad + \int_0^t Ve^{-Vs}\int e^{-\|y-x\|^2/2s}(2\pi s)^{-d/2}\textstyle\sum_n p_n E\sum_{i=1}^n\langle N_{t-s}^{y,i}, \phi\rangle\,dy\,ds, \end{aligned}$$

where $N_{t-s}^{y,i}(\cdot)$ is the number of particles in $\cdot$ coming from the ith offspring of a particle reproducing at position y, at time $t - s$ after its birth, and since

$$E\textstyle\sum_{i=1}^n\langle N_{t-s}^{y,i}, \phi\rangle = nE\langle N_{t-s}^y, \phi\rangle,$$

we have

$$\begin{aligned} E\langle N_t^x, \phi\rangle &= e^{-Vt}\int \phi(y)e^{-\|y-x\|^2/2t}(2\pi t)^{-d/2}\,dy \\ &\quad + m_1\int_0^t Ve^{-Vs}\int E\langle N_{t-s}^y, \phi\rangle e^{-\|y-x\|^2/2s}(2\pi s)^{-d/2}\,dy\,ds. \end{aligned} \tag{16}$$

Define

(17) $$F(s,t)=\iint E\langle N^x_{t-s},\phi\rangle\psi(y)e^{-\|y-x\|^2/2s}(2\pi s)^{-d/2}\,dx\,dy,\quad 0\le s\le t.$$

Multiplying (16) by $\psi(x)$ and integrating we see that $F(s,t)$ satisfies

$$F(0,t)=e^{-Vt}F(t,t)+m_1\int_0^t F(s,t)\,Ve^{-Vs}\,ds,$$

and one can verify that the solution of this equation is

(18) $$F(s,t)=e^{\alpha(t-s)}\iint \phi(x)\psi(y)e^{-\|y-x\|^2/2t}(2\pi t)^{-d/2}\,dx\,dy;$$

in particular, for $s=0$ and $\psi\equiv 1$ we have from (17) and (18)

$$\int E\langle N^x_t,\phi\rangle\,dx=e^{\alpha t}\int\phi(x)\,dx,$$

proving (13) (this proof can be done more directly; the reason for introducing (17) is in the next part).

We will now prove (14) with $s=t$ first. Again by a renewal argument,

(19) $$\begin{aligned}E\langle N^x_t,\phi\rangle\langle N^x_t,\psi\rangle&=e^{-Vt}\int\phi(y)\psi(y)e^{-\|y-x\|^2/2t}(2\pi t)^{-d/2}\,dy\\&\quad+\int_0^t Ve^{-Vs}\int e^{-\|y-x\|^2/2s}(2\pi s)^{-d/2}\textstyle\sum_n p_n\\&\quad\cdot E\textstyle\sum_{i=1}^n\langle N^{y,i}_{t-s},\phi\rangle\sum_{i=1}^n\langle N^{y,i}_{t-s},\psi\rangle\,dy\,ds,\end{aligned}$$

where $N^{y,i}_{t-s}$ is as above, and using independence,

$$\begin{aligned}&E\textstyle\sum_{i=1}^n\langle N^{y,i}_{t-s},\phi\rangle\sum_{i=1}^n\langle N^{y,i}_{t-s},\psi\rangle\\&\qquad=nE\langle N^y_{t-s},\phi\rangle\langle N^y_{t-s},\psi\rangle+n(n-1)E\langle N^y_{t-s},\phi\rangle E\langle N^y_{t-s},\psi\rangle,\end{aligned}$$

hence, integrating (19),

(20) $$\begin{aligned}\int E\langle N^x_t,\phi\rangle\langle N^x_t,\psi\rangle\,dx&=e^{-Vt}\int\phi(x)\psi(x)\,dx\\&\quad+m_1\int_0^t Ve^{-Vs}\int E\langle N^x_{t-s},\phi\rangle\langle N^x_{t-s},\psi\rangle\,dx\,ds\\&\quad+m_2\int_0^t Ve^{-Vs}\int E\langle N^x_{t-s},\phi\rangle E\langle N^x_{t-s},\psi\rangle\,dx\,ds.\end{aligned}$$

In order to obtain a functional equation for $\int E\langle N^x_t,\phi\rangle\langle N^x_t,\psi\rangle\,dx$ we must first compute the last term of (20). From (16) we have

(21) $$\begin{aligned}\int E\langle N^x_t,\phi\rangle E\langle N^x_t,\psi\rangle\,dx&=e^{-2Vt}\iint\phi(y)\psi(z)e^{-\|y-z\|^2/4t}(4\pi t)^{-d/2}\,dy\,dz\\&\quad+m_1e^{-Vt}\int_0^t Ve^{-Vs}\iint(\phi(y)E\langle N^z_{t-s},\psi\rangle\\&\quad+\psi(y)E\langle N^z_{t-s},\phi\rangle)e^{-\|y-z\|^2/2(t+s)}(2\pi(t+s))^{-d/2}\,dy\,dz\,ds\\&\quad+m_1^2\int_0^t\int_0^t Ve^{-Vs}Ve^{-Vr}\iint E\langle N^y_{t-s},\phi\rangle E\langle N^z_{t-r},\psi\rangle\\&\quad\cdot e^{-\|y-z\|^2/2(s+r)}(2\pi(s+r))^{-d/2}\,dy\,dz\,ds\,dr.\end{aligned}$$

Using (17) and (18) in the second term of the right in (21) and performing integrations in s, we obtain

$$\int E\langle N_t^x,\phi\rangle E\langle N_t^x,\psi\rangle\,dx = e^{-2Vt}(2e^{Vm_1t}-1)\iint \phi(y)\psi(z)e^{-\|y-z\|^2/4t}(4\pi t)^{-d/2}\,dy\,dz$$

$$(22)\qquad + m_1^2\int_0^t\int_0^t Ve^{-Vs}Ve^{-Vr}\iint E\langle N_{t-s}^y,\phi\rangle E\langle N_{t-r}^z,\psi\rangle$$

$$\cdot e^{-\|y-z\|^2/2(r+s)}(2\pi(r+s))^{-d/2}\,dy\,dz\,ds\,dr.$$

Let

$$G(r,s,t)=\iint E\langle N_{t-s}^y,\phi\rangle E\langle N_{t-r}^z,\psi\rangle e^{-\|y-z\|^2/2(r+s)}(2\pi(r+s))^{-d/2}\,dy\,dz,\quad 0\le r,s\le t,$$

so (22) becomes

$$G(0,0,t)=e^{-2Vt}(2e^{Vm_1t}-1)G(t,t,t)+m_1^2\int_0^t\int_0^t Ve^{-Vs}Ve^{-Vr}G(r,s,t)\,dr\,ds,$$

and one can verify that the solution of this equation is

$$G(r,s,t)=e^{\alpha(2t-r-s)}\iint \phi(y)\psi(z)e^{-\|y-z\|^2/4t}(4\pi t)^{-d/2}\,dy\,dz,\quad 0\le r,s\le t;$$

in particular, for $s=r=0$ we have the solution of (22):

$$(23)\qquad \int E\langle N_t^x,\phi\rangle E\langle N_t^x,\psi\rangle\,dx = e^{2\alpha t}\iint \phi(y)\psi(z)e^{-\|y-z\|^2/4t}(4\pi t)^{-d/2}\,dy\,dz,$$

and bringing (23) into (20) we obtain a renewal equation for

$$H(t)\equiv\int E\langle N_t^x,\phi\rangle\langle N_t^x,\psi\rangle\,dx,$$

namely:

$$H(t)=e^{-Vt}\bigg(\int\phi(x)\psi(x)\,dx+m_2\int_0^t Ve^{V(2m_1-1)s}$$

$$\cdot\iint\phi(y)\psi(z)e^{-\|y-z\|^2/4s}(4\pi s)^{-d/2}\,dy\,dz\,ds\bigg)+m_1\int_0^t H(t-s)Ve^{-Vs}\,ds,$$

whose solution is

$$H(t)=e^{\alpha t}\bigg(\int\phi(x)\psi(x)\,dx+m_2V\int_0^t e^{\alpha r}\iint\phi(y)\psi(z)e^{-\|y-x\|^2/4r}(4\pi r)^{-d/2}\,dy\,dz\,dr\bigg)$$

proving (14) with $s=t$.

To prove (14) in general we use the fact that

$$\langle N_t^x,\psi\rangle-\int_0^t\langle N_s^x,\tfrac{1}{2}\Delta\psi+\alpha\psi\rangle\,dx,\quad t\ge 0,\quad \psi\in\mathscr{S}(R^d),$$

is a martingale (as in (8) and (9) with $f(x)=x$); hence for $s\le t$, (see (11)),

$$E[\langle N_t^x,\psi\rangle\,|\,\langle N_r^x,\phi\rangle,r\le s,\phi\in\mathscr{S}(R^d)]$$

$$=\langle N_s^x,\psi\rangle+E\bigg[\int_s^t\langle N_r^x,\mathscr{A}^\alpha\psi\rangle dr\,|\,\langle N_r^x,\phi\rangle,r\le s,\phi\in\mathscr{S}(R^d)\bigg],$$

and therefore

$$(24)\qquad E\langle N_s^x,\phi\rangle\langle N_t^x,\psi\rangle = E\langle N_s^x,\phi\rangle\langle N_s^x,\psi\rangle + \int_s^t E\langle N_s^x,\phi\rangle\langle N_r^x,\mathscr{A}^\alpha\psi\rangle dr.$$

Define (see (10))

$$(25)\qquad \psi_t(x) = \mathscr{T}_t^\alpha\psi(x),\quad t\ge 0;$$

hence $\psi_0=\psi$. We will verify that the solution of (24) is

$$(26)\qquad E\langle N_s^x,\phi\rangle\langle N_t^x,\psi\rangle = E\langle N_s^x,\phi\rangle\langle N_s^x,\psi_{t-s}\rangle,\quad s\le t.$$

Indeed, assuming (26),

$$\begin{aligned}\int_s^t E\langle N_s^x,\phi\rangle\langle N_r^x,\mathscr{A}^\alpha\psi\rangle\,dr &= E\langle N_s^x,\phi\rangle\langle N_s^x,\int_s^t \mathscr{A}^\alpha\psi_{r-s}\,dr\rangle\\ &= E\langle N_s^x,\phi\rangle\langle N_s^x,\int_0^{t-s}\mathscr{A}^\alpha\mathscr{T}_r^\alpha\psi\,dr\rangle\\ &= E\langle N_s^x,\phi\rangle\langle N_s^x,\psi_{t-s}-\psi\rangle\quad\text{(using (12))}\\ &= E\langle N_s^x,\phi\rangle\langle N_t^x,\psi\rangle - E\langle N_s^x,\phi\rangle\langle N_s^x,\psi\rangle,\end{aligned}$$

which gives (24). Then from (14) with $s=t$, (25) and integrating (26) we obtain (14). □

COROLLARY 1.

$$(27)\qquad E\langle N_t^T,\phi\rangle = Te^{\alpha t}\int\phi(x)\,dx,\quad t\ge 0.$$

$$(28)\qquad \operatorname{Cov}(\langle N_s^T,\phi\rangle,\langle N_t^T,\psi\rangle) = TC(s,\phi;t,\psi),\quad 0\le s\le t,$$

with $C(s,\phi;t,\psi)$ *given by* (15).

PROOF. (6), (7), (13) and (14). □

COROLLARY 2. $\tilde N^T$ *defined by* (3) *is a* $\mathscr{S}'(R^d\times R^+)$*-valued random field.*

PROOF. To see that (5) is indeed the characteristic functional of an $\mathscr{S}'(R^d\times R^+)$-random field, by the Bochner-Minlos Theorem [9] it suffices to note that (5) is 1 at $\phi=0$, positive-definite, and continuous. The continuity follows from

$$\begin{aligned}&|\log E\exp\{i\langle\tilde N^T,\phi\rangle\} - \log E\exp\{i\langle\tilde N^T,\psi\rangle\}|\\ &\qquad\le T\int E\langle\tilde N^x,|\phi-\psi|\rangle\,dx\\ &\qquad\le T\int_0^\infty\int E\langle N_t^x,|\phi(\cdot,t)-\psi(\cdot,t)|\rangle\,dx\,dt\quad\text{(by (1), (2), (3))}\\ &\qquad= T\int_0^\infty\int e^{\alpha t}|\phi(x,t)-\psi(x,t)|\,dx\,dt\qquad\text{(by (13))}\\ &\qquad\le T\sup_x\sup_{t\ge0}\prod_{j=1}^d(1+|x_j|)^2(1+t)^2e^{\alpha t}|\phi(x,t)\\ &\qquad\qquad-\psi(x,t)|\left(2\int_0^\infty(1+x)^{-2}\,dx\right)^d\int_0^\infty(1+t)^{-2}\,dt\\ &\qquad\le K\|\phi-\psi\|_2,\end{aligned}$$

where K is a constant. □

We can now prove the theorems.

PROOF OF THEOREM 1. By (27) and (28),

$$E\left|T^{-1}\langle N_t^T, \phi\rangle - e^{\alpha t}\int \phi(x)\,dx\right|^2$$

$$= T^{-2}\left\{Te^{\alpha t}\int\left[\phi(x) + m_2 V\int_0^t e^{\alpha r}\mathscr{T}_{2r}\phi(x)\,dr\right]\phi(x)\,dx + T^2 e^{2\alpha t}\left(\int\phi(x)\,dx\right)^2\right\}$$

$$- e^{2\alpha t}\left(\int\phi(x)\,dx\right)^2 \to 0 \text{ as } T\to\infty. \square$$

PROOF OF THEOREM 2. We will show that $M^T \Rightarrow M$ (as $\mathscr{S}'(\mathrm{R}^d)$-valued processes) by proving weak convergence of the finite-dimensional distributions and relative weak compactness. This procedure is simpler in the present situation than the Stroock-Varadhan martingale problem method because of technical complications with the martingales in the non-critical cases ($\alpha \neq 0$). Some special martingales will be used however to establish relative weak compactness.

By (4), (13) and (14),

$$E\exp\{i\textstyle\sum_{j=1}^n u_j\langle M_{t_j}^T, \phi_j\rangle\} = \exp\{-\tfrac{1}{2}\sum_{j,k} u_j u_k C(t_j, \phi_j; t_k, \phi_k)\}$$

$$\cdot\exp\left\{\int T\left[E\exp\{i\textstyle\sum_j u_j T^{-1/2}\langle N_{t_j}^x, \phi_j\rangle\} - 1 - iT^{-1/2}\sum_j u_j E\langle N_{t_j}^x, \phi_j\rangle\right.\right.$$

$$\left.\left. + \tfrac{1}{2}T^{-1}\textstyle\sum_{j,k} u_j u_k E\langle N_{t_j}^x, \phi_j\rangle\langle N_{t_k}^x, \phi_k\rangle\right] dx\right\},$$

where the integrand converges to 0 as $T\to\infty$ and is bounded by $K\sum_j u_j^2 E\langle N_{t_j}^x, \phi_j\rangle^2$, with K a constant (see [3]). Hence by the dominated convergence theorem,

$$\lim_{T\to\infty} E\exp\{i\textstyle\sum_j u_j\langle M_{t_j}^T, \phi_j\rangle\} = \exp\{-\tfrac{1}{2}\sum_{j,k} u_j u_k C(t_j, \phi_j; t_k, \phi_k)\},$$

and therefore by Lévy's continuity theorem $\langle M_{t_1}^T, \phi_1\rangle, \cdots, \langle M_{t_n}^T, \phi_n\rangle$ jointly converge weakly to corresponding centered Gaussian random variables with covariances $C(t_j, \phi_j; t_k, \phi_k)$. Thus we have the weak convergence of the finite-dimensional distributions as $T\to\infty$.

We will prove relative weak compactness by means of a general theorem of Holley and Stroock (Theorem (1.2) in [13]). Our calculations are similar to those in [12], with some different technicalities due to the non-constant centering ($e^{\alpha t}$) in our case.

We have the right-continuous martingale ((8) and (9) with $f(x) = x$)

$$\langle N_t^T, \phi\rangle - \int_0^t \langle N_s^T, (\tfrac{1}{2}\Delta + \alpha)\phi\rangle\,ds, \quad t\geq 0, \tag{29}$$

and since $\langle\lambda, \Delta\phi\rangle = 0$, then

$$\langle N_t^T - e^{\alpha t}T\lambda, \phi\rangle - \int_0^t \langle N_s^T - e^{\alpha s}T\lambda, (\tfrac{1}{2}\Delta + \alpha)\phi\rangle\,ds$$

$$= \langle N_t^T, \phi\rangle - \int_0^t \langle N_s^T, (\tfrac{1}{2}\Delta + \alpha)\phi\rangle\,ds - T\int\phi(x)\,dx, \quad t\geq 0,$$

is also a right-continuous martingale, and therefore

(30) $$\langle M_t^T, \phi \rangle - \int_0^t \gamma_{1,\phi}^T(s)\, ds, \quad t \geq 0,$$

is a right-continuous martingale, where

(31) $$\gamma_{1,\phi}^T(s) = \langle M_s^T, (\tfrac{1}{2}\Delta + \alpha)\phi \rangle, \quad s \geq 0.$$

By (28),

$$E\left(\langle N_t^T, \phi \rangle - \int_0^t \langle N_s^T, (\tfrac{1}{2}\Delta + \alpha)\phi \rangle\, ds \right)^2 < \infty \text{ and } E\left(\langle M_t^T, \phi \rangle - \int_0^t \gamma_{1,\phi}^T(s)\, ds \right)^2 < \infty$$

for all t; therefore the martingales (29) and (30) have Doob-Meyer decompositions [17]. The increasing process of (29) is

$$\int_0^t H_f(\langle N_s^T, \phi \rangle)\, ds \quad \text{where} \quad H_f(x) = (\mathscr{L}f^2 - 2f\mathscr{L}f)(x),$$

with $\mathscr{L}$ given by (8) and $f(x) = x$. Hence,

$$\begin{aligned} H_f(\langle N_s^T, \phi \rangle) &= 2\langle N_s^T, \phi \rangle \langle N_s^T, \tfrac{1}{2}\Delta\phi \rangle + \langle N_s^T, |\nabla\phi|^2 \rangle \\ &\quad + 2\alpha \langle N_s^T, \phi \rangle^2 + V(m_2 - m_1 + 1)\langle N_s^T, \phi^2 \rangle - 2\langle N_s^T, \phi \rangle \langle N_s^T, (\tfrac{1}{2}\Delta + \alpha)\phi \rangle \\ &= \langle N_s^T, |\nabla\phi|^2 + V(m_2 - m_1 + 1)\phi^2 \rangle, \end{aligned}$$

so

$$\left(\langle N_t^T, \phi \rangle - \int_0^t \langle N_s^T, (\tfrac{1}{2}\Delta + \alpha)\phi \rangle\, ds \right)^2 - \int_0^t \langle N_s^T, |\nabla\phi|^2 + V(m_2 - m_1 + 1)\phi^2 \rangle\, ds, \quad t \geq 0,$$

is a martingale, and since (29) is a martingale, then

$$\begin{aligned} &\left(\langle N_t^T - e^{\alpha t} T\lambda, \phi \rangle - \int_0^t \langle N_s^T - e^{\alpha s} T\lambda, (\tfrac{1}{2}\Delta + \alpha)\phi \rangle\, ds \right)^2 \\ &\qquad - \int_0^t \langle N_s^T, |\nabla\phi|^2 + V(m_2 - m_1 + 1)\phi^2 \rangle\, ds \\ &\quad = \left(\langle N_t^T, \phi \rangle - \int_0^t \langle N_s^T, (\tfrac{1}{2}\Delta + \alpha)\phi \rangle\, ds \right)^2 - \int_0^t \langle N_s^T, |\nabla\phi|^2 + V(m_2 - m_1 + 1)\phi^2 \rangle\, ds \\ &\qquad + T^2\left(\int \phi(x)\, dx \right)^2 - 2\left(\langle N_t^T, \phi \rangle - \int_0^t \langle N_s^T, (\tfrac{1}{2}\Delta + \alpha)\phi \rangle\, ds \right) T \int \phi(x)\, dx, \quad t \geq 0, \end{aligned}$$

is a martingale, and therefore (see (31))

$$\left(\langle M_t^T, \phi \rangle - \int_0^t \gamma_{1,\phi}^T(s)\, ds \right)^2 - \int_0^t \gamma_{2,\phi}^T(s)\, ds, \quad t \geq 0,$$

is a martingale, where

$$\gamma_{2,\phi}^T(s) = T^{-1}\langle N_s^T, |\nabla\phi|^2 + V(m_2 - m_1 + 1)\phi^2 \rangle, \quad s \geq 0,$$

giving the Doob-Meyer decomposition of (30).

By Theorem (1.2) in [13], $\{M^T\}_{T \geq 1}$ is relatively weakly compact in $D([0, \infty), \mathscr{S}'(R^d))$ if for each $\tau \in (0, \infty)$ there is a constant $c(\tau)$ such that for all $\phi \in \mathscr{S}(R^d)$,

$$\sup_{T\geq 1} E \sup_{0\leq t\leq \tau} \langle M_t^T, \phi\rangle^2 \leq c(\tau)(\| \phi \|_p^2 + \| \Delta\phi \|_p^2) \tag{32}$$

for some $p > 0$ (since our $\mathscr{S}(R^d)$-norms are equivalent to those in [13], this implies condition (1.3) of the theorem), and if for each $\tau \in (0, \infty)$ and $\phi \in \mathscr{S}(R^d)$,

$$\sup_{T\geq 1} E \sup_{0\leq t\leq \tau} (\gamma_{i,\phi}^T(t))^2 < \infty, \quad i = 1, 2. \tag{33}$$

(Notice that $\{\langle M_0^T, \phi\rangle\}_{T\geq 1}$ is tight).

Instead of checking the condition in the theorem which ensures continuity of the limit, we will prove continuity directly later on.

Doob's inequality applied to (30) gives

$$\left[E \sup_{0\leq t\leq \tau}\left(\langle M_t^T, \phi\rangle - \int_0^t \gamma_{1,\phi}^T(s)\, ds\right)^2\right]^{1/2} \leq 2\left[E\left(\langle M_\tau^T, \phi\rangle - \int_0^\tau \gamma_{1,\phi}^T(s)\, ds\right)^2\right]^{1/2}$$

$$\leq 2[E\langle M_\tau^T, \phi\rangle^2]^{1/2} + 2\left[E\left(\int_0^\tau \gamma_{1,\phi}^T(s)\, ds\right)^2\right]^{1/2}$$

$$\leq 2[E\langle M_\tau^T, \phi\rangle^2]^{1/2} + 2\tau^{1/2}\left[\int_0^\tau E(\gamma_{1,\phi}^T(s))^2\, ds\right]^{1/2}.$$

Now, by (27) and (28), and showing that $\int |\phi(x)\mathscr{T}_{2r}\phi(x)|\, dx \leq \|\phi\|_2^2 \int \prod_{j=1}^d (1 + |x_j|)^{-2}\, dx$,

$$E\langle M_t^T, \phi\rangle^2 = T^{-1}\left[E\langle N_t^T, \phi\rangle^2 - 2e^{\alpha t} T E\langle N_t^T, \phi\rangle \int \phi(x)\, dx + e^{2\alpha t}T^2\left(\int \phi(x)\, dx\right)^2\right]$$

$$= e^{\alpha t}\left[\int \phi(x)^2\, dx + m_2 V \int\int_0^t e^{\alpha r}\mathscr{T}_{2r}\phi(x)\, dr\, \phi(x)\, dx\right]$$

$$\leq e^{\alpha t}[1 + m_2 V(e^{\alpha t} - 1)/\alpha]\|\phi\|_2^2 K,$$

where K is a constant; similarly, for (31)

$$E(\gamma_{1,\phi}^T(s))^2 \leq e^{\alpha s}[1 + m_2 V(e^{\alpha s} - 1)/\alpha]\|\tfrac{1}{2}\Delta\phi + \alpha\phi\|_2^2 K,$$

so

$$\left[E \sup_{0\leq t\leq \tau}\left(\langle M_t^T, \phi\rangle - \int_0^t \gamma_{1,\phi}^T(s)\, ds\right)^2\right]^{1/2}$$

$$\leq 2e^{\alpha\tau/2}[1 + m_2 V(e^{\alpha\tau} - 1)/\alpha]^{1/2}\|\phi\|_2 K^{1/2}$$
$$+ 2\tau^{1/2}[(e^{\alpha\tau} - 1)/\alpha + m_2 V((e^{\alpha\tau} - 1)/\alpha)^2/2]^{1/2}(\|\Delta\phi\|_2/2 + |\alpha|\,\|\phi\|_2)K^{1/2}.$$

Therefore,

$$[E \sup_{0\leq t\leq \tau}\langle M_t^T, \phi\rangle^2]^{1/2} \leq 2\left[E \sup_{0\leq t\leq \tau}\left(\langle M_t^T, \phi\rangle - \int_0^t \gamma_{1,\phi}^T(s)\, ds\right)^2\right]^{1/2}$$

$$+ 2\tau^{1/2}\left[\int_0^\tau E(\gamma_{1,\phi}^T(s))^2\, ds\right]^{1/2} \leq 4e^{\alpha\tau/2}[1 + m_2 V(e^{\alpha\tau} - 1)/\alpha]^{1/2}\|\phi\|_2 K^{1/2}$$

$$+ 4\tau^{1/2}[(e^{\alpha\tau} - 1)/\alpha + m_2 V((e^{\alpha\tau} - 1)/\alpha)^2/2]^{1/2}(\|\Delta\phi\|_2/2 + |\alpha|\,\|\phi\|_2)K^{1/2},$$

which yields (32).

It is clear that the same type or arguments prove (33).

We have proved that $M^T \Rightarrow M$, where M is a generalized centered Gaussian process with

$$\mathrm{Cov}(\langle M_s, \phi\rangle, \langle M_t, \psi\rangle) = C(s, \phi; t, \psi)$$

given by (15). Note that this covariance coincides with the expression given in the theorem (use the Chapman-Kolmogorov equation).

The existence of a norm-continuous version of M can be shown by an extension of the Dudley-Fernique theorem due to Mitoma [18]: to prove the norm-continuity of M it suffices to show that $\{\langle M_t, \phi\rangle\}$ has a continuous version for each $\phi \in \mathscr{S}(R^d)$, which can be done by the ordinary Dudley-Fernique theorem.

Let $\tau \in (0, \infty)$. Then for $0 \le s \le t \le \tau$ we have by (27) and (28)

$$E(\langle M_t, \phi\rangle - \langle M_s, \phi\rangle)^2 = E\langle M_t, \phi\rangle^2 + E\langle M_s, \phi\rangle^2 - 2E\langle M_s, \phi\rangle\langle M_t, \phi\rangle$$

$$= \int \phi(x)[\phi(x)(e^{\alpha s} - e^{\alpha t}) - 2e^{\alpha t}(\mathscr{T}_{t-s}\phi(x) - \phi(x))]\, dx$$

$$+ m_2 V \int \phi(x)\left[(e^{\alpha s} - e^{\alpha t}) \int_0^s e^{\alpha r}\mathscr{T}_{2r}\phi(x)\, dr + e^{\alpha t}\int_s^t e^{\alpha r}\mathscr{T}_{2r}\phi(x)\, dr\right] dx.$$

Using (12) with $\alpha = 0$, and the facts that $\mathscr{T}_t$ is contractive and $e^{\alpha s}$ is Lipschitz on $[0, \tau]$ we see that there is a constant K such that

$$E(\langle M_t, \phi\rangle - \langle M_s, \phi\rangle)^2 \le K(t - s), \quad 0 \le s \le t \le \tau,$$

whence follows by Theorem 7-1 in [7] that $\{\langle M_t, \phi\rangle, 0 \le t \le \tau\}$ has a continuous version. But τ is arbitrary.

The Markov property of M follows from (14) upon integrating (26) in the proof of the lemma, because for a generalized Gaussian process M the Markov property is equivalent to: given $t_0 < t$ and $\phi \in \mathscr{S}(R^d)$ there is a $\hat{\phi} \in \mathscr{S}(R^d)$ such that

$$E(\langle M_t, \phi\rangle - \langle M_{t_0}, \hat{\phi}\rangle)\langle M_s, \psi\rangle = 0 \quad \text{for all} \quad s \le t_0 \text{ and } \psi \in \mathscr{S}(R^d);$$

in fact we have $\hat{\phi} = \mathscr{T}^{\alpha}_{t-t_0}\phi$ in this case.

In order to obtain the Langevin equation we regard M^T and M as generalized space-time random fields. Let

$$\langle \tilde{M}^T, \phi\rangle = T^{-1/2}\left(\langle \tilde{N}^T, \phi\rangle - T\int\int_0^\infty e^{\alpha t}\phi(x, t)\, dt\, dx\right), \quad \phi \in \mathscr{S}(R^d \times R^+),$$

with $\tilde{N}^T$ defined by (2) and (3). (Note that $E\langle \tilde{N}^T, \phi\rangle = T\int\int_0^\infty e^{\alpha t}\phi(x, t)\, dt\, dx$).

Since $\tilde{N}^T$ is an $\mathscr{S}'(R^d \times R^+)$-random field with characteristic functional given by (5), then $\tilde{M}^T$ is an $\mathscr{S}'(R^d \times R^+)$-random field with characteristic functional (similarly as above)

$$E\exp\{i\langle \tilde{M}^T, \phi\rangle\} = \exp\{-\tfrac{1}{2}\tilde{C}(\phi, \phi)\}\exp\left\{\int T\left[E\exp\{iT^{-1/2}\langle \tilde{N}^x, \phi\rangle\}\right.\right.$$

$$\left.\left.- 1 - iT^{-1/2}E\langle \tilde{N}^x, \phi\rangle + \tfrac{1}{2}T^{-1}E\langle \tilde{N}^x, \phi\rangle^2\right] dx\right\},$$

where

$$\tilde{C}(\phi, \psi) = \int_0^\infty\int_0^\infty C(s, \phi(\cdot, s); t, \psi(\cdot, t))\, ds\, dt, \quad \phi, \psi \in \mathscr{S}(R^d \times R^+), \tag{34}$$

with $C(s, \phi; t, \psi)$ given by (15). The integrand in the exponential converges to 0 as $T \to \infty$ and is dominated by an integrable function (as before); therefore

$$\lim_{T\to\infty} E\exp\{i\langle \tilde{M}^T, \phi\rangle\} = \exp\{-\tfrac{1}{2}\tilde{C}(\phi, \phi)\}, \quad \phi \in \mathscr{S}(R^d \times R^+),$$

and then by Lévy's continuity theorem for nuclear spaces [16] we may conclude that $\tilde{M}^T \Rightarrow \tilde{M}$ as $T \to \infty$, where $\tilde{M}$ is a centered Gaussian field on $\mathscr{S}'(R^d \times R^+)$ with covariance functional $\tilde{C}(\phi, \psi)$, provided that such a random field exists. To show that $\tilde{M}$ exists we must have that $\tilde{C}(\phi, \psi)$ is bilinear, positive-definite and continuous (see [9]). Bilinearity and positive-definiteness follow from

$$\mathrm{Cov}(\langle \tilde{N}^T, \phi\rangle, \langle \tilde{N}^T, \psi\rangle) = T\tilde{C}(\phi, \psi),$$

and it is not hard to show that

$$|\tilde{C}(\phi, \psi)| \le K\|\phi\|_2\|\psi\|_2,$$

where K is a constant, which implies the continuity. We have explicitly

$$\begin{aligned}\mathrm{Cov}(\langle \tilde{M}, \phi\rangle, \langle \tilde{M}, \psi\rangle) = \int_0^\infty dt \int_0^t ds\, e^{\alpha t}\Big\{\int \Big[\phi(x, s)\mathscr{T}_{t-s}\psi(x, t) + \psi(x, s)\mathscr{T}_{t-s}\phi(x, t)\\ + m_2 V \int_0^s e^{\alpha r}(\mathscr{T}_{2r}\phi(x, s)\mathscr{T}_{t-s}\psi(x, t)\\ + \mathscr{T}_{2r}\psi(x, s)\mathscr{T}_{t-s}\phi(x, t))\, dr\Big]\, dx\Big\}, \quad \phi, \psi \in \mathscr{S}(R^d \times R^+).\end{aligned} \tag{35}$$

REMARK. The process $M \equiv \{M_t, t \ge 0\}$ can also be constructed from $\tilde{M}$ by a method like Martin-Löf's [15] (page 214).

Now we prove the Langevin equation. We define a noise $\mathscr{W}$ so that the equation is satisfied in the space-time sense, i.e.

$$\langle \mathscr{W}, \phi\rangle = -\langle \tilde{M}, \partial\phi/\partial t + \tfrac{1}{2}\Delta\phi + \alpha\phi\rangle, \quad \phi \in \mathscr{S}(R^d \times R^+); \tag{36}$$

hence $\mathscr{W}$ is a generalized space-time centered Gaussian field, and we must show that it has the right covariance. It suffices to compute $E\langle \mathscr{W}, \phi\rangle^2$. Using (10), (11) and (35) we have (notation: $\phi_t \equiv \partial\phi/\partial t$)

$$\begin{aligned}E\langle \mathscr{W}, \phi\rangle^2 = 2\int_0^\infty dt \int_0^t ds\, e^{\alpha s}\Big\{\int \Big[(\phi_t + \mathscr{A}^\alpha\phi)(x, s)\mathscr{T}^\alpha_{t-s}(\phi_t + \mathscr{A}^\alpha\phi)(x,t)\\ + m_2 V \int_0^s e^{-\alpha r}\mathscr{T}_{2r}(\phi_t + \mathscr{A}^\alpha\phi)(x, s)\, dr\, \mathscr{T}^\alpha_{t-s}(\phi_t + \mathscr{A}^\alpha\phi)(x, t)\Big]\, dx\Big\}.\end{aligned} \tag{37}$$

We will first show that for $\Phi \in \mathscr{S}(R^+)$,

$$\int_0^\infty dt \int_0^t ds\, e^{\alpha s}\Phi(s)\mathscr{T}^\alpha_{t-s}(\phi_t + \mathscr{A}^\alpha\phi)(x, t) = -\int_0^\infty e^{\alpha t}\Phi(t)\phi(x, t)\, dt. \tag{38}$$

For the moment we write $\phi(t) \equiv \phi(x, t)$.

$$\begin{aligned}\int_0^\infty dt \int_0^t ds\, e^{\alpha s}\Phi(s)\mathscr{T}^\alpha_{t-s}\mathscr{A}^\alpha\phi(t) &= -\int_0^\infty dt \int_0^t ds \int_t^\infty du\, e^{\alpha s}\Phi(s)\mathscr{T}^\alpha_{t-s}\mathscr{A}^\alpha\phi_t(u)\\ &= -\int_0^\infty du \int_0^u ds\, e^{\alpha s}\Phi(s) \int_s^u dt\, \mathscr{T}^\alpha_{t-s}\mathscr{A}^\alpha\phi_t(u)\\ &= -\int_0^\infty du \int_0^u ds\, e^{\alpha s}\Phi(s)(\mathscr{T}^\alpha_{u-s}\phi_t - \phi_t)(u) \quad \text{(by (12))},\end{aligned}$$

and integrating by parts,

$$\int_0^\infty du \int_0^u ds\, e^{\alpha s}\Phi(s)\phi_t(u) = -\int_0^\infty du\, \phi(u)e^{\alpha u}\Phi(u).$$

Combining these results we obtain (38).

For computing (37) we apply (38) with

$$\Phi_1(s) = (\phi_t + \mathscr{A}^\alpha\phi)(x, s) \quad \text{and} \quad \Phi_2(s) = \int_0^s e^{-\alpha r}\mathscr{T}_{2r}^\alpha(\phi_t + \mathscr{A}^\alpha\phi)(x, s)\, dr.$$

Computation with Φ_1:

$$S(\Phi_1) \equiv \int_0^\infty dt \int_0^t ds\, e^{\alpha s}\Phi_1(s)\mathscr{T}_{t-s}^\alpha(\phi_t + \mathscr{A}^\alpha\phi)(x, t)$$

$$= -\int_0^\infty e^{\alpha t}\phi(x, t)(\phi_t + \tfrac{1}{2}\Delta\phi + \alpha\phi)(x, t)\, dt$$

(integrating by parts the first term)

$$= -\int_0^\infty e^{\alpha t}\phi(x, t)\frac{1}{2}\Delta\phi(x, t)\, dt - \frac{\alpha}{2}\int_0^\infty e^{\alpha t}\phi(x, t)^2 dt + \frac{1}{2}\phi(x, 0)^2.$$

Computation with Φ_2:

$$S(\Phi_2) \equiv \int_0^\infty dt \int_0^t ds\, e^{\alpha s}\Phi_2(s)\mathscr{T}_{t-s}^\alpha(\phi_t + \mathscr{A}^\alpha\phi)(x, t)$$

$$= -\int_0^\infty e^{\alpha t}\phi(x, t)\int_0^t e^{-\alpha r}\mathscr{T}_{2r}^\alpha(\phi_t + \mathscr{A}^\alpha\phi)(x, t)\, dr\, dt$$

$$= -\frac{1}{2}\int_0^\infty e^{\alpha t}\phi(x, t)\int_0^{2t} e^{-\alpha s/2}\mathscr{T}_s^\alpha(\phi_t + \mathscr{A}^\alpha\phi)(x, t)\, ds\, dt$$

(integrating the inner integral by parts and using (12), and then performing some more integrations)

$$= \frac{1}{2}\int_0^\infty e^{\alpha t}\phi(x, t)^2\, dt - \frac{1}{2}\int_0^\infty \phi(x, t)\mathscr{T}_{2t}^\alpha\phi(x, t)\, dt$$

$$-\frac{\alpha}{2}\int_0^\infty e^{\alpha t}\phi(x, t)\int_0^t e^{-\alpha s}\mathscr{T}_{2s}^\alpha\phi(x, t)\, ds\, dt - \int_0^\infty e^{\alpha t}\phi(x, t)\int_0^t e^{-\alpha s}\mathscr{T}_{2s}^\alpha\phi_t(x, t)\, ds\, dt.$$

Taking $\phi \in \mathscr{S}(R^d \times R^+)$ of the form

(39) $\quad \phi(x, t) = \phi(x)f(t), \phi \in \mathscr{S}(R^d) \quad \text{and} \quad f \in \mathscr{S}(R^+) \quad \text{such that} \quad \int_0^\infty e^{\alpha t}f(t)^2\, dt = 1$

we get

$$S(\Phi_1) = -\left(\phi(x)\frac{1}{2}\Delta\phi(x) + \frac{\alpha}{2}\phi(x)^2\right) + \frac{1}{2}\phi(x)^2 f(0)^2,$$

and integrating by parts the last integral in $S(\Phi_2)$,

$$S(\Phi_2) = \frac{1}{2}\phi(x)^2.$$

Collecting results in (37) we find

(40) $\quad E\langle \mathscr{W}, \phi f\rangle^2 = -2\int \phi(x)\frac{1}{2}\Delta\phi(x)\, dx + (m_2 V - \alpha)\int \phi(x)^2\, dx + f(0)^2\int \phi(x)^2\, dx.$

Now, from the relation (34) between the covariances of the $\mathscr{S}'(R^d)$-valued Gaussian process $\{M_t, t \geq 0\}$ and the $\mathscr{S}'(R^d \times R^+)$-valued Gaussian field $\tilde{M}$ it follows that the two are related by

$$(41) \qquad \langle \tilde{M}_t, \phi \rangle = \int_0^\infty \langle M_t, \phi(\cdot, t) \rangle \, dt, \quad \phi \in \mathscr{S}(R^d \times R^+).$$

The same as (41), the $\mathscr{S}'(R^d \times R^+)$-valued Gaussian field $\tilde{\mathscr{W}}$ defined by (36) and the $\mathscr{S}'(R^d)$-valued Gaussian noise $\{\mathscr{W}_t, t \geq 0\}$ described in the theorem should also be related by

$$(42) \qquad \langle \tilde{\mathscr{W}}, \phi \rangle = \int_0^\infty \langle \mathscr{W}_t, \phi(\cdot, t) \rangle \, dt, \quad \phi \in \mathscr{S}(R^d \times R^+),$$

at least for ϕ of the form (39). To prove (42) note that the covariance of $\{\mathscr{W}_t, t \geq 0\}$ including the initial condition $M_0 = W$ of the Langevin equation is given by

$$(43) \qquad \begin{aligned} \operatorname{Cov}(\langle \mathscr{W}_s, \phi \rangle, \langle \mathscr{W}_t, \phi \rangle) &= \delta(s-t)e^{\alpha t}\left\{ -\int \phi(x)\Delta\phi(x)\, dx + (m_2 V - \alpha) \int \phi(x)^2 dx \right\} \\ &\quad + \delta_2(s, t) \int \phi(x)^2 dx, \quad \phi \in \mathscr{S}(R^d), \end{aligned}$$

where δ_2 is the delta function on R^2 centered at the origin. It is then easy to see from (40) and (43) that for ϕ of the form (39),

$$E\langle \tilde{\mathscr{W}}, \phi f \rangle^2 = \int_0^\infty \int_0^\infty f(s)f(t) \operatorname{Cov}(\langle \mathscr{W}_s, \phi \rangle, \langle \mathscr{W}_t, \phi \rangle)\, ds\, dt,$$

which implies (42).

Proof of Theorem 3. $M^{I,T} \Rightarrow M^I$ and $M^{II,T} \Rightarrow M^{II}$ can be proved the same way as $M^T \Rightarrow M$ in Theorem 2, as well as the Markov property and the norm-continuity of M^I and M^{II}. The covariance and the Langevin equation of M^I are known [15]. The Langevin equation for M^{II} is the difference of those for M and M^I. We only have left to obtain the covariances of the limit processes.

Let $B^x \equiv \{B_t^x, t \geq 0\}$ denote Brownian motion in R^d starting from $x \in R^d$. Consistently with our previous notation we view B^x as an $\mathscr{S}'(R^d)$-process $\hat{N}^x$ defined by $\langle \hat{N}_t^x, \phi \rangle = \phi(B_t^x)$, $\phi \in \mathscr{S}(R^d)$. Thus $N^{I,T} = \sum_i \hat{N}^{x_i}$, where $\{x_i\}_{i=1}^\infty$ are the points of the initial Poisson process with intensity measure Tdx.

Just as in Theorem 2 we find

$$(44) \qquad \begin{aligned} \operatorname{Cov}(\langle M_s^{II}, \phi \rangle, \langle M_t^{II}, \psi \rangle) &= \int E\langle N_s^x - \hat{N}_s^x, \phi \rangle \langle N_t^x - \hat{N}_t^x, \psi \rangle \, dx \\ &= \int E\langle N_s^x, \phi \rangle \langle N_t^x, \psi \rangle \, dx + \int E\langle \hat{N}_s^x, \phi \rangle \langle \hat{N}_t^x, \psi \rangle \, dx \\ &\quad - \int E\langle N_s^x, \phi \rangle \langle \hat{N}_t^x, \psi \rangle \, dx - \int E\langle \hat{N}_s^x, \phi \rangle \langle N_t^x, \psi \rangle \, dx \\ &= C(s, \phi; t, \psi) + \iint \phi(x)\psi(y) e^{-\|x-y\|^2/2(t-s)} (2\pi(t-s))^{-d/2} dx\, dy \\ &\quad - \int E\langle N_s^x, \phi \rangle \langle \hat{N}_t^x, \psi \rangle \, dx - \int E\langle \hat{N}_s^x, \phi \rangle \langle N_t^x, \psi \rangle \, dx, \quad s \leq t, \end{aligned}$$

where $C(s, \phi; t, \psi)$ is given by (15) and the second term is the known covariance of M^I [15].

Similarly,

$$\tag{45} \begin{aligned} \mathrm{Cov}(\langle M_s^I, \phi\rangle, \langle M_t^{II}, \psi\rangle) &= \int E\langle \hat{N}_s^x, \phi\rangle\langle N_t^x - \hat{N}_t^x, \psi\rangle\, dx \\ &= \int E\langle \hat{N}_s^x, \phi\rangle\langle N_t^x, \psi\rangle\, dx - \int\int \phi(x)\psi(y)e^{-\|x-y\|^2/2(t-s)}(2\pi(t-s))^{-d/2}dx\, dy, \quad s \le t. \end{aligned}$$

Hence we need only compute $\int E\langle \hat{N}_s^x, \phi\rangle\langle N_t^x, \psi\rangle\, dx$. Since this is similar to the proof of the lemma we will only give a sketch.

$$H(\phi, \psi; t) \equiv H(t) \equiv \int E\langle \hat{N}_t^x, \phi\rangle\langle N_t^x, \psi\rangle\, dx$$

satisfies the renewal equation

$$\begin{aligned} H(t) &= e^{-Vt}\int \phi(x)\psi(x)\, dx \\ &\quad + \alpha e^{\alpha t}\int \phi(x)\int_0^t e^{-Vm_1 s}\mathscr{T}_{2(t-s)}\psi(x)\, ds\, dx + \int_0^t H(t-s)Ve^{-Vs}ds, \end{aligned}$$

whence

$$\tag{46} H(\phi, \psi; t) = \int \phi(x)\psi(x)\, dx + \alpha\int \phi(x)\int_0^t e^{\alpha r}\mathscr{T}_{2r}\psi(x)\, dr\, dx.$$

Using the martingale (8)–(9) one shows that

$$E\langle \hat{N}_s^x, \phi\rangle\langle N_t^x, \psi\rangle = \begin{cases} E\langle \hat{N}_t^x, \mathscr{T}_{s-t}\phi\rangle\langle N_t^x, \psi\rangle, & t \le s, \\ E\langle \hat{N}_s^x, \phi\rangle\langle N_s^x, e^{\alpha(t-s)}\mathscr{T}_{t-s}\psi\rangle, & s \le t, \end{cases}$$

hence

$$\tag{47} \int E\langle \hat{N}_s^x, \phi\rangle\langle N_t^x, \psi\rangle\, dx = \begin{cases} H(\mathscr{T}_{s-t}\phi, \psi; t), & t \le s, \\ H(\phi, e^{\alpha(t-s)}\mathscr{T}_{t-s}\psi; s), & s \le t. \end{cases}$$

Substituting (46) and (47) into (44) and (45) we obtain $\mathrm{Cov}(\langle M_s^{II}, \phi\rangle, \langle M_t^{II}, \psi\rangle)$, and $\mathrm{Cov}(\langle M_s^I, \phi\rangle, \langle M_t^{II}, \psi\rangle)$ as given in the theorem. □

Remark 4 follows from

$$\begin{aligned} &\lim_{t\to\infty}\mathrm{Cov}(\langle M_t, \phi\rangle, \langle M_t, \psi\rangle) \\ &\quad = \int \phi(x)\psi(x)\, dx + m_2 V\int\int \phi(x)\psi(y)\int_0^\infty e^{-\|x-y\|^2/4r}(4\pi r)^{-d/2}dr\, dx\, dy \\ &\quad = \int \phi(x)\psi(x)\, dx + m_2 V\Gamma(d/2-1)(4\pi)^{-1}\int\int \phi(x)\psi(y)\|x-y\|^{-d+2}dx\, dy \end{aligned}$$

when $d \ge 3$.

Remark 5 can be verified by inspection of the covariances.

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