The Annals of Probability
1983, Vol. 11, No. 2, 277–301

POINTWISE TRANSLATION OF THE RADON TRANSFORM AND THE GENERAL CENTRAL LIMIT PROBLEM

BY MARJORIE G. HAHN,[1] PETER HAHN, AND MICHAEL J. KLASS[2]

Tufts University, Massachusetts General Hospital and University of California at Berkeley

We identify a representation problem involving the Radon transforms of signed measures on $\mathbb{R}^d$ of finite total variation. Specifically, if μ is a pointwise translate of ν (i.e., if for all $\theta \in S^{d-1}$ the projection μ_θ is a translate of ν_θ), must μ be a vector translate of ν? We obtain results in several important special cases. Relating this to limit theorems, let $X_{n1}, \cdots, X_{nk_n}$ be a u.a.n. triangular array on $\mathbb{R}^d$ and put $S_n = X_{n1} + \cdots + X_{nk_n}$. There exist vectors $v_n \in \mathbb{R}^d$ such that $\mathscr{L}(S_n - v_n) \to \gamma$ iff (I) a tail probability condition, (II) a truncated variance condition, and (III) a centering condition hold. We find that condition (III) is superfluous in that (I) and (II) always imply (III) iff the limit law γ has the property that the only infinitely divisible laws which are pointwise translates of γ are vector translates. Not all infinitely divisible laws have this property. We characterize those which do. A physical interpretation of the pointwise translation problem in terms of the parallel beam x-ray transform is also discussed.

1. Introduction. Let $\{X_{n1}, \cdots, X_{nk_n}\}$ be a u.a.n. triangular array of independent d-dimensional random vectors with partial sums S_n. When $d = 1$, the classical central limit theorem is of the form: given an infinitely divisible law γ, there exist $v_n \in \mathbb{R}$ such that $\mathscr{L}(S_n - v_n) \to \gamma$ iff (I) a tail condition and (II) a truncated variance condition hold. By contrast, when $d \geq 2$, the form of the classical central limit theorem is: given an infinitely divisible law γ, there exist $v_n \in \mathbb{R}^d$ such that $\mathscr{L}(S_n - v_n) \to \gamma$ iff (I), (II) and (III) a centering condition hold. Sometimes condition (III) holds automatically in the presence of (I) and (II). This is certainly true when $d = 1$; but it also holds when γ is multivariate normal or even stable of index $\alpha \neq 1$. However, condition (III) is required for some stables of index 1. We investigate the general problem: when does every triangular array satisfying (I) and (II) automatically satisfy (III)?

Theorem 2.8 shows that conditions (I) and (II) are equivalent to tightness of $\{\mathscr{L}(S_n - v_n)\}$ for some $v_n \in \mathbb{R}^d$ together with the property that for any weak subsequential limit η and any direction θ, $\hat{\eta}(t\theta) = \hat{\gamma}(t\theta)e^{itc(\theta)}$ for some constant $c(\theta)$. If $c(\theta) = \langle b, \theta \rangle$ for some $b \in \mathbb{R}^d$, then letting $\tilde{v}_n = v_n + b$, $\mathscr{L}(S_n - \tilde{v}_n) \to \gamma$. Consequently, for (III) to be superfluous, the only infinitely divisible laws η whose 1-dimensional projections on lines through the origin are translates of those of γ must be of the form $\eta = \gamma * \delta_b$, and conversely (Theorem 2.9). (Here δ_b denotes a unit point mass at b.)

The determination of which γ have this property is a special case of a more general problem. A description requires some notation. Let $M(\mathbb{R}^d)$ be the algebra of signed measures on $\mathbb{R}^d$ which have finite total variation. $\mathscr{B}(\mathbb{R}^d)$ will denote the Borel subsets of $\mathbb{R}^d$. If $\lambda_1, \lambda_2 \in M(\mathbb{R}^d)$ then the convolution of λ_1 and λ_2 is defined by $\lambda_1 * \lambda_2(E) =$

Received July 1980; revised July 1982.

[1] Supported in part by NSF Grant MCS-81-01895. This paper was completed while the author was visiting the Department of Statistics, University of California, Berkeley, during the 1981–1982 academic year.

[2] Supported in part by NSF Grant MCS-80-04022.

AMS 1980 *subject classifications*. Primary, 60E10, 44A15; secondary, 60F05, 92A05.

Key words and phrases. Radon transform, signed measures, stable laws, general multivariate central limit theorem, infinitely divisible laws, computerized tomography, radiology, pointwise translation problem.

$\int \lambda_1(E - x)\lambda_2(dx)$ for any $E \in \mathscr{B}(\mathbb{R}^d)$. Let S^{d-1} be the unit sphere in $\mathbb{R}^d$, $S^{d-1} \equiv \{x \in \mathbb{R}^d: \|x\| = 1\}$.

1.1 DEFINITION. If $\theta \in S^{d-1}$ and $\lambda \in M(\mathbb{R}^d)$ then the 1-dimensional projection λ_θ of λ on the line $\{t\theta: t \in \mathbb{R}\}$ is defined by

$$\lambda_\theta(B) = \lambda(x \in \mathbb{R}^d: \langle x, \theta\rangle \in B)$$

for every $B \in \mathscr{B}(\mathbb{R})$. The mapping $\theta \to \lambda_\theta: S^{d-1} \to M(\mathbb{R})$ is called the *Radon Transform* of λ.

1.2 DEFINITION. Let $\lambda, \sigma \in M(\mathbb{R}^d)$. σ is called a *pointwise translate* of λ if there exists a function $c: S^{d-1} \to \mathbb{R}$ such that the Radon transforms of λ and σ agree up to translation in each direction by $c(\theta)$, i.e., for every $\theta \in S^{d-1}$,

$$\sigma_\theta = \lambda_\theta * \delta_{c(\theta)}.$$

The collection of all pointwise translates of λ will be denoted by $\mathscr{M}_\lambda$.

Whenever $\lambda_\theta \neq 0$, the translation function $c: S^{d-1} \to \mathbb{R}$ is continuous at θ. However, if $\lambda_\theta = 0$, $c(\theta)$ is not even uniquely defined. c may be extended to all of $\mathbb{R}^d$ by setting $c(t\theta) = tc(\theta)$ for $t \in \mathbb{R}$ and $\theta \in S^{d-1}$.

1.3 DEFINITION. Let $\lambda \in M(\mathbb{R}^d)$ and let $\mathscr{M}$ denote any subset of $M(\mathbb{R}^d)$. The signed measure λ is said to have the *pointwise translation property with respect to* $\mathscr{M}$ (PTP($\mathscr{M}$)) if $\mathscr{M} \cap \mathscr{M}_\lambda \subset \{\lambda * \delta_b: b \in \mathbb{R}^d\}$. $\langle$PTP($\mathscr{M}$)$\rangle$ will denote the set of all $\lambda \in M(\mathbb{R}^d)$ which have the property PTP($\mathscr{M}$). Thus, interchangeably we can say either λ has PTP($\mathscr{M}$) or $\lambda \in \langle$PTP($\mathscr{M}$)$\rangle$.

λ having PTP($\mathscr{M}$) then means that every $\sigma \in \mathscr{M}$ which is a pointwise translate of λ is actually a vector translate of λ. So $\lambda \in \langle$PTP($\mathscr{M}$)$\rangle$ iff $\lambda \in \langle$PTP($\mathscr{M} \cap \mathscr{M}_\lambda$)$\rangle$. With this terminology, the general problem is

1.4 PROBLEM. Given a nonempty subset $\mathscr{M}$ of $M(\mathbb{R}^d)$, characterize $\langle$PTP($\mathscr{M}$)$\rangle$.

With respect to our centering problem for u.a.n. triangular arrays, we take $\mathscr{M} = \mathscr{I}_d$, the set of all infinitely divisible laws on $\mathbb{R}^d$. Theorem 2.9 then says that condition (III) is superfluous iff $\gamma \in \mathscr{I}_d \cap \langle$PTP($\mathscr{I}_d$)$\rangle$. (Unless needed for clarification we write $\mathscr{I}$ rather than $\mathscr{I}_d$.)

Problem 1.4 can be thought of more generally as a recognition problem. For example, let $\lambda \in M(\mathbb{R}^d)$ and suppose that for every $\theta \in S^{d-1}$, $\lambda_\theta = \delta_{c(\theta)}$. Thus from every direction λ appears to be a point mass. Must λ actually be a point mass $\delta_b (b \in \mathbb{R}^d)$ or might λ be a signed measure with positive and negative mass arranged so that cancellation occurs properly in every direction? If all λ_θ are δ_0 on $\mathbb{R}$, it follows easily from uniqueness of the Radon transform (*vide infra*) that $\lambda = \delta_0$ on $\mathbb{R}^d$. Theorem 5.1(i) asserts that, even if λ_θ is $\delta_{c(\theta)}$, λ must be a δ-function.

More generally, if $\sigma \in \mathscr{M}_\lambda$ but $\lambda \notin \langle$PTP(σ)$\rangle$, then from projections one might "recognize" σ as λ modulo translation, when in fact it is not. This type of recognition procedure has practical applications. Section 6 contains a brief discussion of the implications of this concept for the computerized tomographic scanner.

Our discussion of Problem 1.4 is organized as follows. Section 3 contains characterizations of $\mathscr{I} \cap \langle$PTP($\mathscr{I}$)$\rangle$ (Theorems 3.2 and 3.5) in terms of conditions on the Levy measure. In order for an infinitely divisible law γ with Levy measure μ to have a pointwise translate which is not a vector translate it is necessary and sufficient that there exist a non-zero Levy measure ν with the following properties: (i) $\nu \le \mu$; (ii) there is a set A disjoint from $-A$ with $\nu(A^c) = 0$; and (iii) ν_θ is a symmetric Levy measure on $\mathbb{R}\backslash\{0\}$ for all $\theta \in S^{d-1}$ (Corollary 3.20).

No such ν exists if γ is stable of index $\alpha \neq 1$; consequently, γ has PTP($\mathscr{I}$). On the other hand, if γ is stable of index 1, $u \in S^{d-1}$ and $r > 0$, then $d\nu(ru)$ must be of the form $\Gamma(du) \times dr/r^2$ and condition (iii) can be replaced by the simpler condition $\int_{S^{d-1}} u\,\Gamma(du) = \vec{0}$ (Theorem 3.11). The fact that there are stables of index 1 without PTP($\mathscr{I}$) was first noticed by A. deAcosta (see Remark 3.15). (An earlier version of this paper claimed that if $\mathscr{S}_\alpha$ is the set of stable laws of index α, then $\mathscr{S}_\alpha \subset \langle \mathrm{PTP}(\mathscr{S}_\alpha) \rangle$ for $0 < \alpha \leq 2$. This is not true for $\alpha = 1$. See Remark 2.11 for the source of the error. At the end of this paper we have included a list of several published statements which are incorrect because of this mistake.)

Example 3.16 shows that the spherically symmetric Cauchy on $\mathbb{R}^d$ does not have PTP($\mathscr{I}$).

Propositions 3.21 and 3.22 help to elucidate in more concrete terms some of the probability laws in $\mathscr{I} \cap \langle \mathrm{PTP}(\mathscr{I}) \rangle$. This set is weakly dense in $\mathscr{I}$, but so is its complement $\mathscr{I} \cap \langle \mathrm{PTP}(\mathscr{I}) \rangle^c$ (see Proposition 3.24).

Section 4 uses the results on spherically symmetric stables to sharpen the main theorem in Hahn and Klass (1980b) and to correct an error in Hahn and Klass (1981a). This application inspired the present work.

The focus of the previous sections was on infinitely divisible laws. In Section 5 we consider the pointwise translation problem in a broader context. We solve a few special cases. As indicated previously, we show that $\delta_0 \in \langle \mathrm{PTP}(M(\mathbb{R}^d)) \rangle$, as is every other element of $M(\mathbb{R}^d)$ which is invertible under convolution.

Another result (Theorem 5.3) shows that if

$$\mathscr{E}_1 = \left\{ \lambda \in M(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |x|\,|\lambda|\,(dx) < \infty \right\}$$

then $\lambda \in \mathscr{E}_1$ with $\lambda(\mathbb{R}^d) \neq 0$ implies $\lambda \in \langle \mathrm{PTP}(\mathscr{E}_1) \rangle$. Interestingly, the assumption that $\lambda(\mathbb{R}^d) \neq 0$ is essential, as shown by examples due to L. Shepp (see Example 5.3).

Finally, in Section 6, we discuss our definition of the Radon transform of a measure and the implications which the pointwise translation problem has for computerized tomographic scanning. Also, in Section 6 we reinterpret Example 3.16 to obtain a distinct counter-example to the "hole" problem considered by Perry (1977) and Quinto (1981).

CONVENTIONS AND NOTATION. Let

$$\tilde{M}(\mathbb{R}^d) = \{\sigma\text{-finite signed measures on } \mathbb{R}^d\}$$
$$\tilde{M}^+(\mathbb{R}^d) = \{\lambda \in \tilde{M}(\mathbb{R}^d) : \lambda \geq 0\}$$
$$M(\mathbb{R}^d) = \{\lambda \in \tilde{M}(\mathbb{R}^d) : \lambda \text{ has finite total variation}\}$$
$$M^+(\mathbb{R}^d) = \{\lambda \in \mathrm{M}(\mathbb{R}^d) : \lambda \geq 0\}.$$

Unless specified otherwise, all signed measures will be defined on $\mathscr{B}(\mathbb{R}^d)$, the Borel subsets of $\mathbb{R}^d$. For $\lambda, \sigma \in \tilde{M}(\mathbb{R}^d)$, we write $\lambda \leq \sigma$ iff $\lambda(A) \leq \sigma(A)$ for all $A \in \mathscr{B}(\mathbb{R}^d)$. Define $\bar{\lambda}$ by $\bar{\lambda}(A) = \lambda(-A)$ for all $A \in \mathscr{B}(\mathbb{R}^d)$ and let $\lambda^s = \lambda * \bar{\lambda}$. Thus, if $\mathscr{L}(X) = \lambda$ then X^s has law λ^s. $\lambda^+ = \lambda \vee 0$, $\lambda^- = -\lambda \vee 0$, and $|\lambda| = \lambda^+ + \lambda^-$. The vector integral $\int_{\mathbb{R}^d} x\,\lambda(dx)$ of course means $(\int_{\mathbb{R}^d} x_1\,\lambda(dx), \cdots, \int_{\mathbb{R}^d} x_d\,\lambda(dx))$.

For $\lambda \in M(\mathbb{R}^d)$, the Fourier transforms of λ and λ_θ are related by

$$\hat{\lambda}(t\theta) = \int_{\mathbb{R}^d} \exp(i\langle t\theta, x\rangle)\lambda(dx) = \int_{\mathbb{R}^d} \exp(its)\lambda_\theta(ds) = (\lambda_\theta)\hat{}(t).$$

When λ is a probability measure, of course, this formula $\hat{\lambda}(t\theta) = (\lambda_\theta)\hat{}(t)$ relates the characteristic functions. As is well known, the map $\lambda \to \hat{\lambda}$ is $1 - 1$; moreover, $(\lambda * \sigma)\hat{}(t) = \hat{\lambda}(t)\hat{\sigma}(t)$.

An infinitely divisible law γ on $\mathbb{R}^d$ is uniquely determined by its Levy representation.

We may therefore write $\gamma \sim [a, \Phi, \mu]$ where the characteristic function of γ has the form

$$\hat{\gamma}(x) = \exp\left\{ i\langle a, x\rangle - \frac{1}{2}\Phi(x, x) + \int_{\mathbb{R}^d} \left(e^{i\langle x,u\rangle} - 1 - \frac{i\langle x, u\rangle}{1 + \|u\|^2} \right) \mu(du) \right\}.$$

In this representation, $a \in \mathbb{R}^d$, Φ is a covariance, and μ is a Levy measure. The collection of all Levy measures $\tilde{M}_L^+(\mathbb{R}^d)$ is

$$\tilde{M}_L^+(\mathbb{R}^d) \equiv \left\{ \mu \in \tilde{M}^+(\mathbb{R}^d) \colon \mu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} \min(1, \|x\|^2)\mu(dx) < \infty \right\}.$$

Projections have so far been defined solely for $\lambda \in M(\mathbb{R}^d)$ (Definition 1.1). The definition extends to $\lambda \in \tilde{M}^+(\mathbb{R}^d)$, in which case λ_θ may no longer be σ-finite. However, if λ is a Levy measure, λ_θ will be σ-finite on $\mathbb{R}^d \backslash \{0\}$. By convention we then set $\lambda_\theta(\{0\}) = 0$ so that $\theta \mapsto \lambda_\theta$ takes S^{d-1} to $\tilde{M}_L^+(\mathbb{R})$.

We will reserve μ, ν for Levy measures; γ, η for probability measures; ξ for a difference of two Levy measures (see Section 3); Γ, Λ for finite measures on S^{d-1}; and λ, σ for elements of $\tilde{M}(\mathbb{R}^d)$.

2. Centering and the pointwise translation problem. The existence of centerings in 1-dimensional central limit theorems is guaranteed by tail and variance conditions. This is not, however, the case in higher dimensions where an additional constraint must sometimes be imposed. The need for this constraint is governed by whether the limit law has PTP($\mathscr{I}$), a property which trivially holds in 1-dimension.

Let $X_{n1}, \cdots, X_{nk_n}$ be a u.a.n. triangular array of rowwise independent d-dimensional random vectors with sums $S_n = X_{n1} + \cdots + X_{nk_n}$. Given $v_n \in \mathbb{R}^d$, there are three conditions necessary and sufficient for convergence of $\mathscr{L}(S_n - v_n)$ to an infinitely divisible law $\gamma \sim [a, \Phi, \mu]$. These individually correspond to each of the three parameters a, Φ and μ. The continuity points of μ will be denoted by $\mathscr{C}_\mu$.

Recall that when $d = 1$,

(2.1) there exist constants v_n such that $\mathscr{L}(S_n - v_n) \to \gamma$ iff

(i) for every $y > 0$ with $\pm y \in \mathscr{C}_\mu$

$$\lim_{n\to\infty} | \sum_{j=1}^{k_n} P(X_{nj} \geq y) - \mu([y, \infty)) | = 0,$$

$$\lim_{n\to\infty} | \sum_{j=1}^{k_n} P(X_{nj} \leq -y) - \mu((-\infty, -y]) | = 0$$

and

(ii) $$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} | \sum_{j=1}^{k_n} \operatorname{Var}(X_{nj} I_{(|X_{nj}| \leq \varepsilon)}) - \Phi(1, 1) | = 0.$$

The constants v_n are then governed by the relation

(iii) $$\lim_{n\to\infty} | v_n - \tilde{m}_n(\tau) + a_\tau | = 0 \quad \text{for some } \tau > 0 \text{ with } \pm\tau \in \mathscr{C}_\mu \text{ where}$$

$$\tilde{m}_n(\tau) = \sum_{j=1}^{k_n} EX_{nj} I_{(|X_{nj}| \leq \tau)} \quad \text{and}$$

(2.2) $$a_\tau = a + \int_{|u| \leq \tau} u^3/(1 + u^2)\mu(du) - \int_{|u| > \tau} u/(1 + u^2)\mu(du).$$

(Reference: Gnedenko and Kolmogorov (1968), pages 116–117, 84).

For $d > 1$, given $v_n \in \mathbb{R}^d$, $\mathscr{L}(S_n - v_n) \to \gamma$ iff for each $\theta \in S^{d-1}$

(2.3) $$\mathscr{L}(\langle S_n - v_n, \theta\rangle) \to \gamma_\theta.$$

A continuity argument shows that it suffices to check (2.3) for a countable dense set of θ's. The Levy representation for γ_θ is given by $[a_{(\theta)}, \Phi_{(\theta)}, \mu_{(\theta)}]$ where for $\theta \in S^{d-1}$,

$$\mu_{(\theta)}(E) = \mu_\theta(E\backslash\{0\}) \quad \text{for} \quad E \in \mathscr{B}(\mathbb{R})$$

(2.4)
$$\Phi_{(\theta)}(t, t) = t^2\Phi(\theta, \theta) \quad \text{for} \quad t \in \mathbb{R}$$

$$a_{(\theta)} = \langle a, \theta\rangle + r_\mu(\theta)$$

where

$$r_\mu(\theta) = \int_{\mathbb{R}^d} \langle\theta, u\rangle((1 + \langle\theta, u\rangle^2)^{-1} - (1 + \| u \|^2)^{-1})\mu(du).$$

With this background and terminology, the d-dimensional result can be stated in the following manner:

(2.5) d-DIMENSIONAL TRIANGULAR ARRAY THEOREM. *Letting* $\gamma \sim [a, \Phi, \mu]$ *and* $v_n \in \mathbb{R}^d$, $\mathscr{L}(S_n - v_n) \to \gamma$ iff *the following three conditions hold for all* θ *in a countable dense subset* Θ *of* S^{d-1}:

$(\mathrm{I})_\mu$ $\quad \lim_{n\to\infty} | \sum_{j=1}^{k_n} P(\langle X_{nj}, \theta\rangle \geq y) - \mu_\theta([y, \infty)) | = 0$ *for all* $y \in \mathscr{C}_{\mu_\theta}$ *with* $y > 0$;

$(\mathrm{II})_\Phi$ $\quad \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} | \sum_{j=1}^{k_n} \mathrm{Var}(\langle X_{nj}, \theta\rangle I_{(|\langle X_{nj},\theta\rangle| \leq \varepsilon)}) - \Phi(\theta, \theta) | = 0$;

$(\mathrm{III})_{a_\tau}$ *if* $\tilde{m}_n(\theta, \tau) \equiv \sum_{j=1}^{k_n} E\langle X_{nj}, \theta\rangle I_{(|\langle X_{nj},\theta\rangle| \leq \tau)}$, *then for some* $\tau > 0$ *with* $\pm\tau \in \bigcap_{\theta\in\Theta} \mathscr{C}_{\mu_\theta}$,

$$\lim_{n\to\infty} | \langle v_n, \theta\rangle - \tilde{m}_n(\theta, \tau) + a_\tau(\gamma, \theta) | = 0,$$

where

$$a_\tau(\gamma, \theta) \equiv a_{(\theta)} + \int_{|u|\leq\tau} u^3/(1 + u^2)\mu_\theta(du) - \int_{|u|>\tau} u/(1 + u^2)\mu_\theta(du).$$

2.6 REMARK. It can and will be assumed that Θ contains the standard orthonormal basis $\{e_1, \cdots, e_d\}$. Then since v_n is a vector, (III) may be reexpressed in the equivalent form

$(\mathrm{III}')_{a_\tau}$ $\quad \lim_{n\to\infty} | \sum_{j=1}^{d} \langle\theta, e_j\rangle(\tilde{m}_n(e_j, \tau) - a_\tau(\gamma, e_j)) - \tilde{m}_n(\theta, \tau) + a_\tau(\gamma, \theta) | = 0$

for some $\tau > 0$ with $\pm\tau \in \bigcap_{\theta\in\Theta} \mathscr{C}_{\mu_\theta}$.

2.7 REMARK. If all the μ_θ are continuous then the above three conditions can be replaced by

$(\tilde{\mathrm{I}})_\mu$ $\quad \lim_{n\to\infty} \sup_{\|\theta\|=1} | \sum_{j=1}^{k_n} P(\langle X_{nj}, \theta\rangle \geq y) - \mu_\theta([y, \infty)) | = 0$ for all $y \in C_{\mu_\theta}$ with $y > 0$;

$(\tilde{\mathrm{II}})_\Phi$ $\quad \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \sup_{\|\theta\|=1} |\sum_{j=1}^{k_n} \mathrm{Var}(\langle X_{nj}, \theta\rangle I_{(|\langle X_{nj},\theta\rangle| \leq \varepsilon)}) - \Phi(\theta, \theta) | = 0$;

$(\tilde{\mathrm{III}})_{a_\tau}$ $\quad \lim_{n\to\infty} \sup_{\|\theta\|=1} | \langle v_n, \theta\rangle - \tilde{m}_n(\theta, \tau) + a_\tau(\gamma, \theta) | = 0.$

This is a consequence of the fact that (2.3) holds iff it holds uniformly in θ (see Lemma 1, Hahn and Klass, 1980b).

The higher dimensional statement is not quite analogous to that given in $\mathbb{R}^1$. Conditions (I) and (II) clearly imply the existence of 1-dimensional centering constants $v_n(\theta)$ satisfying (III) (with $v_n(\theta)$ replacing $\langle v_n, \theta\rangle$). However, centering in $\mathbb{R}^d$ requires a vector; so when is $v_n(\theta)$ forced to be asymptotically linear in θ? Equivalently, when do (I) and (II) imply (III)? The answer turns out to depend only on the limit law γ.

An understanding of the function of conditions (I) and (II) in proving weak convergence is required. For a triangular array of rowwise independent d-dimensional random vectors with row sums S_n, conditions (I) and (II) may be expressed in the following two equivalent forms:

2.8 THEOREM. *The following are equivalent*:

(A) $(\mathrm{I})_\mu$ *and* $(\mathrm{II})_\Phi$ *hold.*

(B) *For each* $\theta \in S^{d-1}$, *there exist centering constants* $v_n(\theta)$ *such that* $\mathscr{L}(\langle S_n, \theta\rangle - v_n(\theta)) \to \gamma_\theta$.

(C) *There exist centering vectors* v_n *for which* $\{\mathscr{L}(S_n - v_n)\}$ *is tight and such that all weak subsequential limits are in* $\mathscr{M}_\gamma \cap \mathscr{I}$.

PROOF. As already noted above, (A) $\Leftrightarrow$ (B) is merely the 1-dimensional result.

Now assume (A) and (B) and let $w_n = \sum_{j=1}^{d} v_n(e_j)e_j$. Since $\mathscr{L}(\langle S_n - w_n, e_j\rangle) \to \gamma_{e_j}$, the sequence $\{\mathscr{L}(S_n - w_n)\}$ is tight. Any weak subsequential limit η is obviously infinitely divisible. Moreover, for each $\theta \in S^{d-1}$, the Levy measure ν_θ of η_θ must equal μ_θ because $(\mathrm{I})_\mu$ holds along a subsequence. Similarly, the Gaussian parts of η_θ and γ_θ agree. Therefore, the Levy representations for η_θ and γ_θ differ only in their translation part. Hence $\eta \in \mathscr{M}_\gamma \cap \mathscr{I}$, establishing (C).

Finally, assume (C). Let $\rho(\cdot, \cdot)$ denote the Prohorov distance between two measures. Fix $\theta \in S^{d-1}$ and let $\rho_n \equiv \inf_{v\in\mathbb{R}} \rho(\mathscr{L}(\langle S_n, \theta\rangle - v), \gamma_\theta)$. Take any sequence $(n') \subset (n)$. It suffices to show that there exists a subsequence $(n'') \subset (n')$ such that $\rho_{n''} \to 0$. By (C), there exists $(n'') \subset (n')$ and a law $\eta \in \mathscr{M}_\gamma \cap \mathscr{I}$ such that $\mathscr{L}(S_{n''} - v_{n''}) \to \eta$. Thus, there exists $c(\theta)$ such that $\eta_\theta = \gamma_\theta * \delta_{c(\theta)}$. Hence, $\rho_{n''} \le \rho(\mathscr{L}(\langle S_{n''}, \theta\rangle - (\langle v_{n''}, \theta\rangle + c(\theta))), \gamma_\theta) \to 0$. □

Suppose $(\mathrm{I})_\mu$ and $(\mathrm{II})_\Phi$ hold. Then by slightly modifying v_n if necessary, we may assume that $\mathscr{L}(\langle S_n - v_n, e_j\rangle) \to \gamma_{e_j}$, for $j = 1, \cdots, d$. Now if every pointwise translate of γ is a vector translate, then, using (C), every weak subsequential limit of $\mathscr{L}(S_n - v_n)$ equals γ. Hence $\mathscr{L}(S_n - v_n) \to \gamma$.

We have thus proven sufficiency in the following theorem.

2.9 THEOREM. *Let* $\gamma \sim [a, \Phi, \mu]$ *on* $\mathbb{R}^d$. *In order that* $(\mathrm{I})_\mu$ *and* $(\mathrm{II})_\Phi$ *imply* $(\mathrm{III}')_{a_\tau}$ *for any* u.a.n. *triangular array of rowwise independent random vectors, it is necessary and sufficient that* $\gamma \in \mathrm{PTP}(\mathscr{I})$.

PROOF. (Necessity). Assume $(\mathrm{I})_\mu$ and $(\mathrm{II})_\Phi$ imply $(\mathrm{III})_{a_\tau}$. Let $\eta \in \mathscr{M}_\gamma \cap \mathscr{I}$. Thus, there exists $c(\theta)$ such that $\eta_\theta = \gamma_\theta * \delta_{c(\theta)}$ for all $\theta \in S^{d-1}$. Let $[b, \Psi, \nu]$ be the Levy representation for η. Since $\eta \in \mathscr{I}$, there exist rowwise i.i.d. random vectors $X_{n1}, \cdots, X_{nn}$ and $c_n \in \mathbb{R}^d$ such that $\mathscr{L}(X_{n1} + \cdots + X_{nn} - c_n) \to \eta$. Hence $(\mathrm{I})_\nu$, $(\mathrm{II})_\Psi$ and $(\mathrm{III})_{a_\tau(\eta)}$ hold (for some $\tau > 0$). Since $\eta \in \mathscr{M}_\gamma$, $\nu_\theta = \mu_\theta$ and $\Psi(\theta, \theta) = \Phi(\theta, \theta)$, whence $\cap_{\theta\in\Theta} \mathscr{C}_{\mu_\theta} = \cap_{\theta\in\Theta} \mathscr{C}_{\nu_\theta}$ and both $(\mathrm{I})_\mu$ and $(\mathrm{II})_\Phi$ hold. By our hypothesis, it follows that $(\mathrm{III})_{a_\tau(\gamma)}$ holds. Therefore, using the same τ, we have for $\lambda = \eta$ and $\lambda = \gamma$ and all $\theta \in \Theta$,

$$\lim_{n\to\infty} \left| \sum_{j=1}^{d} \langle \theta, e_j\rangle(\tilde{m}_n(e_j, \tau) - a_\tau(\lambda, e_j)) - \tilde{m}_n(\theta, \tau) + a_\tau(\lambda, \theta) \right| = 0. \tag{2.10}$$

Consequently,

$$\sum_{j=1}^{d} \langle \theta, e_j\rangle(a_\tau(\eta, e_j) - a_\tau(\gamma, e_j)) = a_\tau(\eta, \theta) - a_\tau(\gamma, \theta).$$

Direct calculation shows that

$$a_\tau(\eta, \theta) = a_\tau(\gamma, \theta) + c(\theta).$$

Therefore, $c(\theta) = \sum_{j=1}^{d} \langle \theta, e_j\rangle c(e_j)$ as required. □

2.11 REMARK. Conditions $(\mathrm{I})_\mu$ and $(\mathrm{II})_\Phi$ make sense for families of 1-dimensional Levy measures $\{\mu(\theta): \theta \in \Theta\}$ and families of constants $\{\Phi(\theta, \theta): \theta \in \Theta\}$ without presupposing the existence of a Levy measure μ and a covariance Φ from which they are derived. In fact, their existence can be inferred. (The argument is essentially sentences 2–6 in the proof of Theorem 2.8.) However, this is not to say that a Levy measure μ when restricted to $\mathbb{R}^d\backslash\{0\}$ is determined by its Radon transform on $\mathbb{R}^d\backslash\{0\}$. This would be the case if μ were a finite measure. (Such a supposition for σ-finite measures led to an erroneous proof that every infinitely divisible law has $\mathrm{PTP}(\mathscr{I})$ in an earlier version of this paper. We are grateful to a sharp-eyed referee for questioning this point.) Were it true in general, then any two infinitely divisible laws γ and η which are pointwise translates would have the same Levy measure. Since their Gaussian component is determined by projection, it follows that γ and η must be vector translates. We show in Example 3.16 that this need not be the case.

In the absence of a given Levy measure μ and covariance Φ from which the families $\{\mu(\theta):\theta \in \Theta\}$ and $\{\Phi(\theta, \theta):\theta \in \Theta\}$ are derived, we write (I) and (II) as $(\mathrm{I})_{\mu(\theta)}$ and $(\mathrm{II})_{\Phi(\theta)}$, respectively.

It being unnecessary to assume the existence of γ at the outset, we can state a slightly stronger form of Theorem 2.9.

2.12 THEOREM. *Let $\{\mu(\theta):\theta \in \Theta\}$ and $\{\Phi(\theta):\theta \in \Theta\}$ be families of σ-finite Levy measures and positive constants, respectively. For all* u.a.n. *triangular arrays of rowwise independent d-dimensional random vectors satisfying* $(\mathrm{I})_{\mu(\theta)}$ *and* $(\mathrm{II})_{\Phi(\theta)}$, *there exist centering vectors v_n (depending on the triangular array) and a law to which $\mathscr{L}(S_n - v_n)$ converges weakly* iff *some (and hence any) possible weak subsequential limit of $\mathscr{L}(S_n - v_n)$ has* PTP($\mathscr{I}$), *in which case there exists a unique Levy measure μ and a unique covariance Φ with projections $\mu(\theta)$ and $\Phi(\theta)$ respectively.*

PROOF. Sufficiency is an immediate consequence of Theorem 2.9.

For necessity, tightness follows as in Theorem 2.8. Now take $\gamma \in \mathscr{I}$ which does not have PTP($\mathscr{I}$) (the existence of such γ is shown in Example 3.16). Then there exists $c(\theta)$ nonlinear and $\eta \in \mathscr{I}$ such that $\eta_\theta = \gamma_\theta * \delta_{c(\theta)}$. γ_θ and η_θ have the same one-dimensional Levy measure $\mu(\theta)$ and the same Gaussian part $\Phi(\theta)$. Let $\{X_{n1}, \cdots, X_{nn}\}$ be a triangular array of rowwise i.i.d. random vectors such that $\mathscr{L}(S_{2n} - v_{2n}) \to \gamma$ and $\mathscr{L}(S_{2n+1} - v_{2n+1}) \to \eta$. Now clearly, $(\mathrm{I})_{\mu(\theta)}$ and $(\mathrm{II})_{\Phi(\theta)}$ hold. However, (III′) must fail since $\mathscr{L}(S_n - v_n)$ does not actually converge weakly. □

2.13 REMARK. Assuming that $\{X_{nj}^s\}$, rather than $\{X_{nj}\}$, is u.a.n. may appear to lead to a slight generalization in each of the above theorems. This is, however, not the case. For any random vector X, if $P(|X^s| > \varepsilon) < \delta$ then there exists $a \in \mathbb{R}^d$ such that $P(|X - a| > \varepsilon) < \delta$. Thus if $\{X_{nj}\}$ has the property that $\{X_{nj}^s\}$ is u.a.n. then there exist a_{nj} so that $\{X_{nj} - a_{nj}\}$ is u.a.n. In this case the above theorems apply to the array $\{X_{nj} - a_{nj}\}$ rather than $\{X_{nj}\}$. Gnedenko and Kolmogorov (1968) call such arrays asymptotically constant.

3. Characterizations of $\mathscr{L} \cap$ PTP($\mathscr{I}$).

Let the infinitely divisible laws γ and η have Levy representations $[a, \Phi, \mu]$ and $[b, \Psi, \nu]$ respectively. When are γ and η pointwise translates and when are they vector translates? Clearly, $\eta \in \mathscr{M}_\gamma$ iff $\Psi(\theta, \theta) = \Phi(\theta, \theta)$ and the measures ν_θ and μ_θ agree on $\mathbb{R}\backslash\{0\}$ for all $\theta \in S^{d-1}$ (use (2.4), the representation of the projections). Two Gaussian components with the same projections are equal, so in fact, $\Psi = \Phi$ whenever $\eta \in \mathscr{M}_\gamma$. Finally the pointwise translates η and γ are actually vector translates iff in addition, $\nu = \mu$.

When $\eta_\theta = \gamma_\theta * \delta_{c(\theta)}$, $c(\theta)$ is given by

$$c(\theta) = \langle b - a, \theta \rangle + r_\nu(\theta) - r_\mu(\theta) \tag{3.1}$$

where for $\lambda \in \tilde{\mathscr{M}}_L^+(\mathbb{R}^d)$,

$$r_\lambda(\theta) = \int_{\mathbb{R}^d} \langle x, \theta \rangle((1 + \langle x, \theta \rangle^2)^{-1} - (1 + \|x\|^2)^{-1})\lambda(dx).$$

To verify (3.1) it suffices to assume $\Psi = \Phi = 0$. Using (2.4),

$$\exp\left\{it(\langle b, \theta \rangle + r_\nu(\theta)) + \int_{\mathbb{R}} (e^{ity} - 1 - ity/(1 + y^2))\nu_\theta(dy)\right\} = (\eta_\theta)\hat{}(t) = (\gamma_\theta)\hat{}(t)\exp(itc(\theta))$$

$$= \exp\left\{it(\langle a, \theta \rangle + r_\mu(\theta) + c(\theta)) + \int_{\mathbb{R}} (e^{ity} - 1 - ity/(1 + y^2))\mu_\theta(dy)\right\}.$$

Since $\nu_\theta = \mu_\theta$ on $\mathbb{R}\backslash\{0\}$, solving for $c(\theta)$ gives (3.1).

Owing to our representation of $c(\theta)$ it is possible to determine whether $\eta \in \mathscr{M}_\gamma$ is a vector translate of γ without directly verifying that $\nu = \mu$. Clearly η is a vector translate of γ iff there is $c \in \mathbb{R}^d$ such that $c(\theta) = \langle c, \theta \rangle$. This in turn is equivalent to $(r_\nu - r_\mu)(\theta)$ being linear. In actuality, $(r_\nu - r_\mu)(\theta)$ is non-linear iff $r_\nu \neq r_\mu$. To see this, first note that $r_\nu \neq r_\mu$ implies $\nu \neq \mu$, whence η and γ are not vector translates. Second, if $\nu \neq \mu$ (but $\nu_\theta = \mu_\theta$ on $\mathbb{R}\backslash\{0\}$), then since η and γ are not vector translates it follows that $(r_\nu - r_\mu)(\theta)$ is non-linear and so $r_\nu - r_\mu \not\equiv 0$. Thus, we have proven the following theorem.

3.2 THEOREM. *Let γ have Levy representation $[a, \Phi, \mu]$. Then $\gamma \notin \langle \mathrm{PTP}(\mathscr{I})\rangle$* iff *there exists a Levy measure ν such that*

(i) $\nu_\theta = \mu_\theta$ *on* $\mathbb{R}\backslash\{0\}$ *for all* $\theta \in S^{d-1}$

and

(ii) $r_\nu \not\equiv r_\mu$.

In general, it is not easy to check whether $r_\mu = r_\nu$. Seeking an alternative method for determining whether $\gamma \in \langle \mathrm{PTP}(\mathscr{I})\rangle$, we consider the difference between two Levy measures. To insure that this difference is well defined and not indeterminate, we restrict the sets on which these measures act.

Let $\mathscr{C}^d = \cup_{n=1}^{\infty} \mathscr{B}(\{x \in \mathbb{R}^d : \|x\| \geq 1/n\})$ and $\tilde{M}_L^+(\mathbb{R}^d, \mathscr{C}^d) = \{\tilde{\mu} : \tilde{\mu}$ is the restriction of a Levy measure μ to $\mathscr{C}^d\}$. The *restriction map F* with $F(\mu) = \tilde{\mu}$ is $1-1$ and linear from $\tilde{M}_L^+(\mathbb{R}^d)$ to $\tilde{M}_L^+(\mathbb{R}^d, \mathscr{C}^d)$. The set $\mathscr{G} = \{(\mu, \nu) : \mu, \nu \in \tilde{\mathscr{M}}_L^+(\mathbb{R}^d)\}$ is endowed with a natural equivalence relation: $(\mu_1, \nu_1) \sim (\mu_2, \nu_2)$ iff $\tilde{\mu}_1 - \tilde{\nu}_1 = \tilde{\mu}_2 - \tilde{\nu}_2$.

3.3 DEFINITION. A *Levy s-measure* ξ is a signed measure on $(\mathbb{R}^d\backslash\{0\}, \mathscr{C}^d)$ defined by $\xi = \tilde{\mu} - \tilde{\nu}$ for some $(\mu, \nu) \in \mathscr{G}$. $\tilde{M}_{Ls}(\mathbb{R}^d\backslash\{0\}, \mathscr{C}^d)$ will denote the vector space of Levy s-measures on $(\mathbb{R}^d\backslash\{0\}, \mathscr{C}^d)$.

Since the same Levy s-measure ξ is obtained for each element of a single equivalence class in $\mathscr{G}$, we may *always assume* that (μ, ν) is the unique representative in which μ and ν are mutually singular and we then write $\xi^+ = \tilde{\mu}$ and $\xi^- = \tilde{\nu}$ and let $|\xi| = \xi^+ + \xi^-$. Notice that $\mathscr{C}^d$ is not a σ-algebra and in general ξ cannot be defined on $\mathscr{B}(\mathbb{R}^d)$ unless $F^{-1}\xi^+(\mathbb{R}^d) \wedge F^{-1}\xi^-(\mathbb{R}^d) < \infty$. In the latter case, ξ^+, ξ^- and $|\xi|$ all agree with their customary definitions.

We interpret integration with respect to ξ as follows: for

$$f \in L^1(F^{-1}\xi^+) \cup L^1(F^{-1}\xi^-), \int f\,d\xi \equiv \int f\,dF^{-1}\xi^+ - \int f\,dF^{-1}\xi^-.$$

In actuality, this definition of integration extends the domain of a Levy s-measure to all Borel sets E in $\mathbb{R}^d$ for which $F^{-1}\xi^+(E) \wedge F^{-1}\xi^-(E) < \infty$. Besides being a vector space, $\tilde{M}_{Ls}(\mathbb{R}^d\backslash\{0\}, \mathscr{C}^d)$ is a Banach space with the norm of ξ defined by $\|\xi\| = \int (1 \wedge \|x\|^2)\,|\xi|(dx)$.

The Radon transform of a Levy s-measure ξ is the map $\theta \to \xi_\theta : S^{d-1} \to \tilde{M}_{Ls}(\mathbb{R}\backslash\{0\}, \mathscr{C}^1)$ defined by $\xi_\theta = (\xi^+)_\theta - (\xi^-)_\theta$. Notice that for $E \in \mathscr{C}^1$, $\xi_\theta(E) = (F^{-1}\xi^+)_\theta(E) - (F^{-1}\xi^-)_\theta(E)$. Consequently, since $(F^{-1}\xi^+)_\theta$ and $(F^{-1}\xi^-)_\theta$ are σ-finite measures on $\mathbb{R}\backslash\{0\}$ and $\mathscr{B}(\mathbb{R}\backslash\{0\})$ is the σ-algebra generated by $\mathscr{C}^1$, we see than

$$\xi_\theta = 0 \text{ (on } \mathscr{C}^1\text{) iff } (F^{-1}\xi^+)_\theta = (F^{-1}\xi^-)_\theta \text{ on } \mathbb{R}\backslash\{0\}. \tag{3.4}$$

Moreover, a computation verifies that $(\xi_1 + \xi_2)_\theta = (\xi_1)_\theta + (\xi_2)_\theta$.

A characterization of $\mathscr{I} \cap \langle \mathrm{PTP}(\mathscr{I})\rangle$ can be given in terms of Levy s-measures.

3.5 THEOREM. *Let γ have Levy representation $[a, \Phi, \mu]$ on $\mathbb{R}^d$. Then there exists an infinitely divisible pointwise translate of γ which is not a vector translate* iff *there exists a non-zero Levy s-measure ξ on $\mathbb{R}^d\backslash\{0\}$ such that*

(i) $\xi_\theta = 0$ *on* $\mathscr{C}^1$ *for all* $\theta \in S^{d-1}$

and

(ii) $\xi^- \leq \tilde{\mu}$.

PROOF. Suppose $\gamma \notin \langle \mathrm{PTP}(\mathscr{I}) \rangle$. Then there exists $\eta \sim [b, \Psi, \nu]$ with $\nu \neq \mu$ and $\nu_\theta = \mu_\theta$ on $\mathbb{R}\backslash\{0\}$. Since $(\nu, \mu) \in \mathscr{G}$, a Levy s-measure is defined by letting $\xi = \tilde{\nu} - \tilde{\mu}$. Clearly $\xi \neq 0$ and $\xi_\theta = 0$ on $\mathscr{C}^1$ for all $\theta \in S^{d-1}$. Furthermore, $\xi^- = (\tilde{\nu} - \tilde{\mu})^- = (\tilde{\mu} - \tilde{\nu})^+ \leq \tilde{\mu}$.

Conversely, suppose there exists a non-zero Levy s-measure ξ satisfying (i) and (ii). By (ii), $\tilde{\mu} + \xi = (\tilde{\mu} - \xi^-) + \xi^+$ is a non-negative Levy s-measure. Hence $\nu \equiv F^{-1}(\tilde{\mu} + \xi)$ is a Levy measure. Using the commutativity of F and projection on $\{t\theta : t \in \mathbb{R}\}$, together with linearity, $(\nu_\theta)^\sim = (\tilde{\nu})_\theta = (\tilde{\mu})_\theta + \xi_\theta = (\tilde{\mu})_\theta = (\mu_\theta)^\sim$ (on $\mathscr{C}^1$). Since the restriction map on $\mathbb{R}^1$ is $1-1$, $\nu_\theta = \mu_\theta$ on $\mathbb{R}\backslash\{0\}$. Consequently, if $\eta \sim [0, \Phi, \nu]$ then $\eta \in \mathscr{M}_\gamma$. However, η cannot be a vector translate of γ since $\nu \neq \mu$. □

Define the vector space $\mathscr{Z}_d \equiv \{\xi \in \tilde{M}_{Ls}(\mathbb{R}^d, \mathscr{C}^d) : \xi_\theta = 0 \text{ on } \mathscr{C}^1\ \forall\, \theta \in S^{d-1}\}$. By Theorem 3.5, $\gamma \in \langle \mathrm{PTP}(\mathscr{I}) \rangle$ iff $\xi^- \leq \tilde{\mu}$ for some $\xi \in \mathscr{Z}_d\backslash\{0\}$. Whether a Levy s-measure is in $\mathscr{Z}_d$ can be determined by integration:

3.6 PROPOSITION. *Let* $\xi \in M_{Ls}(\mathbb{R}^d, \mathscr{C}^d)$.

(i) $\xi \in \mathscr{Z}_d$ iff $\int (e^{it\langle x, \theta\rangle} - 1 - it\langle x, \theta\rangle/(1 + \langle x, \theta\rangle^2))\xi(dx) = 0$ *for all* $t \in \mathbb{R}$ *and* $\theta \in S^{d-1}$.

(ii) $\xi = 0$ iff $\xi \in \mathscr{Z}_d$ *and* $r_\xi(\theta) \equiv r_{\xi^+}(\theta) - r_{\xi^-}(\theta) = 0$ (see (3.1)).

(iii) $\xi \in \mathscr{Z}_d$ *implies* $\xi^+(E) = \xi^-(-E)$ *for all* $E \in \mathscr{C}^d$ *and there exists* $A \in \mathscr{B}(\mathbb{R}^d)$ *such that* $A \cap (-A) = \phi$ *and* $F^{-1}\xi^+(A^c) = 0$. *As a consequence, if* $\xi \in \mathscr{Z}_d$ *and* $\xi^- = 0$, *then* $\xi = 0$.

PROOF. (i) For each t, the integrand is in $L^1(\xi^+) \cap L^1(\xi^-)$. If $\xi \in \mathscr{Z}_d$, $\xi_\theta = 0$ on $\mathscr{C}^1$; so by change of variables and (3.4) the integral is 0. The converse follows from the fact that two Levy measures μ and ν on $\mathbb{R}$ coincide iff $\int f_t(y)\mu(dy) = \int f_t(y)\nu(dy)$, $\forall\, t \in \mathbb{R}$, where $f_t(y) = e^{ity} - 1 - ity/(1 + y^2)$ and from the commutativity of F with projection on the direction θ.

(ii) A uniqueness argument as in (i) shows that $\xi = 0$ iff $\int g_{t,\theta}(x)\xi(dx) = 0$ for all $t \in \mathbb{R}$ and $\theta \in S^{d-1}$ where $g_{t,\theta}(x) = e^{it\langle x, \theta\rangle} - 1 - it\langle x, \theta\rangle/(1 + \|x\|^2)$. Since $\int g_{t,\theta}(x)\xi(dx) = \int f_t(\langle x, \theta\rangle)\xi(dx) - itr_\xi(\theta)$, the conclusion follows immediately from (i).

(iii) Let $\xi \in \mathscr{Z}_d$. Since $\mathscr{Z}_d$ is a vector space, $\xi_s \equiv \xi + \bar{\xi}$ is also in $\mathscr{Z}_d$. The linearity of $\xi \to r_\xi$ shows that $r_{\bar{\xi}} = r_\xi + r_{\bar{\xi}} = r_\xi - r_\xi = 0$. Hence, by (ii), $\xi^s = 0$. So, $0 = \xi + \bar{\xi} = \xi^+ - \xi^- + \bar{\xi}^+ - \bar{\xi}^-$, which implies

$$\xi^+ + \bar{\xi}^+ = \xi^- + \bar{\xi}^-. \tag{3.7}$$

Now suppose there exists $E \in \mathscr{C}^d$ such that $\xi^+(E) \neq \xi^-(-E)$. Without loss of generality we may assume $\xi^+(E) > \xi^-(-E)$. There are disjoint sets C and D with $C \cup D = \mathbb{R}^d$ such that $F^{-1}\xi^+(D) = 0$ and $F^{-1}\xi^-(C) = 0$. So using (3.7),

$$\begin{aligned} \xi^+(E \cap C) &= \xi^+(E) > \xi^-(-E) \geq \xi^-(-(E \cap C)) \\ &= \xi^-(E \cap C) + \xi^-(-(E \cap C)) = \xi^+(E \cap C) + \bar{\xi}^+(E \cap C) \\ &\geq \xi^+(E \cap C), \end{aligned}$$

a contradiction. Thus, $\xi^+(E) = \xi^-(-E)$ for every $E \in \mathscr{C}^d$, which establishes the first part of (iii). Let $A = C \cap (-D)$. Then $A \cap (-A) = \phi$ and $F^{-1}\xi^+(A^c) = F^{-1}\xi^+(A^c \cap C) = F^{-1}\xi^+((-D)^c \cap C) = F^{-1}\xi^+((-D^c) \cap C) \leq F^{-1}\xi^+(-D^c) = F^{-1}\xi^-(D^c) = F^{-1}\xi^-(D^c \cap D) = 0$. Thus (iii) holds. □

As an application of the above results, we characterize those stables having $\mathrm{PTP}(\mathscr{I})$. Recall that γ is a stable measure on $\mathbb{R}^d$ of index α, $0 < \alpha \leq 2$, iff there exist vectors $b_k \in \mathbb{R}^d$ such that for every positive integer k,

$$\hat{\gamma}^k(t) = \hat{\gamma}(k^{1/\alpha}t)e^{i\langle t, b_k\rangle}. \tag{3.8}$$

The case $\alpha = 2$ corresponds to a multivariate normal, which, having a degenerate Levy measure, has $\mathrm{PTP}(\mathscr{I})$ by Theorem 3.5. We therefore restrict our attention to $0 < \alpha < 2$.

The Levy measure μ of a stable of index α, $0 < \alpha < 2$, has a special form. If $r = \| x \|$, $u = x/\| x \| \in S^{d-1}$, then the polar decomposition of μ is given by

$$\mu(dx) = \Gamma(du) \times dr/r^{1+\alpha}, \qquad r > 0,$$

where Γ is a finite measure on S^{d-1}. Γ is called the *spectral measure* and is determined from μ via the formula

$$\Gamma(A) = \mu\{x \in \mathbb{R}^d : x/\| x \| \in A, \| x \| \geq 1\} \quad \text{for any} \quad A \in \mathscr{B}(S^{d-1}). \tag{3.9}$$

Kuelbs (1973) observes that the linear spans of the supports of Γ and γ are the same.

The most general stable γ of index $\alpha \neq 2$ has Levy representation $[a, 0, \mu] = [a, 0, \Gamma(du) \times dr/r^{1+\alpha}]$. After integration of the radial component, the characteristic function of γ acquires the following form (ref. Kuelbs, 1973),

$$\hat{\gamma}(x) = \exp\left\{ i\langle a, x\rangle - \int_{S^{d-1}} |\langle x, u\rangle|^\alpha \Gamma(du) + i\omega_\Gamma(\alpha, x) \right\} \tag{3.10}$$

where

$$\omega_\Gamma(\alpha, x) = \begin{cases} \tan\dfrac{\pi\alpha}{2} \displaystyle\int_{S^{d-1}} \langle x, u\rangle |\langle x, u\rangle|^{\alpha-1}\Gamma(du), & \text{if} \quad \alpha \neq 1, \\ \dfrac{2}{\pi} \displaystyle\int_{S^{d-1}} \langle x, u\rangle \ln|\langle x, u\rangle| \Gamma(du) & \text{if} \quad \alpha = 1. \end{cases}$$

The above representation will be called the *Levy spectral representation* for γ.

Moreover, for each $0 < \alpha < 2$, *every* finite measure Γ generates a Levy measure μ by setting $\mu(A) = \iint_{A\setminus\{0\}}\Gamma(du)\, dr/r^{1+\alpha}$. To show that such a Γ is the spectral measure of some stable, let $\gamma \sim [a, 0, \mu]$. Then upon integrating the radial component, $\hat{\gamma}$ satisfies (3.10). Any such $\hat{\gamma}$ is easily seen to verify (3.8), consequently γ is stable. Furthermore, for each α, distinct Γ's correspond to distinct Levy measures by (3.9). Hence, unicity of Levy measures implies that each stable law has a unique spectral measure.

Kuelbs (1973) proves that $\phi(x) \equiv \exp\{\int_{S^{d-1}} |\langle x, u\rangle|^\alpha \Lambda(du)\}$ is the characteristic function of a symmetric stable law γ of index α, $0 < \alpha < 1$ or $1 < \alpha < 2$, whenever Λ is a finite measure on S^{d-1}. In our terminology, ϕ is the Levy spectral representation of γ iff

$$\int_{S^{d-1}} \langle x, u\rangle |\langle x, u\rangle|^{\alpha-1}\Lambda(du) = 0.$$

Thus, uniqueness of the spectral measure implies that $(\Lambda + \bar{\Lambda})/2$ is the spectral measure of a symmetric law with characteristic function ϕ.

We are now ready for the characterization of those stables which have PTP($\mathscr{I}$).

3.11 THEOREM. *Let γ be a stable measure on $\mathbb{R}^d$ of index α, $0 < \alpha \leq 2$, with spectral measure Γ if $\alpha \neq 2$.*

(a) *If $\alpha \neq 1$ or $d = 1$, $\gamma \in \langle \text{PTP}(\mathscr{I})\rangle$.*

(b) *If $\alpha = 1$ and $d \geq 2$, $\gamma \notin \langle \text{PTP}(\mathscr{I})\rangle$ iff there exists a Borel set $A \subseteq S^{d-1}$ and a non-zero positive measure λ on S^{d-1} such that*

i) $A \cap (-A) = \phi$;

ii) $\lambda(A^c) = 0$;

iii) $\int_{S^{d-1}} u\lambda(du) = \vec{0}$;

iv) $\lambda \leq \Gamma$.

In this case, if η is stable of index 1 with spectral measure $\Lambda = \Gamma - \lambda + \bar{\lambda}$, then η is a pointwise translate of γ but not a vector translate.

PROOF. We may assume $0 < \alpha < 2$ and $d \geq 2$. It suffices to consider $\gamma \sim [0, 0, \Gamma(du) \times dr/r^{1+\alpha}]$ and any $\eta \in \mathscr{M}_\gamma \cap \mathscr{I}$. Thus $\eta_\theta = \gamma_\theta * \delta_{c(\theta)}$ for $\theta \in S^{d-1}$ and some $c(\theta) \in \mathbb{R}$. Since an infinitely divisible law is stable of index α iff all of its 1-dimensional projections are stable of index α, η is stable of index α (see Giné and Hahn, 1982). Hence $\eta \sim [b, 0, \Lambda(du) \times dr/r^{1+\alpha}]$ for some finite measure Λ and some $b \in \mathbb{R}^d$. Without loss of generality, we may assume $b = 0$.

(a) If $\alpha \neq 1$, letting $x = t\theta$, in (3.10) where $\theta \in S^{d-1}$ and $t \in \mathbb{R}$, we see that

$$\exp\left\{-|t|^\alpha \int_{S^{d-1}} |\langle\theta, u\rangle|^\alpha \Lambda(du) + it|t|^{\alpha-1}\omega_\Lambda(\alpha, \theta)\right\}$$

$$= \hat{\eta}_\theta(t) = \hat{\gamma}_\theta(t)e^{itc(\theta)}$$

$$= \exp\left\{itc(\theta) - |t|^\alpha \int_{S^{d-1}} |\langle\theta, u\rangle|^\alpha \Gamma(du) + it|t|^{\alpha-1}\omega_\Gamma(\alpha, \theta)\right\}.$$

Varying t, the coefficients of t in the two characteristic functions must be equal (similarly for those of $|t|^\alpha$ and $t|t|^{\alpha-1}$). This requires $c(\theta) = 0 = \langle 0, \theta\rangle$. Thus $\gamma \in \langle \mathrm{PTP}(\mathscr{S})\rangle$.

(b) If $\alpha = 1$ and $d \geq 2$,

$$\exp\left\{-|t| \int_{S^{d-1}} |\langle\theta, u\rangle|\Lambda(du) + i\frac{2}{\pi}t \int_{S^{d-1}} \langle\theta, u\rangle(\log|t| + \log|\langle\theta, u\rangle|)\Lambda(du)\right\}$$

$$= \hat{\eta}_\theta(t) = \hat{\gamma}_\theta(t)e^{itc(\theta)} = \exp\left\{itc(\theta) - |t| \int_{S^{d-1}} |\langle\theta, u\rangle|\Gamma(du)\right.$$

$$\left. + i\frac{2}{\pi}t \int_{S^{d-1}} \langle\theta, u\rangle(\log|t| + \log|\langle\theta, u\rangle|)\Gamma(du)\right\}.$$

Varying t, it follows that for each (and hence all) $\theta \in S^{d-1}$,

$$c(\theta) = \frac{2}{\pi}\int_{S^{d-1}} \langle\theta, u\rangle \log|\langle\theta, u\rangle|(\Lambda - \Gamma)(du), \tag{3.12}$$

$$\int_{S^{d-1}} \langle\theta, u\rangle(\Lambda - \Gamma)(du) = 0, \tag{3.13}$$

and

$$\int_{S^{d-1}} |\langle\theta, u\rangle|(\Lambda - \Gamma)(du) = 0. \tag{3.14}$$

Suppose first that $\gamma \notin \langle \mathrm{PTP}(\mathscr{S})\rangle$ and that η is not a vector translate of γ. Then by uniqueness of the spectral representation, $\Lambda \neq \Gamma$. Hence, the restriction to $\mathscr{C}^d$ of $(\Lambda - \Gamma)(du) \times dr/r^2$ is the polar decomposition of a non-zero element of $\mathscr{Z}_d$. It follows from (iii) of Proposition 3.6 that $\lambda = (\Gamma - \Lambda)^+$ satisfies (i), (ii) and (iv) since $\bar{\lambda} = (\Gamma - \Lambda)^-$. Moreover, since $\Gamma - \Lambda = \lambda - \bar{\lambda}$, (3.13) implies that

$$0 = \int u(\lambda - \bar{\lambda})(du) = 2\int u\lambda(du)$$

which gives (iii).

Conversely, if there exist λ and A such that (i)-(iv) hold, let Λ be defined by $\Lambda(E) = \Gamma(E) + \lambda(-E) - \lambda(E)$ for all $E \in \mathscr{B}(S^{d-1})$. By (iv), Λ is a finite measure, hence the Levy spectral measure of some stable law η of index 1. Since, by (i) and (ii), $\Lambda(-A) = \Gamma(-A) + \lambda(A) - \lambda(-A) = \Gamma(-A) + \lambda(A) > \Gamma(-A)$, η is not a vector translate of γ. From (iii) it follows as above that

$$\int_{S^{d-1}} \langle\theta, u\rangle(\Lambda - \Gamma)(du) = 0,$$

verifying (3.13). Finally, since

$$\int_{S^{d-1}} |\langle \theta, u\rangle| \, d\lambda(-u) = \int_{S^{d-1}} |\langle \theta, u\rangle| \, d\lambda(u),$$

we may conclude that (3.14) holds. Hence $\hat{\eta}_\theta(t) = \hat{\gamma}_\theta(t)e^{itc(\theta)}$, where $c(\theta)$ is given by (3.12). Since $\eta \in \mathscr{M}_\gamma$ but η is not a vector translate of γ, the conclusion is that $\gamma \notin \langle \mathrm{PTP}(\mathscr{S})\rangle$.

3.15 HISTORICAL REMARK. A. deAcosta (private communication) was the first to provide examples and methods of constructing stables of index 1 on $\mathbb{R}^2$ without PTP($\mathscr{S}$). He considered non-symmetric finite measures Γ on S^1 with vector mean 0 and $\Lambda = (\Gamma + \bar{\Gamma})/2$ for which $r_{\Lambda-\Gamma}(\theta) \neq 0$. $r_{\Lambda-\Gamma}(\theta) \not\equiv 0$ whenever $\Gamma \neq \Lambda$, whence the pointwise translation function c is nonlinear. In fact, this construction works in $\mathbb{R}^d$. DeAcosta's note was the catalyst for the above theorem.

Even the spherically symmetric Cauchy on $\mathbb{R}^d$ fails to have PTP($\mathscr{S}$), as we now indicate.

3.16 EXAMPLE. First consider $d = 2$. The spectral measure m_2 of a spherically symmetric stable of index 1 is uniform on S^1 with $m_2(S^1) = 2\pi$. Define Γ_2 on S^1 by $\Gamma_2(E) = m_2(E \cap A)$ where

$$A = \cup_{k=0}^{2} \left\{(\cos t, \sin t) : 2k\frac{\pi}{3} \le t < (2k+1)\frac{\pi}{3}\right\}.$$

Γ_2 is nonsymmetric with vector mean 0 on S^1 and clearly satisfies both $(\Gamma_2 + \bar{\Gamma}_2)/2 = m_2$ and $r_{m_2-\Gamma_2} \not\equiv 0$. Hence, the spherically symmetric Cauchy on $\mathbb{R}^2$ is one of the deAcosta-type examples which does not have PTP($\mathscr{S}$).

Next assume $d \ge 3$. In polar coordinates a vector $u \in S^{d-1}$ may be written as

$$u = (\cos\theta_1, \sin\theta_1\cos\theta_2, \cdots, \sin\theta_1 \cdots \sin\theta_{d-2}\cos\theta_{d-1}, \sin\theta_1 \cdots \sin\theta_{d-1}),$$

where $\theta_1 \in [0, \pi]$ and $\theta_k \in [0, 2\pi]$ for $2 \le k \le d-1$. Define the positive measure Γ on S^{d-1} by

$$\int f(u)\Gamma(du) = \int_0^\pi \int_0^{2\pi} \cdots \int_0^{2\pi} f(\cos\theta_1, \cdots, \sin\theta_1 \cdots \sin\theta_{d-1}) \, d\theta_{d-1} \cdots d\theta_1.$$

Γ is the spectral measure of a spherically symmetric stable γ of index 1. We will construct a positive measure λ on S^{d-1} and a measurable set $A \subseteq S^{d-1}$ which satisfy (i)–(iv) of Theorem 3.11. Let

$$B = \left\{0 \le \theta < \pi/2 : \sin\theta \le 1 - \frac{1}{\sqrt{2}}\right\} \cup \left\{\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}\right\} \quad \text{and} \quad A = \{u \in S^{d-1} : \theta_1 \in B\}.$$

Now $A \cap -A = \phi$ since $u \in A \Rightarrow \theta_1 \in B$ whence $\pi - \theta_1 \notin B$ and therefore $-\mu$, which has first coordinate $-\cos\theta_1 = \cos(\pi - \theta_1)$, is not in A. Define $\lambda(E) = \Gamma(E \cap A)$. Then $\lambda(A^c) = 0$ and $\lambda \le \Gamma$. It remains only to verify (iii).

$$\int_{S^{d-1}} u\lambda(du) = \int_B \int_0^{2\pi} \cdots \int_0^{2\pi} (\cos\theta_1, \sin\theta_1 \cos\theta_2, \cdots, \sin\theta_1 \cdots \sin\theta_{d-1}) \, d\theta_{d-1} \cdots d\theta_1$$

$$= (2\pi)^{d-2} \int_B (\cos\theta_1, 0, \cdots, 0) \, d\theta_1 = 0.$$

Whence, Theorem 3.11 implies that $\gamma \notin \langle \mathrm{PTP}(\mathscr{S})\rangle$. □

There are, however, stables of index 1 which do have PTP($\mathscr{S}$).

3.17 EXAMPLE of a stable of index 1 having PTP($\mathscr{I}$). Let $\Gamma = \sum_{i=1}^{d} a_i\delta_{u_i}$ where $u_1, \cdots, u_d$ are linearly independent unit vectors and $a_1, \cdots, a_d$ are all positive. We will show that there is no non-zero measure λ on S^{d-1} satisfying (i)–(iv) of Theorem 3.11.

Suppose such a λ does exist. Then there are $c_i \geq 0$ with $\lambda = \sum_{i=1}^{d} c_i\delta_{u_i}$. Since, by assumption (iii),

$$0 = \int_{S^{d-1}} u\lambda(du) = \textstyle\sum_{i=1}^{d} c_i u_i,$$

linear independence of $u_1, \cdots, u_d$ implies that $c_1 = \cdots = c_d = 0$. Consequently, $\lambda \equiv 0$, a contradiction. □

The fact that the Levy measure alone governs whether an infinitely divisible law has PTP($\mathscr{I}$), partitions all infinitely divisible laws into two sets of equivalance classes. Let

$$\begin{aligned} \mathscr{H}_d &= \{u \in \tilde{M}_L^+(\mathbb{R}^d) : [0, 0, \mu] \in \langle \text{PTP}(\mathscr{I})\rangle\}, \\ \langle \mathscr{H}_d\rangle &= \{\gamma \in \mathscr{I} : \gamma \sim [a, \Phi, \mu] \text{ with } \mu \in \mathscr{H}_d\} \\ \mathscr{N}_d &= \{\mu \in \tilde{M}_L^+(\mathbb{R}^d) : [0, 0, \mu] \notin \langle \text{PTP}(\mathscr{I})\rangle\}, \\ \langle \mathscr{N}_d\rangle &= \{\gamma \in \mathscr{I} : \gamma \sim [a, \Phi, \mu] \text{ with } \mu \in \mathscr{N}_d\}. \end{aligned}$$

$\langle \mathscr{H}_d\rangle$ and $\langle \mathscr{N}_d\rangle$ are partitioned into equivalence classes via the Levy measures in $\mathscr{H}_d$ and $\mathscr{N}_d$ respectively. Clearly, $\langle \mathscr{H}_d\rangle = \mathscr{I} \cap \langle \text{PTP}(\mathscr{I})\rangle$ and $\langle \mathscr{N}_d\rangle = \mathscr{I} \backslash \langle \text{PTP}(\mathscr{I})\rangle$, both of which are non-empty by our previous examples.

As a result of Theorem 3.5, the Levy s-measures in $\mathscr{Z}_d$ determine which Levy measures lie in $\mathscr{H}_d$ and which in $\mathscr{N}_d$. For instance, Theorem 3.11 characterizes all $\xi \in \mathscr{Z}_d$ which can be used to produce pointwise translates of stables of index 1. This subset of $\mathscr{Z}_d$ consists of $\{(\lambda - \bar{\lambda})(du) \times dr/r^2 : \lambda$ satisfies (i)–(iii) of Theorem 3.11$\}$. In fact, a simple set of properties characterizes all elements of $\mathscr{Z}_d$.

3.18 THEOREM. $\xi \in \mathscr{Z}_d \backslash \{0\}$ iff *there exists a non-zero Levy measure* ζ *on* $\mathbb{R}^d$ *and a measurable set* $A \subset \mathbb{R}^d$ *such that*

i) $A \cap (-A) = \phi$;
ii) $\zeta(A^c) = 0$;
iii) ζ_θ *is symmetric about* 0 *for all* $\theta \in S^{d-1}$;
iv) $\xi = \tilde{\zeta} - (\bar{\tilde{\zeta}})$.

PROOF. Let ζ be a non-zero Levy measure on $\mathbb{R}^d$ which satisfies (i)–(iv). Then ξ is a non-zero Levy s-measure. Applying (i), (ii) and (iv), $\xi_\theta \equiv (\xi^+)_\theta - (\xi^-)_\theta = (\tilde{\zeta})_\theta - (\bar{\tilde{\zeta}})_\theta$ which is 0 by (iii). Thus $\xi \in \mathscr{Z}_d \backslash \{0\}$.

For the converse, let $\xi \in \mathscr{Z}_d \backslash \{0\}$ and $\tilde{\zeta} = \xi^+$. By Proposition 3.6(iii) there is a set $A \subset \mathbb{R}^d$ with $A \cap (-A) = \phi$, $F^{-1}\xi^+(A^c) = 0$ and $F^{-1}\xi^+(-A) = F^{-1}\xi^-(A)$. Therefore, $\zeta(A^c) = 0$ and $\bar{\tilde{\zeta}} = \xi^-$; so ζ and A satisfy (i), (ii) and (iv). By Proposition 3.6(iii), $\xi = \tilde{\zeta} - \bar{\tilde{\zeta}}$ and $\tilde{\zeta} \neq 0$. Hence $\zeta \neq 0$, $(\tilde{\zeta})_\theta - (\bar{\tilde{\zeta}})_\theta = \xi_\theta = 0$ and so for any $E \in \mathscr{C}^1$, $(\bar{\tilde{\zeta}})_\theta(E) = (\bar{\tilde{\zeta}})_\theta(E) = (\bar{\tilde{\zeta}})_\theta(E) = (\tilde{\zeta})_\theta(-E)$, whence (iii) holds. Necessity of (i)–(iv) is therefore proven. □

3.19 REMARK. If ζ is a Levy measure on $\mathbb{R}^d$ such that ζ_θ is symmetric about 0 for all $\theta \in S^{d-1}$, then $\xi \equiv \tilde{\zeta} - \bar{\tilde{\zeta}}$ is in $\mathscr{Z}_d$. It follows that ζ is also non-zero iff $\zeta \neq \bar{\zeta}$. Furthermore, $\zeta = \xi^+$ iff conditions (i) and (ii) hold for a measurable set $A \subset \mathbb{R}^d$. Similarly, (i) and (ii) in Theorem 3.11 can be replaced by $\lambda \neq \bar{\lambda}$.

Theorems 3.5 and 3.18 can be combined.

3.20 COROLLARY. *Let* γ *have Levy representation* $[a, \Phi, \mu]$ *on* $\mathbb{R}^d$. *Then* $\gamma \notin \langle \text{PTP}(\mathscr{I})\rangle$ iff *there exists a non-zero Levy measure* λ *with the following properties*:

i) $\lambda \le \mu$;
ii) *there is a measurable set* $A \subset \mathbb{R}^d\backslash\{0\}$ *with* $A \cap -A = \phi$ *and* $\lambda(A^c) = 0$;
iii) λ_θ *is symmetric about* 0 *for all* $\theta \in S^{d-1}$.

We now identify further properties of elements of $\mathscr{Z}_d$ which may aid in their construction. Some of these properties will subsequently be applied to identify various subclasses of $\langle \mathscr{H}_d \rangle$ and $\langle \mathscr{N}_d \rangle$.

3.21 PROPOSITION (*properties of* $\mathscr{Z}_d$).
i) $\mathscr{Z}_d$ *is a Banach space with* $\| \xi \| = \int_{\mathbb{R}^d} (1 \wedge \| x \|^2) | \xi |(dx)$.
ii) (*Closure under operators.*) *If* A *is an invertible bounded linear operator on* $\mathbb{R}^d$, *then* $A\mathscr{Z}_d = \mathscr{Z}_d$ *where* $A\xi(E) = \xi(A^{-1}E)$ *for* $E \in \mathscr{C}^d$.
iii) $\xi \in \mathscr{Z}_d\backslash\{0\} \Rightarrow F^{-1}\xi^+(\mathbb{R}^d) = \infty = F^{-1}\xi^-(\mathbb{R}^d)$.
iv) $\xi \in \mathscr{Z}_d\backslash\{0\} \Rightarrow \int_{\|x\|\ge 1} \| x \| \, | \xi |(dx) = \infty$.
v) *If* E *is a j-dimensional hyperplane which does not pass through the origin* $(0 \le j \le d-1)$ *and* $\xi \in \mathscr{Z}_d\backslash\{0\}$, *then* $\xi(E) = 0$.

PROOF. (i) Let $\{\xi_n\}$ be a Cauchy sequence in $\mathscr{Z}_d$. By completeness of $\tilde{M}_{Ls}(\mathbb{R}^d)$, it converges to a Levy s-measure ξ. To see that $\xi \in \mathscr{Z}_d$ note that

$$\| \xi_\theta \| = \|(\xi_n)_\theta - \xi_\theta\| = \|(\xi_n - \xi)_\theta \| \le \| \xi_n - \xi \| \to 0.$$

Hence $\xi_\theta = 0$ for all $\theta \in S^{d-1}$.

(ii) If A is an invertible bounded linear operator on $\mathbb{R}^d$ and $\xi \in \mathscr{Z}_d$, then $A\xi \in \tilde{M}_{Ls}(\mathbb{R}^d, \mathscr{C}^d)$. For $E \in \mathscr{C}^1$

$$\begin{aligned}(A\xi)_\theta(E) &= A\xi(x \in \mathbb{R}^d : \langle x, \theta \rangle \in E) = \xi(A^{-1}x : \langle x, \theta \rangle \in E)\\ &= \xi(y : \langle Ay, \theta \rangle \in E) = \xi\left(y : \left\langle y, \frac{A^*\theta}{\| A^*\theta \|}\right\rangle \in E \, \| A^*\theta \|^{-1}\right)\\ &= \xi_{A^*\theta/\|A^*\theta\|}(E \, \| A^*\theta \|^{-1}) = 0\end{aligned}$$

since $\xi \in \mathscr{Z}_d$. Hence $A\xi \in \mathscr{Z}_d$.

For $\zeta \in \mathscr{Z}_d$, $A^{-1}\zeta \in \mathscr{Z}_d$ so $\zeta = A(A^{-1}\zeta) \in A\mathscr{Z}_d$. Thus, $A\mathscr{Z}_d = \mathscr{Z}_d$.

(iii). Let $\xi \in \mathscr{Z}_d\backslash\{0\}$. Suppose $F^{-1}\xi^+(\mathbb{R}^d) < \infty$. By Proposition 3.6(iii), $F^{-1}\xi^-(\mathbb{R}^d) = F^{-1}\xi^+(-\mathbb{R}^d) < \infty$. Moreover, $(F^{-1}\xi^+)_\theta = (F^{-1}\xi^-)_\theta$ on $\mathbb{R}\backslash\{0\}$ and $(F^{-1}\xi^+)_\theta(\{0\}) = F^{-1}\xi^+(x \in \mathbb{R}^d : \langle x, \theta \rangle = 0) = F^{-1}\xi^-(-x \in \mathbb{R}^d : \langle x, \theta \rangle = 0) = (F^{-1}\xi^-)_\theta(\{0\})$. Hence $(F^{-1}\xi^+)_\theta = (F^{-1}\xi^-)_\theta$ on $\mathbb{R}$. Uniqueness of the Radon transform on $\dot{M}(\mathbb{R})$ implies $F^{-1}\xi^+ = F^{-1}\xi^-$. Since F is $1-1$, $\xi^+ = \xi^-$, so that $\xi = 0$, a contradiction. Similarly, $F^{-1}\xi^-(\mathbb{R}^d) = \infty$.

(iv). Let $\xi \in \mathscr{Z}_d\backslash\{0\}$ and suppose, to obtain a contradiction, that $\int_{\|x\|\ge 1} \|x\| \, |\xi|(dx) < \infty$. Let $\mu_1 = F^{-1}\xi^+$ and $\mu_2 = F^{-1}\xi^-$. If X_i, $i = 1, 2$, are random vectors with Levy representation $[0, 0, \mu_i]$, then X_1 is a pointwise but not a vector translate of X_2. Hence, there nonlinear $c(\theta)$ such that for $\theta \in S^{d-1}$, $\mathscr{L}(\langle X_1, \theta \rangle) = \mathscr{L}(\langle X_2, \theta \rangle + c(\theta))$. Our assumption insures that the characteristic function of each $\langle X_i, \theta \rangle$ is differentiable at $t = 0$. Therefore, $E|\langle X_i, \theta \rangle| < \infty$ for $i = 1, 2$ and $\theta \in S^{d-1}$. Hence $c(\theta)$ may be identified as $c(\theta) = E\langle X_1 - X_2, \theta \rangle$ which is linear in θ, a contradiction.

(v). Suppose there exists $\xi \in \mathscr{Z}_d\backslash\{0\}$ such that for some hyperplane E in $\mathbb{R}^d\backslash\{0\}$, $\xi(E) \ne 0$. Then E may be chosen to be maximal in the sense that for any hyperplane H in $\mathbb{R}^d\backslash\{0\}$ with $\dim(H) > \dim(E)$ we have $\xi(H) = 0$. Since $\xi_\theta = 0$ on $\mathscr{C}^1$, it is obvious that $f \equiv \dim(E) \le d - 2$. There are uncountably many $(f + 1)$-dimensional hyperplanes F on $\mathbb{R}^d\backslash\{0\}$ which contain E. For any two such, F_1 and F_2, $F_1\backslash E$ and $F_2\backslash F$ are disjoint and have measure

$$\xi(F_i\backslash E) = \xi(F_i) - \xi(E) = -\xi(E) \ne 0.$$

Hence, $|\xi|$ is not σ-finite, which gives a contradiction. Consequently, $\xi(E) = 0$ for any hyperplane E in $\mathbb{R}^d\backslash\{0\}$. In particular, ξ has no atoms. □

The properties of $\mathscr{Z}_d$ aid in identifying certain subsets of Levy measures which lie in $\mathscr{H}_d$.

3.22 PROPOSITION (*Sufficient conditions for membership in* $\mathscr{H}_d$). *Let* $\mu, \mu_1 \in \tilde{M}_L^+(\mathbb{R}^d)$.

i) $\mu(\mathbb{R}^d) < \infty \Rightarrow \mu \in \mathscr{H}_d$.

ii) μ atomic $\Rightarrow \mu \in \mathscr{H}_d$.

iii) $\mu_1 \in \mathscr{H}_d$ *and* $\mu \leq \mu_1 \Rightarrow \mu \in \mathscr{H}_d$.

iv) $\mu^s \in \mathscr{H}_d \Rightarrow \mu \in \mathscr{H}_d$ *where* $\mu^s(E) \equiv (\mu(E) + \bar{\mu}(E))/2$.

v) $\mu \in \mathscr{H}_d$ *and A an invertible bounded linear operator on* $\mathbb{R}^d \Rightarrow A\mu \in \mathscr{H}_d$ *(in fact,* $A\mathscr{H}_d = \mathscr{H}_d$ *and similarly* $A\mathscr{N}_d = \mathscr{N}_d$*).*

vi) $\int_{\|x\|\geq 1} \| x \| \mu(dx) < \infty \Rightarrow \mu \in \mathscr{H}_d$.

PROOF. To obtain a contradiction for (i)–(iv), suppose that $\mu \notin \mathscr{H}_d$ and therefore there exists $\xi \in \mathscr{Z}_d \backslash \{0\}$ with $\xi^- \leq \tilde{\mu}$.

i) If $\mu(\mathbb{R}^d) < \infty$, then $F^{-1}\xi^-(\mathbb{R}^d) < \infty$, which contradicts Proposition 3.21(iii). Hence $\mu \in \mathscr{H}_d$.

ii) If μ is atomic, then either ξ^- is atomic, contradicting Proposition 3.21(v), or $\xi^- = 0$ which by Proposition 3.6(iii) implies $\xi = 0$, again a contradiction.

iii) If $\mu_1 \in \mathscr{H}_d$ and $\mu \leq \mu_1$, then $\xi^- \leq \tilde{\mu}_1$ contradicting $\mu_1 \in \mathscr{H}_d$.

iv) If $\mu^s \in \mathscr{H}_d$, then $(2^{-1}\xi)^- \leq (\mu^s)^\sim$ contradicting $\mu^s \in \mathscr{H}_d$.

v) Let $\mu \in \mathscr{H}_d$ and let A be an invertible linear operator on $\mathbb{R}^d$.

Let $[0, 0, \nu]$ be a pointwise translate of $[0, 0, A\mu]$. Calculations as in the proof of Proposition 3.21(ii) show that for $E \in \mathscr{B}(\mathbb{R}\backslash\{0\})$,

$$\begin{aligned}(A^{-1}\nu)_\theta(E) &= \nu(y : \langle y, A^{-1*}\theta/\| A^{-1*}\theta \|\rangle \in \| A^{-1*}\theta \|^{-1}E)\\ &= A\mu(y : \langle y, A^{-1*}\theta/\| A^{-1*}\theta\|\rangle \in \| A^{-1*}\theta \|^{-1}E)\\ &= \mu_\theta(E).\end{aligned}$$

Consequently, $[0, 0, A^{-1}\nu]$ is a pointwise translate of $[0, 0, \mu]$. Since $\mu \in \mathscr{H}_d$, it follows that $A^{-1}\nu = \mu$ and therefore $\nu = A\mu$. Hence $A\mu \in \mathscr{H}_d$, so $A\mathscr{H}_d \subseteq \mathscr{H}_d$.

Now if $\mu \in \mathscr{H}_d$ then $A^{-1}\mu \in \mathscr{H}_d$ and thus $\mu = A(A^{-1}\mu) \in A\mathscr{H}_d$. Hence $\mathscr{H}_d = A\mathscr{H}_d$. Similar reasoning shows $A\mathscr{N}_d = \mathscr{N}_d$.

vi) For any random vector Z on $\mathbb{R}^d$, $\langle Z, \theta\rangle = \sum_{j=1}^d \langle\theta, e_j\rangle\langle Z, e_j\rangle$. By linearity of expectations $E\langle Z, \theta\rangle$ is linear in θ whenever $E|\langle Z, e_j\rangle| < \infty$ for each $j = 1, \cdots, d$. Let $X \sim [0, 0, \mu]$ where μ satisfies (vi). Then $E\|X\| < \infty$ as in (iv) of Proposition 3.21. Let Y be a pointwise translate of X. Then there exists a continuous $c(\theta)$ such that $\mathscr{L}(\langle Y, \theta\rangle) = \mathscr{L}(\langle X, \theta\rangle + c(\theta))$. Since $E\|Y\| < \infty$, $c(\theta) = E\langle Y, \theta\rangle - E\langle X, \theta\rangle$ is linear in θ. Thus, $[0, 0, \mu] \in \langle\mathrm{PTP}(\mathscr{I})\rangle$, which yields $\mu \in \mathscr{H}_d$. □

3.23 REMARK. Analogous to (iii), if $\mu_1 \in \mathscr{N}_d$ and $\mu \geq \mu_1$ then $\mu \in \mathscr{N}_d$. The implication is that whenever an infinitely divisible measure γ has a convolution factor $\eta \in \langle\mathscr{N}_d\rangle$, then $\gamma \in \langle\mathscr{N}_d\rangle$ also. In this sense, $\langle\mathscr{N}_d\rangle$ is absorbing. However, $\langle\mathscr{H}_d\rangle$ is not absorbing.

For example, let η_i, $i = 1, 2$, be spherically symmetric stables on $\mathbb{R}^d$ of indices α_1 and α_2 respectively with $0 < \alpha_1 < 1 < \alpha_2 < 2$. By Theorem 3.11, each $\gamma_i \in \langle\mathscr{H}_d\rangle$. There exist positive constants c_i so that the Levy measures μ_i corresponding to η_i are of the form

$$\mu_i(dx) = r^{-\alpha_i-1}\, dr \times c_i\Gamma(du)$$

where Γ is Lebesgue measure on S^{d-1}. Let $0 < c \leq c_1 \wedge c_2$. There exists $\xi \in \mathscr{Z}_d\backslash\{0\}$ such that $d|\xi|(x) = r^{-2}\, dr \times c\Gamma(du)$. If $\gamma = \eta_1 * \eta_2$ then $\mu_\gamma = \mu_1 + \mu_2$, hence $\xi^- \leq \tilde{\mu}_\gamma$. Thus, by Theorem 3.5, $\mu_\gamma \in \mathscr{N}_d$ which implies $\gamma \in \langle\mathscr{N}_d\rangle$.

We have seen that $\langle\mathscr{H}_d\rangle$ and $\langle\mathscr{N}_d\rangle$ are both nonempty. In fact, each collection is sizable.

3.24 PROPOSITION. $\langle\mathscr{H}_d\rangle$ *and* $\langle\mathscr{N}_d\rangle$ *are each weakly dense in* $\mathscr{I}_d$.

PROOF. First we show that $\langle \mathscr{H}_d \rangle$ is dense. Let $\gamma \in \mathscr{I}_d$, so $\gamma \sim [a, \Phi, \mu]$. Define $\gamma_n \sim [a, \Phi, \mu_n]$ where $\mu_n(E) = \mu(E \cap \{x : \|x\| \geq n^{-1}\})$. Each $\gamma_n \in \langle \mathscr{H}_d \rangle$ by (i) of Proposition 3.22. Moreover, $\hat{\gamma}_n(t) \to \hat{\gamma}(t)\ \forall\, t \in \mathbb{R}^d$ so by the Levy Continuity Theorem, $\gamma_n \to \gamma$ weakly.

Next we show that $\langle \mathscr{N}_d \rangle$ is dense. Let $\gamma \in \mathscr{I}_d$, so $\gamma \sim [a, \Phi, \mu]$. We know there exists a non-zero $\zeta \in \mathscr{N}_d$. In view of Remark 3.23, $\gamma_n \sim [a, \Phi, \mu + n^{-1}\zeta] \in \langle \mathscr{N}_d \rangle$. Since $\gamma_n \to \gamma$ weakly, $\langle \mathscr{N}_d \rangle$ is dense in $\mathscr{I}_d$. □

3.25 REMARK. There are examples of $\gamma \notin \mathscr{I}_d$ but such that $\gamma_\theta \in \mathscr{I}_1$ for all $\theta \in S^{d-1}$ (see Linnik-Ostrovskii, 1977). This suggests that $\mathscr{M}_\gamma$ may not be contained in $\mathscr{I}_d$, in which case $\mathscr{I}_d \cap \langle \mathrm{PTP}(\mathscr{I}_d) \rangle \neq \langle \mathrm{PTP}(\mathscr{I}_d) \rangle$.

4. Application to spherically symmetric stable limits. We utilize the results in Section 2 and Theorem 3.11 to refine a limit theorem for affinely normed random vectors in Hahn and Klass (1980b).

Let $X, X_1, X_2, \cdots$ be i.i.d. d-dimensional random vectors with law $\mathscr{L}(X)$ and nth partial sum S_n. Whenever $\{\langle S_n, \theta \rangle\}$ has greatly varied growth rates along at least two different unit directions θ, affine transformations are required to normalize S_n in such a way that the weak limit distribution is *full* (i.e., concentrated on no $(d-1)$-dimensional subspace). See Hahn and Klass (1980a), (1980b), (1981a), (1981b) or Hahn (1979) for further discussion and examples. This situation leads naturally to the following definition: X is said to be in the *generalized domain of attraction* (GDOA) of a full law γ if and only if there exist affine transformations A_n such that $\mathscr{L}(A_n S_n) \to \gamma$.

The GDOA of every spherically symmetric law, which is of necessity spherically symmetric stable, has been characterized.

4.1 THEOREM (*Hahn and Klass* (1980b, *Theorem* 1) *and* (1981a, *pages* 212–217)). *Let* $X, X_1, X_2, \cdots$ *be* i.i.d. *full d-dimensional random vectors. If* $E\|X\| < \infty$ *assume* $EX = \vec{0}$. *Let* $S_n = X_1 + \cdots + X_n$. *Then there exist a full spherically symmetric law* γ, *linear transformations* T_n, *and vectors* v_n *such that*

$$\mathscr{L}(T_n(S_n - v_n)) \to \gamma$$

iff *there exist orthonormal bases* $\{\theta_{n1}, \cdots, \theta_{nd}\}_{n \geq 1}$ *such that*

(A) *there exists* $0 < \alpha \leq 2$ *such that*

$$\lim_{t\to\infty}\sup_{\theta\in S^{d-1}}\left|\frac{t^2 P(\langle X, \theta\rangle > t)}{E(\langle X, \theta\rangle^2 \wedge t^2)} - \frac{2-\alpha}{4}\right| = 0;$$

(B) $$\lim_{n\to\infty}\sup_{\theta\in S^{d-1}}|\, a_n^2(\theta)/(\textstyle\sum_{j=1}^d \langle \theta, \theta_{nj}\rangle^2 a_n^2(\theta_{nj})) - 1\,| = 0$$

where $a_n(\theta) \equiv \sup\{a : c^{-1} n E(\langle X, \theta\rangle^2 \wedge a^2) \geq a^2\}$ *for some* $c > 0$;

and

(C) $$\lim_{n\to\infty}\sup_{\theta\in S^{d-1}}|\, m_n(\theta) - \textstyle\sum_{j=1}^d \langle \theta, \theta_{nj}\rangle m_n(\theta_{nj})|/a_n(\theta) = 0$$

where $m_n(\theta) \equiv nE\langle X, \theta\rangle I_{(|\langle X,\theta\rangle| \leq a_n(\theta))}$.

Moreover, whenever (A)–(C) *hold*, γ *is spherically symmetric stable of index* α. (*This dependence will be denoted by writing* $\gamma^{(\alpha)}$.) *Thus for all unit vectors* θ, $(\gamma^{(\alpha)})\hat{}(t\theta) = \exp(\tilde{c}\,|t|^\alpha)$ *where*

$$\tilde{c} = \begin{cases} \dfrac{c}{2}\dfrac{\Gamma(3-\alpha)}{\alpha-1}\cos\dfrac{\pi\alpha}{2} & \text{if } 0<\alpha<1 \text{ or } 1<\alpha<2 \\[2ex] -\dfrac{c\pi}{4} & \text{if } \alpha = 1 \end{cases}$$

for some $c > 0$. *When the constant c determining* $\tilde{c}$ *is the same as that determining* $a_n(\theta)$

(*see* (B)), *the linear transformations* T_n *and vectors* v_n *may be chosen to satisfy*

(D) $T_n x = \sum_{j=1}^{d} (\langle x, \theta_{nj}\rangle / a_n(\theta_{nj}))e_j$, $\langle v_n, \theta_{nj}\rangle = m_n(\theta_{nj})$ for $j = 1, \cdots, d$.

Furthermore, (A)–(C) *hold* iff *they hold with* θ_{nj} *chosen in the following manner*:

(E) $a_n(\theta_{n1}) \equiv \inf_{\theta \in S^{d-1}} a_n(\theta)$, $a_n(\theta_{n,j+1}) \equiv \inf_{\theta \in \Gamma_j^\perp} a_n(\theta)$ for $j = 1, \cdots, d - 1$, *where* $\Gamma_j^\perp = \{\theta \in S^{d-1} : \langle \theta, \theta_{nk}\rangle = 0$ for $k = 1, \cdots, j\}$.

4.2 REMARKS. Condition (A) expresses the fact that the usual 1-dimensional condition must hold uniformly in every direction. For the standard multivariate normal limit, this condition alone is necessary and sufficient (see Hahn and Klass, 1980a). However, if $0 < \alpha < 2$, Example 4 of Hahn and Klass (1980b) shows that Condition (B) must be assumed. Condition (B) basically insures that at stage n, the 1-dimensional growth rates of $\langle S_n, \theta\rangle$ for various directions θ, as expressed by the 1-dimensional norming constants $a_n(\theta)$ are determined from those along a preferred orthonormal basis $\{\theta_{n1}, \cdots, \theta_{nd}\}$. This is automatically the case when $\alpha = 2$.

Finally, Condition (C) concerns centering. Reflect for a moment on 1-dimensional centering when considering convergence to a 1-dimensional symmetric stable of index α, call it $\lambda^{(\alpha)}$. Now $\lambda^{(\alpha)}$ symmetric implies that $a_\tau = 0$ (see (2.2)). Thus by Condition (2.1)(iii), the 1-dimensional centering constants v_n must satisfy $\lim_{n\to\infty} |v_n - \tilde{m}_n(\tau)| = 0$. Recall that $\tilde{m}_n(\tau) = nE(X/a_n)I_{(|X| \le \tau a_n)}$. This quantity tends to 0 for $\alpha \ne 1$, thus no further centering is needed. However, if $\alpha = 1$, $\tilde{m}_n(\tau)$ may not go to 0, in which case centering is required. These 1-dimensional considerations suggest that for $d \ge 2$ no centering condition is required iff $\alpha \ne 1$.

(In Hahn and Klass (1981), Theorem 5.4, page 210, is incorrectly stated, omitting Condition (C) in the case $\alpha = 1$. This is due to the false claim in the earlier version of this paper that all symmetric stables are in PTP($\mathscr{I}$). See the parenthetical discussion in Remark 2.11. However, Theorem 5.4 contains a correct proof that Condition (B) only needs to be checked for the preferred orthonormal basis constructed in (E). That the same is true for Condition (C) is proved in the Appendix.)

Our proof requires several facts from Hahn and Klass (1980b).

4.3 FACT (pages 72–73). Condition (A) implies both

$$\lim_{n\to\infty} \sup_{\theta \in S^{d-1}} \left| nP(\langle X, \theta\rangle > y a_n(\theta)) - \frac{c}{4}(2 - \alpha) y^{-\alpha} \right| = 0$$

and

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \sup_{\theta \in S^{d-1}} | c^{-1} nE((\langle X, \theta\rangle / a_n(\theta))^2 \wedge \varepsilon^2) - I_{(\alpha=2)} | = 0.$$

4.4 FACT (pages 70–71). Suppose there exist orthonormal bases $\{\theta_{n1}, \cdots, \theta_{nd}\}_{n\ge1}$ and functions $b_n(\theta)$ with $b_n(\theta) = (\sum_{j=1}^{d} \langle \theta, \theta_{nj}\rangle^2 b_n^2(\theta_{nj}))^{1/2}$, such that for any sequence of unit vectors ψ_n,

$$\mathscr{L}(\langle S_n - v_n, \psi_n\rangle / b_n(\psi_n)) \to \lambda^{(\alpha)}).$$

Then

$$\lim_{n\to\infty} \sup_{\|\theta\|=1} | a_n(\theta)/b_n(\theta) - 1 | = 0. \tag{4.5}$$

Moreover, (A) and (B) hold if in addition to (4.5) both

$$\lim_{n\to\infty} \sup_{\|\theta\|=1} \left| nP(\langle X, \theta\rangle > y a_n(\theta)) - \frac{c}{4}(2 - \alpha) y^{-\alpha} \right| = 0$$

and

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \sup_{\|\theta\|=1} \left| \frac{n}{a_n^2(\theta)} \operatorname{Var}(\langle X, \theta\rangle I_{(|\langle X, \theta\rangle| \le \varepsilon a_n(\theta))}) - cI_{(\alpha=2)} \right| = 0.$$

4.6 THEOREM. *If $\alpha \neq 1$, any random vector on $\mathbb{R}^d$ satisfying* (A) *and* (B) *satisfies* (C). *If $\alpha = 1$ and $d > 1$, there exists a random vector on $\mathbb{R}^d$ satisfying* (A) *and* (B) *but not* (C).

PROOF. $\alpha = 2$ is known (see Remark 4.2). First assume X satisfies (A) and (B) with $\alpha \neq 1$ or 2. Defining $T_n x = \sum_{j=1}^d \langle x, \theta_{nj} \rangle e_j / a_n(\theta_{nj})$, (B) implies that

$$\lim_{n\to\infty} \sup_{\|\theta\|=1} | \, \| T_n^{*-1} \theta \| / a_n(\theta) - 1 \, | = 0.$$

Now utilizing in order Fact 4.4 and then Fact 4.3,

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{\theta \in S^{d-1}} \left| n\, P(\langle T_n X, \theta \rangle > y) - \frac{c}{4}(2-\alpha) y^{-\alpha} \right| \\
&= \lim_{n\to\infty} \sup_{\gamma \in S^{d-1}} \left| n\, P(\langle T_n X, T_n^{*-1}\gamma / \| T_n^{*-1}\gamma \| \rangle > y) - \frac{c}{4}(2-\alpha) y^{-\alpha} \right| \\
&= \lim_{n\to\infty} \sup_{\gamma \in S^{d-1}} \left| n\, P(\langle X, \gamma \rangle > y \, \| T_n^{*-1}\gamma \|) - \frac{c}{4}(2-\alpha) y^{-\alpha} \right| \\
&= \lim_{n\to\infty} \sup_{\gamma \in S^{d-1}} \left| n\, P(\langle X, \gamma \rangle > y\, a_n(\gamma)) - \frac{c}{4}(2-\alpha) y^{-\alpha} \right| \\
&= 0.
\end{aligned}$$

Thus, $(\mathrm{I})_\mu$ holds where $\mu_\theta([y, \infty)) = (c/4)(2-\alpha) y^{-\alpha}$. Similarly,

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \sup_{\theta \in S^{d-1}} n\, E(\langle T_n X, \theta \rangle^2 \wedge \varepsilon^2) \\
&\quad = \lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \sup_{\gamma \in S^{d-1}} n\, E(\langle T_n X, T_n^{*-1}\gamma / \| T_n^{*-1}\gamma \| \rangle^2 \wedge \varepsilon^2) \\
&\quad = \lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \sup_{\gamma \in S^{d-1}} n\, E((\langle X, \gamma \rangle / a_n(\gamma))^2 \wedge \varepsilon^2) \\
&\quad = 0.
\end{aligned}$$

Thus, $(\mathrm{II})_{\Phi=0}$ holds.

Since $\gamma^{(\alpha)} \in \langle \mathrm{PTP}(\mathscr{I}) \rangle$ if $\alpha \neq 1$, it follows from Theorem 2.12 that there exist $v_n \in \mathbb{R}^d$ such that $\mathscr{L}(T_n(S_n - v_n)) \to \gamma^{(\alpha)}$. Consequently, by Theorem 4.1, (A)-(C) must hold. Thus, (A) and (B) $\Rightarrow$ (C).

Now consider $\alpha = 1$. Let $\mathscr{L}(X) \sim [0, 0, \Lambda \times dr/r^2]$ and $\gamma^{(\alpha)} \sim [0, 0, \Gamma \times dr/r^2]$ where Λ and Γ are the measures on S^{d-1} defined in Example 3.16. Now, $\mathscr{L}(X) = \mathscr{L}(\sum_{i=1}^n X_i/n)$ and the Levy measures μ_θ for $\langle X, \theta \rangle$ are all continuous. So, by Theorem 2.8 and Remark 2.7,

$$\lim_{n\to\infty} \sup_{\|\theta\|=1} \left| n\, P(\langle X, \theta \rangle > yn) - \frac{c}{4} y^{-1} \right| = 0.$$

and

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \sup_{\|\theta\|=1} \left| n \operatorname{Var}\left(\frac{\langle X, \theta \rangle}{n} I_{(|\langle X, \theta \rangle| \le \varepsilon n)} \right) \right| = 0.$$

By Fact 4.4,

$$\lim_{n\to\infty} \sup_{\|\theta\|=1} | a_n(\theta)/n - 1 | = 0.$$

Thus (A) and (B) hold. It therefore remains to show that (C) fails.

Suppose (C) holds. Then Theorem 4.1 implies the existence of $v_n \in \mathbb{R}^d$ such that

$$\gamma^{(\alpha)} = \lim_{n\to\infty} \mathscr{L}\left(\frac{S_n}{n} - v_n \right) = \lim_{n\to\infty} \mathscr{L}(X - v_n).$$

This can only occur if $\lim_{n\to\infty} v_n$ exists, call it v. Then, $\mathscr{L}(X - v) = \gamma^{(\alpha)}$, which contradicts the fact that $\Lambda \neq \Gamma$. □

5. The pointwise translation problem for signed measures. Heretofore, we have confined our attention to infinitely divisible laws which are pointwise translates of a given (infinitely divisible) probability law. We now expand our investigations to signed measures. For certain classes of signed measures λ we find that $\mathscr{M}_\lambda$ is no larger than the collection of vector translates of λ.

5.1 THEOREM. (i) $\mathscr{M}_{\delta_0} = \{\delta_b : b \in \mathbb{R}^d\}$, *so* $\delta_0 \in \langle \mathrm{PTP}(\delta_0)\rangle$. (ii) *If* γ *has an inverse under convolution in* $M(\mathbb{R}^d)$, *then* $\mathscr{M}_\gamma = \{\gamma * \delta_b : b \in \mathbb{R}^d\}$, *so* $\gamma \in \langle \mathrm{PTP}(\mathscr{M}_\gamma)\rangle$.

PROOF. Part (ii) clearly reduces to (i). Thus we prove (i). The result is trivial for $d = 1$. Suppose $d = 2$ and $\lambda \in \mathscr{M}_{\delta_0}$. There exists a continuous function $c : S^{d-1} \to \mathbb{R}$ such that $\lambda_\theta = \delta_{c(\theta)}$. Note that $\hat{\lambda}(t\theta) = (\lambda_\theta)\hat{}(t) = \int e^{its}\delta_{c(\theta)}(ds) = \exp(itc(\theta))$, whence $(\bar{\lambda})\hat{}(t\theta) = \hat{\lambda}(-t\theta) = \exp(-itc(\theta))$. Therefore, $(\lambda * \bar{\lambda})\hat{} \equiv 1 = \hat{\delta}_0$ which implies $\lambda * \bar{\lambda} = \delta_0$. If λ were continuous it would be impossible to convolve it with another signed measure to obtain a non-zero discrete measure. Hence there exists $b \in \mathbb{R}^2$ such that $\lambda(\{b\}) \neq 0$. Replacing λ by $\lambda * \delta_{-b}$ if necessary, it suffices to assume that $\lambda(\{0\}) \neq 0$.

For each $\theta \in S^{d-1}$, let $A_\theta = \{x \in \mathbb{R}^d \backslash \{0\} : \langle x, \theta\rangle = 0\}$ and put $Q = \{\theta \in S^{d-1} : \lambda(A_\theta) = 0\}$. Since $d = 2$, A_θ is an uncountable family of disjoint sets. By the σ-finiteness of λ, $S^1 \backslash Q$ is at most countable; thus, Q is dense in S^1. For $\theta \in Q$, λ_θ has an atom at zero. But in fact, $\lambda_\theta = \delta_{c(\theta)}$. Thus $c(\theta) = 0$ for $\theta \in Q$. By continuity of c, we may conclude that $c(\theta) \equiv 0$, whence $\hat{\lambda} = 1$ and $\lambda = \delta_0$. The case $d > 2$ follows from the case $d = 2$ by means of the following lemma.

5.2 LEMMA. *Let* $\lambda, \sigma \in M(\mathbb{R}^d)$, $d \geq 2$, *with* $\sigma_\theta \neq 0$ *for every* $\theta \in S^{d-1}$. *Assume that for every one- or two-dimensional subspace* $V \subseteq \mathbb{R}^d$, *there exists* $\beta_V \in V$ *such that* $P_V\lambda = (P_V\sigma) * \delta_{\beta_V}$ (P_V *is the operator denoting perpendicular projection onto* V). *Then* λ *is a translate of* σ.

PROOF. Whenever $V = \mathrm{span}\{\theta\}$, $\theta \in S^{d-1}$, write $\beta_V = c(\theta)\theta$. Define $\beta = c(e_1)e_1 + \cdots + c(e_d)e_d$ and let $K = \{z \in \mathbb{R}^d : c(z/\|z\|) = \langle \beta, z/\|z\|\rangle$ for $z \neq 0\}$. A calculation, together with the observation that the only translation invariant signed measure in $M(\mathbb{R})$ is the zero measure, show that K is a linear subspace. Since $e_j \in K$ for $j = 1, \cdots, d$, $K = \mathbb{R}^d$. This shows that for all $\theta \in S^{d-1}$, $c(\theta) = \langle \beta, \theta\rangle$. Consequently, λ and $\sigma * \delta_\beta$ possess the same Radon transform, so the result follows. □

Many of the signed measures observed in applications are either bounded or have at least a first moment. By linearity of expectations, probability measures with finite mean have $\mathrm{PTP}(M^+(\mathbb{R}^d))$. Perhaps surprisingly, the pointwise translation property can fail to hold for signed measures λ with $\lambda(\mathbb{R}^d) = 0$ even if $\int_{\mathbb{R}^d} |x|^k |\lambda|(dx) < \infty$ for all $k \geq 1$.

5.3 THEOREM. *Let* $\mathscr{E}_k = \{\gamma \in M(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^k |\gamma|(dx) < \infty\}$, *and let* $\mathscr{E}_\infty = \cap_{k=1}^\infty \mathscr{E}_k$.

i) *If* $\gamma \in \mathscr{E}_1$ *with* $\gamma(\mathbb{R}^d) \neq 0$, *then* $\mathscr{E}_1 \cap \mathscr{M}_\gamma = \{\gamma * \delta_b : b \in \mathbb{R}^d\}$ *so that* $\gamma \in \langle \mathrm{PTP}(\mathscr{E}_1)\rangle$.

ii) *There exists a non-zero* $\gamma \in \mathscr{E}_\infty$ *with* $\gamma(\mathbb{R}^d) = 0$ *but* $\mathscr{M}_\gamma \cap \mathscr{E}_\infty \neq \{\gamma * \delta_b : b \in \mathbb{R}^d\}$. *Thus,* $\gamma \notin \langle \mathrm{PTP}(\mathscr{E}_\infty)\rangle$.

iii) *There exists* $\gamma \in \mathscr{E}_\infty$ *and* $\eta \in \mathscr{M}_\gamma \cap \mathscr{E}_\infty$ *such that* $\eta^+ \notin \mathscr{M}_{\gamma^+}$ *and* $\eta^- \notin \mathscr{M}_{\gamma^-}$.

PROOF. (i) Let $\gamma \in \mathscr{E}_1$ with $\gamma(\mathbb{R}^d) \neq 0$ and $\eta \in \mathscr{E}_1 \cap \mathscr{M}_\gamma$. Then $\eta_\theta = \gamma_\theta * \delta_{c(\theta)}$. Direct computation shows that

$$\int_{\mathbb{R}^d} \langle x, \theta\rangle \eta(dx) = \int_{\mathbb{R}^d} \langle x, \theta\rangle \gamma(dx) + c(\theta)\gamma(\mathbb{R}^d)$$

so that

$$c(\theta) = (\gamma(\mathbb{R}^d))^{-1} \int_{\mathbb{R}^d} \langle x, \theta\rangle (\eta - \gamma)(dx).$$

Hence c is linear which implies $\gamma \in \langle \mathrm{PTP}(\mathscr{E}_1)\rangle$.

(iii) follows from (i) and (ii). Just take a γ having the properties in (ii). Since $\gamma \not\equiv 0$ and $\gamma(\mathbb{R}^d) = 0$, $\gamma^+(\mathbb{R}^d) = \gamma^-(\mathbb{R}^d) \neq 0$. Moreover, $\gamma \in \mathscr{E}_1$ so both γ^+ and γ^- are also in $\mathscr{E}_1$. Let $\eta \in \mathscr{M}_\gamma$ be such that $\eta_\theta = \gamma_\theta * \delta_{c(\theta)}$ with c non-linear. Since $\eta_\theta^+ = \gamma_\theta^+ * \delta_{c(\theta)}$ and $\eta_\theta^- = \gamma_\theta^- * \delta_{c(\theta)}$, the nonlinearity of $c(\theta)$ implies that $\eta^+ \notin \mathscr{M}_{\gamma^+}$ and $\eta^- \notin \mathscr{M}_{\gamma^-}$. Hence (iii) holds provided (ii) holds.

The fact that there are examples satisfying property (ii) is a consequence of Example 5.4 below. □

The class of examples to be constructed was derived by simply abstracting the ideas in the construction of a specific example due to Larry Shepp (private communication). The proof of this generalization is essentially identical to Shepp's original proof.

Let $\mathscr{C}^{(j)}$ denote the j times continuously differentiable functions on $[0, 2\pi]$.

5.4 EXAMPLE. Fix $k \in \{1, 2, \cdots, \infty\}$. Let $\psi(w)$ be any real function on $[0, \infty)$ which satisfies

(a) $\psi \in \mathscr{C}^{(k+3)}$;

(b) $\psi^{(i)} \equiv (d^i/dw^i)\psi \in L^1([0, \infty), w\, dw)$ for $i = 0, \cdots, k+3$;

(c) $\psi^{(i)}(0) = 0$ for $i = 0, \cdots, k+1$;

(d) $\psi(w) \not\equiv 0$ on a dense subset of $[0, \infty)$.

Let $\mathscr{C}_\pi^{(j)} = \{c = c(\theta) : 0 \leq \theta \leq 2\pi, c \in \mathscr{C}^{(j)}, c(\theta + \pi) = -c(\theta)\}$. For $c \in \mathscr{C}_\pi^{(k+3)}$ define

$$\psi_c(w, \theta) = \psi(w)e^{-iwc(\theta)}, \qquad w \in [0, \infty), \qquad 0 \leq \theta \leq 2\pi. \tag{5.5}$$

Then there exist two signed measures μ_c and μ_0 on $\mathbb{R}^2$ with finite kth moments for which $\hat{\mu}_c(w\theta) = \psi_c(w, \theta)$ and $\hat{\mu}_0(w\theta) = \psi(w)$. Furthermore, if c is nonlinear then μ_c is not a vector translate of μ_0 even though $(\mu_c)_\theta = (\mu_0)_\theta * \delta_{c(\theta)}$ for every θ. Necessarily $\mu_c(\mathbb{R}^2) = 0$.

PROOF. Throughout, polar coordinates will be denoted by (w, θ) and rectangular coordinates by (x, y) where $x = w\cos\theta$, $y = w\sin\theta$. Notice that if $c \equiv 0$ then $\psi_c = \psi_0 = \psi$. Let

$$f_c(x, y) = \int_0^{2\pi} d\theta \int_0^\infty w\, dw\, \psi_c(w, \theta)e^{iw(x\cos\theta + y\sin\theta)} \tag{5.6}$$

$$= \int_{-\infty}^\infty dy' \int_{-\infty}^\infty dx'\, \psi_c(x', y')e^{i(xx'+yy')}. \tag{5.7}$$

Suppose, for the moment, f_c and f_0 are real in $L^1(\mathbb{R}^2)$. Fourier inversion then implies that ψ_c and ψ_0 are Fourier transforms of signed measures μ_c and μ_0. In fact,

$$\psi_c(x, y) = (2\pi)^{-2} \int_{-\infty}^\infty dy' \int_{-\infty}^\infty dx'\, f_c(x', y')^{e - i(xx'+yy')}$$

so that $d\mu_c(x', y') = (2\pi)^{-2}f_c(-x', -y')dx'dy'$. Equation (5.5) then implies that $(\mu_c)_\theta = (\mu_0)_\theta * \delta_{-c(\theta)}$. Furthermore, if $\|(x, y)\|^k f_c(x, y) \in L^1(\mathbb{R}^2)$, then μ_c has a finite kth moment. Finally, if c is nonlinear then μ_c cannot be a translate of μ_0 because if $g(x, y) = f_0(x - a, y - b)$, then

$$\hat{g}(w, \theta) = \hat{f}_0(w, \theta)e^{-iw(a\cos\theta + b\sin\theta)} = (2\pi)^2\psi(w)e^{-iw(a\cos\theta + b\sin\theta)}$$

and $\psi(w) \neq 0$ on a dense subset of $[0, \infty)$ by (d).

Thus, we must show that (a)–(c) imply that f_c is real and $\|(x, y)\|^k f_c(x, y) \in L^1(\mathbb{R}^2)$. Note that (c) does imply that $\mu_c(\mathbb{R}^2) = \psi_c(0) = 0$ as is necessary.

For all $c \in \mathscr{C}_\pi^{(k+3)}$, f_c is real because f_c can be rewritten as

$$f_c(x, y) = 2\int_0^{\pi} d\theta \int_0^{\infty} w\, dw\, \psi(w) \cos(w(x\cos\theta + y\sin\theta - c(\theta))).$$

For the existence of kth moments of μ, let $1 \le p \le k + 3$. Integrating by parts p times on x' in (5.7) we get for $x \neq 0$,

$$f_c(x, y) = (-1)^p \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx'\, D_1^p \psi_c(x', y') \frac{e^{i(xx'+yy')}}{(ix)^p} \tag{5.8}$$

where $D_1^p = \partial^p/(\partial x')^p$ provided that $D_1^k\psi_c(x', y') \in L^1(\mathbb{R}^2)$ for $k = 0, \cdots, p$. For verification that this does indeed follow from (a)–(c) we need a few observations.

Utilizing the chain rule

$$\frac{\partial}{\partial x'} = \frac{\partial w}{\partial x'}\frac{\partial}{\partial w} + \frac{\partial\theta}{\partial x'}\frac{\partial}{\partial\theta}, \qquad \frac{\partial w}{\partial x'} = \cos\theta, \qquad \frac{\partial\theta}{\partial x'} = -\frac{\sin\theta}{w},$$

a direct calculation shows that $D_1\psi_c = (\partial/\partial x')\psi_c$ is a complex linear combination of ψ and ψ' times $e^{-iwc(\theta)}$ (the coefficients being bounded functions of c, c', $\cos\theta$, $\sin\theta$). In order to obtain $D_1^2\psi_c$, notice that $(\partial w/\partial x')(\partial/\partial w)$ converts ψ and ψ' to ψ' and $\psi^{(2)}$ respectively. Furthermore, $(\partial\theta/\partial x')(\partial/\partial\theta)$ converts ψ and ψ' to terms involving ψ, ψ/w and ψ', ψ'/w respectively.

Inductively, this procedure yields for general $j \ge 1$, that $D_1^j\psi_c$ is a bounded complex linear combination of

$$\begin{cases} \psi/w^k, & k = 0, \cdots, j-1 \quad \text{if} \quad c \not\equiv 0 \\ \text{and} & \\ \psi^{(r)}/w^s, & r = 1, \cdots, j \quad \text{and} \quad s = 0, \cdots, j - r. \end{cases} \tag{5.9}$$

Integrability of $D_1^k\psi_c$ for $k = 0, \cdots, p$ will therefore follow from integrability of the terms in (5.9) when $j = p$. Consequently, consider

$$A_{rs} = \int_0^{2\pi}\int_0^{\infty} |\psi^{(r)}/w^{s-1}|\, dw\, d\theta.$$

Condition (b) implies the inner integral is finite near ∞ for $s = 0$ and hence for $s \ge 1$. The Taylor series expansion

$$\psi^{(r)}(w) = \textstyle\sum_{d=0}^{s-2} w^d\psi^{(r+d)}(0)/d! + w^{s-1}\psi^{(r+s-1)}(\xi)/(s-1)!, \qquad 0 \le \xi \le w$$

and (c) imply finiteness of the inner integral near 0, for $0 \le r \le k + 3$ and $0 \le s \le k + 3 - r$. Thus, utilizing (a) we see that $A_{rs} < \infty$ for $0 \le r \le k + 3$ and $0 \le s \le k + 3 - r$, so (a)–(c) do indeed imply integrability of the terms in (5.9). (Actually, (b)–(c) are equivalent to integrability of the terms in (5.9) for $\psi \in C^{(k+3)}$.)

A similar verification yields

$$f_c(x, y) = (-1)^p \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx'\, D_2^p \psi_c(x', y') \frac{e^{i(xx'+yy')}}{(iy)^p} \tag{5.10}$$

where $D_2^p = \partial^p/(\partial y')^p$.

For $0 \le p \le k + 3$, let

$$M_{p,c} = \max\left(\iint |\psi_c|, \iint |D_1^p\psi_c|, \iint |D_2^p\psi_c|\right).$$

Then from (5.8) and (5.10), we see that

$$|f_c(w, \theta)| = |f_c(x, y)| \le M_{p,c}(1 \wedge |x|^{-p} \wedge |y|^{-p}) \le 2^{p/2}M_{p,c}(1 \wedge w^{-p})$$
$$\equiv N_{p,c}(1 \wedge w^{-p}).$$

Finally, letting $p = k + 3$, $\| (x + y) \|^k f_c \in L^1(\mathbb{R}^2)$ since

$$\int_0^{2\pi} \int_0^{\infty} w^k |f_c(w, \theta)| \, w \, dw \, d\theta \leq N_{k+3,c} \int_0^{2\pi} \int_0^{\infty} w^{k+1}(1 \wedge w^{-k-3}) \, dw \, d\theta$$

$$= N_{k+3,c} \int_0^{2\pi} \left[\int_0^1 + \int_1^{\infty} \right] w^{k+1}(1 \wedge w^{-k-3}) \, dw \, d\theta$$

$$= N_{k+3,c} \int_0^{2\pi} [(k + 2)^{-1} + 1] \, d\theta$$

$$= 2\pi N_{k+3,c}(k + 3)/(k + 2) < \infty.$$

The proof is now complete. □

5.11 REMARK. Shepp's specific example uses $\psi(w) = (e^{-(1/w^2)-w^2})/w$, $k = \infty$, and $c(\theta) = \sin 3\theta$.

6. Applications connected with the Radon transform. The notion of the Radon transform of a function is widely utilized in tomography and partial differential equations. For an element $f \in L^1(\mathbb{R}^d)$, the Radon transform of f is the function $(\theta, t) \to f_\theta(t): S^{d-1} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f_\theta(t) = \int_{\langle y, \theta \rangle = t} f(y) m(dy)$$

where m is the $(d - 1)$-dimensional Lebesgue measure on the hyperplane $\langle y, \theta \rangle = t$ in $\mathbb{R}^d$ (see e.g. Smith, Solomon, and Wagner, 1977). More explicitly, if U_θ is an orthogonal linear transformation on $\mathbb{R}^d$ taking the vector $e_d = (0, 0, \cdots, 0, 1)$ to θ, then since $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$,

$$f_\theta(t) = \int_{\mathbb{R}^d} f(U_\theta(y + te_d)) \, dy.$$

Recall that we defined the Radon transform of $\lambda \in M(\mathbb{R}^d)$ as the mapping $\theta \to \lambda_\theta: S^{d-1} \to M(\mathbb{R})$. Since $(\lambda_\theta)\hat{}(t) = \hat{\lambda}(t\theta)$, unicity of the Radon transform follows from unicity of the Fourier transform. If λ has a signed density $f \in L^1(\mathbb{R}^d)$, then a calculation verifies that $\lambda_\theta(A) = \int_A f_\theta(t) \, dt$. Thus, the Radon transform of a measure is a natural generalization of the Radon transform of a function.

We now proceed to a brief discussion of the physical meaning of Problem 1.4 in the setting of the computerized tomographic (CT) scanner. Inside the unit disc D in the plane is placed a two-dimensional object of non-uniform opacity (density) f to x-rays. From the point p on $\partial D (= S^1)$ a thin beam of x-rays of known intensity I_0 is directed towards a detector at another point q on ∂D. The beam is attenuated from having passed through the x-ray absorbing object; and the intensity $I(pq)$ measured at q is $I_0 \exp(-\int_{pq} f ds)$. $\int_{pq} f ds$ is the line integral of f with respect to arc length over the line segment from p to q. As the line segment pq varies over all parallel lines, $-\ln(I(pq)/I_0)$ gives a projection f_θ of the density f on the line $t\theta$, where θ is a unit vector perpendicular to the parallel segments pq. Thus the x-ray transform permits calculation of the Radon transform of f, which can be inverted to give f (Shepp and Kruskel, 1977).

If one imagines the density of f to be a section through the head or body of an obtunded or uncooperative patient, it is not unrealistic to suspect that the introduction of translation artifacts is a fact of life for radiologists who interpret CT scans made on the basis of the principle outlined in the paragraph above. They recognize such artifacts as streaks indicating non-uniform density where uniform density is expected. Translation by a nonlinear $c(\theta)$ relative to a known density can never be a "real" finding; the densities f are supported on D, so that our discussion of measures with finite first moments applies (Theorem 5.1(i)). (See also Kolwalski and Wagner, 1977.)

The argument in Theorem 5.1 requires that $\int_D f \neq 0$. Signed measures, however, never arise in the conventional applications of the CT scanner. One might imagine an x-ray source within D, such as an α-ray-emitting isotope remaining in the patient's circulation from a previous procedure, contributing to the beam detected at q. However, it is easy to see that this concept leads to a model in which the resulting intensity $I(pq) \neq I(qp)$, so that the result is not a Radon transform of any density at all, even one taking negative values. It is possible to envision a different model in which an emitter produces α-rays proportionally to the number of x-rays impinging on it, generating a chain reaction. The CT scanner would then report $f - g$, where g is a measure of the distribution of emitter. However, we know of no practical application in which this type of interaction occurs, so we omit the details.

Contrasting generalizations of the Radon transform of a measure are possible. Hertle (1980) uses a definition equivalent to setting $\int_{\mathbb{R}^d} f dR\lambda = \int_{S^{d-1}} \int_{\mathbb{R}} f(\theta, t)\, d\lambda_\theta(t) m(d\theta)$, where again m is Lebesgue measure on S^{d-1}. Thus, $R\lambda$ is a single measure on $S^{d-1} \times \mathbb{R}$, rather than a function on S^{d-1} taking values among signed measures on $\mathbb{R}$. Hertle's generalization is natural from the point of view that $R\lambda$ is a function on $S^{d-1} \times \mathbb{R}$, rather than a function from S^{d-1} to functions on $\mathbb{R}$.

We choose to regard the Radon transform as an indexed family of functions or measures. This usage is in keeping with the mechanical operation of CT scanners in which the x-ray source and detector assembly discretely rotates or "indexes" around the patient. Our concept is perhaps more convenient for expressing ideas which pertain to projection on individual directions. Moreover, our definition easily generalizes to signed measures on Banach spaces (see Hahn and Hahn, 1981).

The definition of Radon transform naturally extends to σ-finite measures which do not give infinite mass to any $(d-1)$-dimensional hyperplane. Levy measures are not generally of this form. However, if a Levy measure is considered as defined on $\mathbb{R}^d \backslash \{0\}$ then μ_θ is a Levy measure on $\mathbb{R} \backslash \{0\}$. Hence, the mapping $\theta \to \mu_\theta : S^{d-1} \to M_L^+(\mathbb{R} \backslash \{0\})$ might be called the Radon transform of a Levy measure. Example 3.16 shows that the Radon transform for Levy measures is not unique. This non-uniqueness gives rise to additional counterexamples and to a general "hole theorem" for the Radon transform.

Perry (1977) and Quinto (1981) consider the question: If the Radon transform of a function g is zero on the exterior of a ball centered at the origin, must g be zero outside of that ball? This "hole theorem" is not true in general. If f_Γ and f_Λ are defined as in Example 3.16, then for $u \in S^{d-1}$, $r \in \mathbb{R}$,

$$g(ru) = (f_\Gamma - f_\Lambda)(u) r^{-2}$$

is a non-zero function on $\mathbb{R}^d \backslash \{0\}$ whose Radon transform is zero outside of every ball centered at the origin.

Acknowledgments. We want to thank Marek Kanter for helpful and stimulating discussion at the beginning of our investigation. The examples provided by Alex deAcosta and Larry Shepp together with some questions from the referee furnished the impetus to extend our investigations well beyond our original pursuit. We are grateful to them for their contributions.

APPENDIX

As explained previously in Problem 1.4 and Remarks 2.11 and 4.2, an earlier unpublished version of this paper contained a fundamental error asserting that every infinitely divisible law has PTP($\mathscr{I}$). As we have seen (Example 3.16), this is false. Consequently, several false statements appear in Hahn and Hahn (1981) and in Hahn and Klass (1981b), where reference is made to this result.

Thus, on page 177 of Hahn and Hahn (1981), Case 5 is specifically contradicted by Example 3.16 if $\alpha \neq 1$ and the generalization of Case 4 is therefore false. Similarly, in the preceding paragraph on page 177 the discussion requires $\alpha \neq 1$. Theorem 2.6, Remark 2.7, and Remark 3.3 also rely on Case 4. However, the techniques used in the derivation of these results may be of some independent interest.

In Hahn and Klass (1981b), Theorem 5.4 on page 210 requires a centering condition if $\alpha \neq 1$. Consequently, the correct statement is as in Theorem 4.1 of the current paper. The only omission from the proof of that result is verification that if

$$\text{(III)} \qquad \lim_{n\to\infty} \sup_{\theta \in S^{d-1}} | m_n(\theta) - \textstyle\sum_{j=1}^{d} \langle \theta, \gamma_{nj} \rangle m_n(\gamma_{nj}) | / a_n(\theta) = 0$$

holds for some ONB$\{\gamma_{nj}\}$ then it also holds for the preferred ONB$\{\theta_{nj}\}$ chosen as in (E). The verification is as follows:

By (III) there exists $\varepsilon_n(\theta)$ with $\varepsilon_n \equiv \sup_{\theta \in S^{d-1}} | \varepsilon_n(\theta) | \to 0$ such that

$$m_n(\theta) = \textstyle\sum_{j=1}^{d} \langle \theta, \gamma_{nj} \rangle m_n(\gamma_{nj}) + \varepsilon_n(\theta) a_n(\theta).$$

Thus,

$$\begin{aligned} | m_n(\theta) - \textstyle\sum_{j=1}^{d} \langle \theta, \theta_{nj} \rangle m_n(\theta_{nj}) | &= | \textstyle\sum_{i=1}^{d} \langle \theta, \gamma_{ni} \rangle m_n(\gamma_{ni}) + \varepsilon_n(\theta) a_n(\theta) \\ &\quad - \textstyle\sum_{j=1}^{d} \langle \theta, \theta_{nj} \rangle (\textstyle\sum_{i=1}^{d} \langle \theta_{nj}, \gamma_{ni} \rangle m_n(\gamma_{ni}) + \varepsilon_n(\theta_{nj}) a_n(\theta_{nj})) | \\ &= | \varepsilon_n(\theta) a_n(\theta) - \textstyle\sum_{j=1}^{d} \langle \theta, \theta_{nj} \rangle \varepsilon_n(\theta_{nj}) a_n(\theta_{nj}) | \\ &\leq \varepsilon_n (a_n(\theta) + (d \textstyle\sum_{j=1}^{d} \langle \theta, \theta_{nj} \rangle^2 a_n^2(\theta_{nj}))^{1/2}) \\ &\leq 2d\, \varepsilon_n a_n(\theta) \quad \text{uniformly in } \theta \text{ for all } n \text{ large.} \end{aligned}$$

Consequently, (III) holds with θ_{nj} replacing γ_{nj}.

We also note the following two misprints in the definitions of the norming constants on page 198 of Hahn-Klass (1981b):

Line 15 should read:

$$a_n^2(\theta) = nE(\langle X, \theta \rangle^2 \wedge a_n^2(\theta)).$$

Line 17 should read:

$$2a_n^2(\theta) = \textstyle\sum_{i=1}^{k_n} E(\langle X_{ni}^s, \theta \rangle^2 \wedge a_n^2(\theta)).$$

REFERENCES

FLEMING, W. H. (1965). *Functions of Several Variables.* Addison-Wesley, Reading.

GINÉ, E. and HAHN, M. G. (1982). On stability of probability laws with univariate stable marginals. Unpublished manuscript.

GNEDENKO, B. V. and KOLMOGOROV, A. N. (1968). *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, Reading.

HAHN, M. G. (1978). The generalized domain of attraction of a Gaussian law on Hilbert space. Proceedings of the Second International Conference on Probability in Banach Spaces. *Lecture Notes in Math.* **709** 125–144. Springer-Verlag, Berlin.

HAHN, M. G. and HAHN, P. (1981). The pointwise translation problem for the Radon transform in Banach spaces. Proceedings of Third International Conference on Probability in Banach Spaces. *Lecture Notes in Math.* **860** 176–186. Springer-Verlag, Berlin.

HAHN, M. G. and KLASS, M. J. (1980a). Matrix norming of sums of i.i.d. random vectors in the domain of attraction of the multivariate normal. *Ann. Probability* **8** 262–280.

HAHN, M. G. and KLASS, M. J. (1980b). The generalized domain of attraction of spherically symmetric stable laws on $\mathbb{R}^d$. Proceedings of Conference on Probability on Vector Spaces II, Poland. *Lecture Notes in Math.* **828** 52–81. Springer-Verlag, Berlin.

HAHN, M. G. and KLASS, M. J. (1981a). A survey of generalized domains of attraction and operator norming methods. Proceedings of Third International Conference on Probability in Banach Spaces. *Lecture Notes in Math.* **860** 187–218. Springer-Verlag, Berlin.

HAHN, M. G. and KLASS, M. J. (1981b). The multidimensional central limit theorem for arrays normed by affine transformations. *Ann. Probability* **9** 611–623.

HERTLE, A. (1980). The Radon transform of functions and measures on $\mathbb{R}^d$. Thesis, Erlangen.

KOWALSKI, G. and WAGNER, W. (1977). Artifacts in CT pictures. *Medicamundi* **22** 13–17.

KUELBS, J. (1973). A representation theorem for symmetric stable processes and stable measures on H. *Z. Wahrsch. verw. Gebiete* **26** 259–271.

LINNIK, J. V. and OSTROVSKII, I. V. (1977). Decomposition of random variables and vectors. *Translations of Mathematical Monographs* **48**. American Mathematical Society, Providence, RI.

PERRY, R. M. (1977). On reconstructing a function on the exterior of a disk from its Radon transform. *J. Math. Analysis and Appl.* **59** 324–341.

QUINTO, E. T. (1981). The invertibility of rotation invariant Radon transforms. Unpublished manuscript.

SHEPP, L. A. and KRUSKEL, J. B. (1977). Computerized tomography: the new medical x-ray technology. *Math. Monthly* **85** 420–439.

SMITH, K. T., SOLOMON, D. C., and WAGNER, S. L. (1977). Practical and mathematical aspects of the problem of reconstructing objects from radiographs. *Bull. Amer. Math. Soc.* **83** 1227–1270.

MARJORIE G. HAHN
DEPARTMENT OF MATHEMATICS
TUFTS UNIVERSITY
MEDFORD, MASSACHUSETTS 02155

PETER HAHN, M.D.
DEPARTMENT OF RADIOLOGY
MASSACHUSETTS GENERAL HOSPITAL
FRUIT STREET
BOSTON, MASSACHUSETTS 02114

MICHAEL J. KLASS
DEPARTMENTS OF STATISTICS AND MATHEMATICS
UNIVERSITY OF CALIFORNIA, BERKELEY
BERKELEY, CALIFORNIA 94720