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A CONDITIONED LIMIT THEOREM FOR RANDOM WALK AND BROWNIAN LOCAL TIME ON SQUARE ROOT BOUNDARIES

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We count the number of times a random walk exits from a square root boundary and show that the normalized counting process and the normalized random walk converge jointly in law to a "local time," whose inverse is a stable subordinator of a known index, and a Brownian motion. The study of this limit process leads to some precise sample path properties of Brownian motion. These properties improve earlier results of Dvoretsky and Kahane on the existence of small oscillations in the Brownian path.

1. Introduction. Consider a Brownian motion, B_t, and a random time set M. Think of Y_t as the process obtained by resetting B to zero at times in M, and suppose that M is a regenerative random set, in the sense of Maisonneuve (1974), for Y. A familiar example of such a set is

$$M = \{t \mid B_t = \inf_{s \le t} B_s\}.$$

In this case Lévy (1948) showed that

$$Y_t = B_t - \inf_{s \le t} B(s)$$

is a reflecting Brownian motion and $-\inf_{s \le t} B_s$ is L_t, the local time of Y at 0 (or equivalently the local time of M), and also the functional inverse of a stable subordinator of index ½. Intuitively the excursions of Y are (unsigned) Brownian excursions and when Y would exit from $[0, \infty)$, its local time increases to keep it positive.

Suppose that we replace "when Y would exit from $[0, \infty)$" with "when Y would exit from a moving boundary." A suitable class of boundaries is of the form $I(t) = [c_1\sqrt{t}, c_2\sqrt{t}](-\infty \le c_1 \le 0 \le c_2 \le \infty)$. (Note that $c_1 = 0$, $c_2 = \infty$ gives the case described above.) We would like to construct Y by running B_t until it hits $I(t)$, at which time we reset it to zero and then repeat this procedure. The set M would be the set of "renewal times" for Y and the local time, L, of M would increase an infinitesimal amount at each point in M. Strictly speaking this is nonsense since B_t will exit from $I(t)$ immediately. Nonetheless we will show that by using random walk and an invariance principle it is possible to construct processes Y_t, L_t and a Brownian motion, B_t, such that B_t stays inside the square root boundaries on the intervals of constancy of L (Theorem 13). Although we are unable to construct Y and L directly from B (as Lévy does in the case $(c_1, c_2) = (0, \infty)$), several other properties of the "Lévy case" do hold. Implicit in the above is the fact that there are times t such that $B(t + h) - B(t)$ stays inside $I(h)$ for small h. It has been shown by Kahane (1974) that for large $|c_1|$ and c_2 this is true. We will not use Kahane's results but will refine them by our methods (see Theorems 1, 17 and 19).

We obtain L and Y as the weak limit of discrete processes, L_n, Y_n, defined as described above, but from normalized random walks $S_{[nt]}n^{-1/2}$ which converge weakly to B_t (the heuristic definitions of L and Y are easily made precise in the discrete setting). The

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convergence of $\{L_n\}$ is established by first showing that if

$$T = \min\{n \mid S_n \notin [c_1\sqrt{n}, c_2\sqrt{n}]\}$$

then (Theorem 5)

(1) $\lim_{n\to\infty} P(S_n n^{-1/2} \le y \mid T > n) = Q((-\infty, y])$

(2) $P(T > n) = n^{-\lambda_0(c_1,c_2)}\pi(n)$, where π is slowly varying (i.e., $\lim_{n\to\infty}\pi(tn)\pi(n)^{-1} = 1$, for all $t > 0$). Hence, T is in the domain of attraction of a stable law of index $\min(2, \lambda_0(c_1, c_2))$.

Here $-\lambda_0(c_1, c_2)$ is the largest eigenvalue of

$$A\Psi = -\lambda\Psi, \qquad \Psi(c_i) = 0 \quad \text{if} \quad |c_i| < \infty, \qquad (i = 1, 2),$$

where $A = \frac{1}{2}(d^2/dx^2 - x d/dx)$, $Q(dy) = \Psi_0(y)e^{-y^2/2}\,dy$, and Ψ_0 is the eigenfunction corresponding to $-\lambda_0(c_1, c_2)$. As L_n is the properly normalized number of times the random walk exits from a square root boundary, (2) implies that the functional inverses of L_n converge weakly to a stable subordinator of index $\lambda_0(c_1, c_2)$, assuming $\lambda_0(c_1, c_2) < 1$, and hence L_n converges weakly to the inverse of this subordinator. Results of the form (2) have also been found recently by Novikov (1981 and personal communications). Breiman (1965) proves (1) and a stronger version of (2) with $\pi(n)$ constant in the case $c_1 = -c_2$ under slightly stronger hypotheses than those in Theorem 5. The proof in [6], however, seems to be incomplete since it uses Lemma 9 below without proof and a Tauberian theorem without verification of its hypotheses. A complete proof of (1) and (2) is therefore given in Section 2. We follow Breiman's approach in our use of the eigenfunction expansion of the transition density of the Ornstein-Uhlenbeck process. We also use some estimates from Uchiyama (1980) (in the case $c_1 = -\infty$). Actually, (1) and (2) are established for asymptotically square root boundaries and a partial converse is stated (Theorem 11) which shows that these are essentially the only boundaries for which (1) and (2) can hold. An explicit formula for both Q and $\lambda_0(c_1, c_2)$ is given only in the familiar setting $(c_1, c_2) = (0, \infty)$. In this case $Q(dy) = ye^{-y^2/2}\,dy\ (y > 0)$, $\lambda_0(0, \infty) = \frac{1}{2}$ and our result includes the one-dimensional version of a result of Iglehart (1974) and Bolthausen (1976, Corollary 10).

In Section 3 we show that $(A_n, Y_n)(t)$ converges weakly to a homogeneous Markov process (A, Y), whose transition probabilities are described explicitly in terms of Wiener measure. Here $A_n(t)$, $(A(t))$ is the time since the last renewal of Y_n, (Y). From this follows the joint convergence of $(A_n, Y_n, S_{[n\cdot]}n^{-1/2}, L_n)$ to (A, Y, B, L) (Theorem 13). This theorem is the heart of the paper and suggests two avenues for further study.

One is the limit process (A, Y, B, L) itself. Some properties are established in Theorem 13 and the generalization of the aforementioned results of Lévy are proved in Section 4. Nonetheless a fundamental question that remains unanswered is:

Is L a measurable function of B?

The answer is known only when $(c_1, c_2) = (0, \infty)$ or $(-\infty, 0)$, in which case Y and L are functions of B, as discussed above.

A second topic is the application of Theorem 13 to obtain further information about the random walk. This, unlike the results of Section 5, requires the full strength of Theorem 13 and is pursued in a subsequent paper [14].

In Section 5, the construction of (A, Y, B, L) is used to establish the following theorem on the sample path behaviour of Brownian motion:

THEOREM 1. *Let B be a one-dimensional Brownian motion.*

(a) $$\gamma \equiv \inf_{t\ge 0}\limsup_{h\to 0^+} |B(t+h) - B(t)|\,h^{-1/2} = 1 \quad \text{a.s.}$$

(b) $$\gamma_0 \equiv \inf_{t\ge 0, B(t)=0}\limsup_{h\to 0^+} |B(t+h)|\,h^{-1/2}$$

is a.s. *the unique positive root of*

$$\textstyle\sum_{n=1}^{\infty} (x^2/2)^n((2n-1)n!)^{-1} = 1$$

(*and therefore* $\gamma_0 \approx 1.3069$). □

Note that (a) states that for $c > 1$ there exist times t and $\delta > 0$ such that $|B(t + h) - B(t)| \le c\sqrt{h}$ for $0 \le h \le \delta$, but for $c < 1$ there are no such times. This refines results of Dvoretsky (1963), ($\gamma > 1/4$) and Kahane (1974) ($\gamma < \infty$). Part (b) is also concerned with the existence of times t for which $|B(t + h) - B(t)| \le c\sqrt{h}$ ($0 \le h \le \delta$), only now t must be in the zero set B. This improves a result of Kahane (1976) ($0 < \gamma_0 < \infty$). In fact we prove versions of (a) and (b) which hold for asymmetric square root boundaries (Theorems 17 and 19).

Finally, let us mention that our original proofs of Theorems 1 and 13 used nonstandard analysis. The intuitive description of the construction of L from the Brownian path becomes a rigorous definition when one considers the Brownian motion as the standard part of a random walk with infinitesimal step size (see Anderson, 1976). This nonstandard viewpoint is used in the appendix, where the tightness of $(A_n, Y_n, S_{[n\cdot]}n^{-1/2}, L_n)$ is established.

2. A Conditioned Limit Theorem. Throughout most of this work we assume the following hypotheses:

(H) (i) $\{X_i \mid i \in N\}$ *are* i.i.d. *random variables such that* $E(X_i) = 0$, $E(X_i^2) = 1$, *and* $E(X_i^2 \log^+(X_i)) < \infty (\log^+ x = \max(0, \log(x))$.

(ii) $f_i : N \cup \{0\} \to [-\infty, \infty]$, $i = 1, 2$ *satisfy* $f_1 \le 0 \le f_2$ *and* ***one*** *of the following* 3 *conditions*:

(H_a) $f_1(n)$ *is non-increasing for large* n, $f_2(n)$ *is nondecreasing for large* n *and* $\lim_{n\to\infty} f_i(n)n^{-1/2} = c_i$, *where* $-\infty < c_1 < c_2 < \infty$.

(H_b) $\lim_{n\to\infty} f_2(n)n^{-1/2} = c_2 < \infty$ *and* $f_1 \equiv -\infty \equiv c_1$

(H_c) $\lim_{n\to\infty} f_1(n)n^{-1/2} = c_1 > -\infty$ *and* $f_2 \equiv +\infty \equiv c_2$.

(iii) *If* $S_n = \sum_{i=1}^n X_i$, *and*

$$T = \min\{n \mid S_n \notin [f_1(n), f_2(n)]\},$$

then $P(T < n) > 0$ *for each* $n \in N$.

In this section we wish to establish (1) and (2). Define

$$Q_n(B) = P(S_n n^{-1/2} \in B \mid T > n) \quad \text{for } B \text{ a Borel set.}$$

An Ornstein-Uhlenbeck process $U(t)$ is a diffusion with scale function $s(x) = \int_0^x e^{y^2/2}\,dy$ and speed measure $m(dx) = 2e^{-x^2/2}\,dx$. Such a process may be obtained from a one dimensional Brownian motion, denoted always by $B(t)$, by

$$U(t) = e^{-t/2}B(e^t - 1), \qquad (t \ge 0)$$

(see Knight [21, page 96–98]). Square root boundaries for B become constant boundaries for U. The notation P_x is used to denote a measure for which $P_x(B(0) = x) = P_x(U(0) = x) = 1$, and

$$\rho(c_1, c_2) = \inf\{t \mid U(t) \notin [c_1, c_2]\}.$$

We now collect a few results from the theory of o.d.e.'s.

PROPOSITION 2. *Let* $A = \frac{1}{2}(d^2/dx^2 - x\,d/dx)$ *and consider the Sturm-Liouville equation*

$$A\Psi = -\lambda\Psi \quad \text{on} \quad (c_1, c_2), \qquad \Psi(c_i) = 0 \quad \text{if} \quad |c_i| < \infty, \tag{3}$$

where $-\infty \le c_1 < c_2 \le \infty$ *and* $\min(|c_1|, |c_2|) < \infty$.

(a) *There is a sequence of simple eigenvalues* $0 \ge -\lambda_0(c_1, c_2) > \cdots > -\lambda_n(c_1, c_2) > \cdots$ *whose corresponding eigenfunctions* $\{\psi_n(c_1, c_2)\}$ *form a complete orthonormal system with respect to* $m(dx)$.

(b) $\lambda_1(-\infty, c_2) > \lambda_0(-\infty, c_2) + \frac{1}{2}$

(c) $\lambda_0(c_1, c_2)$ *is strictly positive, jointly continuous on*

$$C = \{(c_1, c_2) \in [-\infty, \infty]^2 \mid c_1 < c_2, \min(|c_1|, |c_2|) < \infty\},$$

strictly increasing in $c_1 \in (-\infty, c_2]$ *for* $c_2 \leq \infty$, *and strictly decreasing in* $c_2 \in [c_1, \infty)$, *for* $c_1 \geq -\infty$. *Also,*

$$\lim_{(c_1,c_2)\to 0}\lambda_0(c_1, c_2) = \infty, \qquad \lim_{(c_1,c_2)\to(-\infty,\infty)}\lambda_0(c_1, c_2) = 0,$$

$$\lambda_0(-1, 1) = 1, \qquad \lambda_0(-\infty, 0) = \tfrac{1}{2}, \qquad \psi_0(0, \infty, x) = (2\pi)^{-1/4}x.$$

(d) *We may (and shall) choose* $\psi_0(c_1, c_2)$ *such that for each* $\varepsilon > 0$,

$$\inf\{\psi_0(c_1, c_2, x) \mid x \in (c_1 + \varepsilon, c_2 - \varepsilon)\} > 0.$$

PROOF. (a) If $\max(|c_1|, |c_2|) < \infty$, this is a classical result which may be found in Coddington and Levinson [8, Chapter 7] (see also Tricomi [35, pages 117–119, 124]). If $\max(|c_1|, |c_2|) = \infty$, the result is in the appendix of Uchiyama [36].

(b) See Proposition 1.1 of Uchiyama [36].

(c) Recall that A is the infinitesimal generator of the Ornstein-Uhlenbeck process, $U(t)$, killed at $\{c_1, c_2\}$ (see Knight [21, Theorem 4.3.3] for a precise description of the domain of the generator). Let $R_\alpha^{c_1,c_2}$ denote the resolvent of this killed process ($\alpha \geq 0$). If $\max(|c_1|, |c_2|) < \infty$ and f is continuous on $[c_1, c_2]$, then

$$R_0^{c_1,c_2}f(x) = \int_{c_1}^{c_2} G_0(c_1, c_2)(x, y)f(y)m(dy),$$

where

$$G_0(c_1, c_2)(x, y) = \begin{cases} (s(x) - s(c_1))(s(c_2) - s(y))(s(c_2) - s(c_1))^{-1} & \text{if } c_1 \leq x \leq y \leq c_2 \\ G_0(c_1, c_2)(y, x) & \text{if } c_1 \leq y \leq x \leq c_2 \\ 0 & \text{if } (x, y) \notin [c_1, c_2]^2 \end{cases}$$

(Knight [21, Lemma 4.3.4]). Note that

$$\lim_{c_1\to-\infty} G_0(c_1, c_2)(x, y) = \begin{Bmatrix} s(c_2) - s(x \vee y) & \text{if } x \vee y \leq c_2 \\ 0 & \text{otherwise} \end{Bmatrix} \equiv G_0(-\infty, c_2)(x, y),$$

$$\lim_{c_2\to+\infty} G_0(c_1, c_2)(x, y) = \begin{Bmatrix} s(x \wedge y) - s(c_1) & \text{if } x \wedge y \geq c_1 \\ 0 & \text{otherwise} \end{Bmatrix} \equiv G_0(c_1, \infty)(x, y),$$

$$\lim_{(c_1,c_2)\to(c,c),c_1<c_2} G_0(c_1, c_2)(x, y) = 0 \equiv G_0(c, c)(x, y).$$

For each (x, y), $G_0(c_1, c_2)(x, y)$ is strictly decreasing in $c_1 \in [-\infty, c_2]$ and strictly increasing in $c_2 \in [c_1, \infty]$. A direct integration shows that $G_0(c_1, c_2) \in L^2(m \times m)(\min(|c_1|, |c_2|) < \infty)$. Therefore

$$G_0: \{(c_1, c_2) \in [-\infty, \infty]^2 \mid c_1 \leq c_2, \quad \min(|c_1|, |c_2|) < \infty\} \to L^2(m \times m)$$

is continuous by the monotonicity established above and the dominated convergence theorem. If $f \geq 0$ is bounded and continuous on $\mathbb{R}$, then

$$R_0^{c_1,c_2}f(x) = \int_{c_1}^{c_2} G_0(c_1, c_2)(x, y)f(y)m(dy)$$

holds for all (c_1, c_2) in C by the monotone convergence theorem. Therefore we may consider $R_0^{c_1,c_2}$ as a Hilbert-Schmidt operator on $L^2(m)$. As the eigenvalues of $R_0^{c_1,c_2} = -A^{-1}$ are $\{\lambda_k(c_1, c_2)^{-1}\}$, we see that $\| R_0^{c_1,c_2} \| = \lambda_0(c_1, c_2)^{-1}$. The strict positivity of λ_0 on C is immediate. The continuity of λ_0 on C follows from

$$\| R_0^{c_1,c_2} - R_0^{c_1',c_2'} \| \leq \| G_0(c_1, c_2) - G_0(c_1', c_2') \|_{L^2(m\times m)}.$$

Since $\lim_{(c_1,c_2)\to 0}\| R_0^{c_1,c_2} \| = 0$, we see that $\lim_{(c_1,c_2)\to 0}\lambda_0(c_1, c_2) = \infty$. Also

$$\lim_{(c_1,c_2)\to(-\infty,\infty)}\lambda_0(c_1, c_2) \leq k \lim_{(c_1,c_2)\to(-\infty,\infty)}\| R_0^{c_1,c_2}(1) \|_{L^2(m)}^{-1} = 0 \qquad (k \in (0, \infty)).$$

Breiman [6, Theorem 1] shows $\lambda_0(-1, 1) = 1$, Uchiyama [36, Proposition 1.1] shows $\lambda_0(-\infty, 0) = ½$, and $\psi_0(0, \infty, x) = (2\pi)^{-1/4}x(x \geq 0)$ follows from a short computation.

Suppose that $-\infty < c_1 < c_1' < c_2 \leq \infty$. Then if $\psi_0(c_1', c_2)$ is 0 on $(-\infty, c_1']$,

$$\begin{aligned}\lambda_0(c_1', c_2)^{-2} &= \| R_0^{c_1', c_2}\psi_0^{c_1', c_2}\|^2_{L^2(m)} \\ &= \int_{c_1}^{c_2}\left[\int_{c_1'}^{c_2} G_0(c_1', c_2)(x, y)\psi_0(c_1', c_2, y)m(dy)\right]^2 m(dx) \\ &< \lambda_0(c_1, c_2)^{-2}\end{aligned}$$

by the strict monotonicity of $G_0(\cdot, c_2)(x, y)$.

This shows that $c_1 \to \lambda_0(c_1, c_2)$ is strictly increasing on $(-\infty, c_2]$. Similarly $c_2 \to \lambda_0(c_1, c_2)$ is strictly decreasing on $[c_1, \infty)$ for $c_1 \geq -\infty$.

(d) If $\max(|c_1|, |c_2|) < \infty$, the result follows from Theorem 2.1 of Chapter 8 in Coddington and Levinson [8]. Assume therefore that $c_1 = -\infty$, $c_2 < \infty$. Theorem 1.1 of Uchiyama [36] shows that $\psi_0(-\infty, c_2) \geq 0$. A calculus argument shows that $d/dx(\psi_0(-\infty, c_2, x)) < 0$ for x large negative. Therefore if the statement of (d) is false there is an $x_0 < c_2$ such that $\psi_0(-\infty, c_2, x_0) = 0$ and $\psi_0(-\infty, c_2, x) > 0$ for $x < x_0$. Therefore $\psi_0(-\infty, c_2) \in L^2(m)$, $A\,\psi_0(-\infty, c_2)(x) = -\lambda_0(-\infty, c_2)\psi_0(-\infty, c_2, x)$ on $(-\infty, x_0)$, $\psi_0(-\infty, c_2, x_0) = 0$. By (c), $\lambda_0(-\infty, x_0) > \lambda_0(-\infty, c_2)$ and hence $\psi_0(-\infty, x_0)$ and $\psi_0(-\infty, c_2)$ are orthogonal with respect to m on $(-\infty, x_0)$ since they are eigenfunctions of the same Sturm-Liouville equation, with different eigenvalues. As $\psi_0(-\infty, c_2) > 0$ on $(-\infty, x_0)$, this implies $\psi_0(-\infty, x_0) = 0$, a contradiction. □

NOTATION. $\theta(c_1, c_2) = \int_{c_1}^{c_2} \psi_0(c_1, c_2, x)m(dx)$.

$\to_w$ denotes weak convergence.

Although the following Lemma is well-known, we sketch a proof for the sake of completeness. In what follows we suppress dependence on (c_1, c_2) whenever possible.

LEMMA 3. (a) *Let $q(t, x, y)$ denote the transition density (with respect to $m(dy)$) of an Ornstein-Uhlenbeck process U killed at $\{c_1, c_2\}$. If $t > 0$ is fixed, then*

$$q(t, x, y) = \textstyle\sum_{k=0}^{\infty} e^{-\lambda_k t}\psi_k(x)\psi_k(y),$$

where the series converges absolutely, uniformly for (x, y) in compact subsets of $[c_1, c_2]^2$. The convergence also holds in $L^2((c_1, c_2)^2, m(dx)m(dy))$.

(b) *If $t \in (0, 1)$, then*

$$\int_{c_1}^{c_2} P_x(\rho > \log t^{-1})\psi_0(x)m(dx)\theta^{-1} = t^{\lambda_0}.$$

PROOF. (a) We only consider the case when $c_1 = -\infty$ (see Itô and McKean [17, Chapter 4.11] for the case when $[c_1, c_2]$ is bounded) and suppress dependence on c_2. Let R_α and $G_\alpha(x, y)$ (respectively $R_\alpha^{(n)}$ and $G_\alpha^{(n)}(x, y)$) denote the resolvent and Green's function for an Ornstein-Uhlenbeck process killed at $\rho \equiv \rho(-\infty, c_2)$ (respectively, $\rho(-n, c_2)$). That is, if $f \geq 0$ is defined on $(-\infty, c_2]$,

$$R_\alpha f(x) = E_x\left(\int_0^\rho f(U(t))e^{-\alpha t}\,dt\right) = \int_{-\infty}^{c_2} G_\alpha(x, y)f(y)m(dy).$$

Clearly $G_\alpha^{(n)}(x, y)$ is non-decreasing in n for each (x, y) and it follows easily that $\lim_{n\to\infty} G_\alpha^{(n)}(x, y) = G_\alpha(x, y)$. Using the explicit formula for $G_\alpha^{(n)}$ given by Itô and McKean [17, page 128–129], it is easy to see that G_α is continuous and symmetric. Moreover, $G_\alpha \in L^2(m \times m)$ because $G_\alpha \leq G_0$ and $G_0 \in L^2(m \times m)$ (see the proof of Proposition 1). The eigenvalues of R_α are $\{(\alpha + \lambda_k)^{-1}\}$ and the corresponding eigenfunctions are $\{\psi_k\}$ (recall

$R_\alpha = (\alpha - A)^{-1}$). Although Mercer's theorem (Riesz-Nagy [39, page 245]) applies to finite intervals, one can use the transformation $y = m([-\infty, x])$ to transform $L^2((-\infty, c_2], m)$ into $L^2([0, m(-\infty, c_2]], dz)$ and apply virtually the same arguments as in the proof of Mercer's Theorem in Riesz-Nagy [39] to see that

$$G_\alpha(x, y) = \sum_{k=0}^{\infty} (\alpha + \lambda_k)^{-1}\psi_k(x)\psi_k(y),$$

where the series converges absolutely, uniformly on compact sets, and in $L^2((c_1, c_2)^2, m \times m)$. If $t > 0$, let

$$q'(t, x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t}\psi_k(x)\psi_k(y),$$

where the series converges in the same sense as the previous one since $e^{-\lambda_k t} \le (1 + \lambda_k)^{-1}$ for large k. Taking Laplace Transforms in t and noting that the absolute convergence allows us to interchange summation and integration, we see that

$$\int_0^\infty e^{-\alpha t} q'(t, x, y)\, dt = G_\alpha(x, y).$$

This identifies $q'(t, x, y)$ as the transition density with respect to $m(dy)$ of $U(t)$, killed at $\{c_2\}$, and completes the proof of (a).

(b) $$\int_{c_1}^{c_2} P_x(\rho > \log t^{-1})\psi_0(x)m(dx)\theta^{-1}$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} t^{\lambda_k} \int_{c_1}^{c_2}\int_{c_1}^{c_2} \psi_k(x)\psi_k(y)\psi_0(x)m(dx)m(dy)\theta^{-1},$$

by the L^2 convergence in (a),

$$= t^{\lambda_0}\int_{c_1}^{c_2} \psi_0(y)m(dy)\theta^{-1} = t^{\lambda_0}. \square$$

The next proposition now follows easily from some estimates of Uchiyama [36, Theorem 1.1].

PROPOSITION 4. (a) *If $y, c_2 \in \mathbb{R}$ and $\varepsilon > 0$, then*

$$\lim_{t\to\infty} \sup_{-\varepsilon^{-1}t^{1/2} \le x \le c_2-\varepsilon} \Big| P_x(U(t) \le y \,|\, \rho(-\infty, c_2) > t) - \int_{-\infty}^{y} \psi_0(-\infty, c_2, z)m(dz)\theta(-\infty, c_2)^{-1} \Big| = 0.$$

(b) *If $c_1 < c_2$, $\max(|c_1|, |c_2|) < \infty$, $\varepsilon > 0$ and $y \in \mathbb{R}$, then*

$$\lim_{t\to\infty} \sup_{c_1+\varepsilon \le x \le c_2-\varepsilon} \Big| P_x(U(t) \le y \,|\, \rho(c_1, c_2) > t) - \int_{-\infty}^{y} \psi_0(c_1, c_2, z)m(dz)\theta(c_1, c_2)^{-1} \Big| = 0.$$

PROOF. (a) By Theorem 1.1 of Uchiyama [36], for each $\delta > 0$ there is a $c_\delta > 0$ such that

(4) $$P_x(\rho(-\infty, c_2) > t) = e^{-\lambda_0(-\infty, c_2)t}(\theta(-\infty, c_2)\psi_0(-\infty, c_2, x) + r(t, x)),$$

where

$$|r(t, x)| \le c_\delta e^{-(\lambda_1-\lambda_0)(-\infty, c_2)t} e^{\delta x^2}.$$

Lemma 3 and (4) give us

$$P_x(U(t) \le y \mid \rho > t) = \int_{-\infty}^{y} \psi_0(z)m(dz)(\theta + r(t, x)\psi_0(x)^{-1})^{-1}$$

(5)

$$+ \int_{-\infty}^{y} (\textstyle\sum_{n=1}^{\infty} e^{-\lambda_n t} \psi_n(x)\psi_n(z))m(dz)e^{\lambda_0 t}(\theta\psi_0(x) + r(t, x))^{-1}.$$

As in the proof of Theorem 1.1 in Uchiyama [36],

$$\int_{-\infty}^{c_2} |\textstyle\sum_{n=1}^{\infty} e^{-\lambda_n t}\psi_n(x)\psi_n(z)|m(dz) \le c_\delta e^{-\lambda_1 t}e^{\delta x^2}.$$

Fix $\varepsilon > 0$ and choose $\delta > 0$ so that $\delta\varepsilon^{-2} < \frac{1}{2}$. Then for large t,

$$\sup_{-\varepsilon^{-1}t^{1/2} \le x \le c_2 - \varepsilon} \int_{-\infty}^{c_2} |\textstyle\sum_{n=1}^{\infty} e^{-\lambda_n t}\psi_n(x)\psi_n(z)|m(dz)e^{\lambda_0 t}(\theta\psi_0(x) + r(t, x))^{-1}$$

$$\le K \exp\{-(\lambda_1 - \lambda_0)t + \delta\varepsilon^{-2}t\}((\inf_{-\varepsilon^{-1}t^{1/2} \le x \le c_2 - \varepsilon} \theta\psi_0(x) - c_\delta \exp\{-(\lambda_1 - \lambda_0 - \delta\varepsilon^{-2})t\})^+)^{-1}$$

$$\le K' \exp\{-t(\tfrac{1}{2} - \delta\varepsilon^{-2})\}; \text{ (Proposition 2(b), (d)).}$$

Similarly

$$\sup_{-\varepsilon^{-1}t^{1/2} \le x \le c_2 - \varepsilon} |r(t, x)|\psi_0(x)^{-1} \le K'' \exp\{-t(\tfrac{1}{2} - \delta\varepsilon^{-2})\}.$$

The result follows by using these estimates in (5).

(b) Following the proof of Theorem 1.1 in Uchiyama [36], one obtains (for $\max(|c_1|, |c_2|) < \infty$).

$$P_x(\rho > t) = e^{-\lambda_0 t}(\theta\psi_0(x) + r(t, x)), \tag{6}$$

where

$$|r(t, x)| \le c \exp\{-(\lambda_1 - \lambda_0)t\}.$$

The proof now proceeds as in (a). □

We are ready to state the main result of this section.

THEOREM 5. *Assume the hypotheses* (H). *Recall that* $Q_n(A) = P(S_n n^{-1/2} \in A \mid T > n)$.

(a) $Q_n \to_w Q_\infty$, *where*

$$Q_\infty(B) = \int_B \psi_0(c_1, c_2, y)m(dy)\theta(c_1, c_2)^{-1}.$$

(b) $P(T > n) = n^{-\lambda_0(c_1, c_2)}\pi(n)$, *where* π *is slowly varying.*

We first prove a sequence of lemmas. The hypotheses (H) *of Theorem* 5 *are assumed in Lemmas* 6 *to* 9.

NOTATION. (i) If $m \le n$, let

$$P_x(m, n, y) = P(S_n n^{-1/2} \le y, \quad S_k \in [f_1(k), f_2(k)] \text{ for } k = m, \cdots, n \mid S_m m^{-1/2} = x)$$

$$Q_x(m, n, y) = P_x(m, n, y)P_x(m, n, \infty)^{-1}.$$

We sometimes write $P_x(m, n, A)$ for $\int_A P_x(m, n, dy)$.

(ii) If $f: \mathbb{N} \cup \{0\} \to \mathbb{R}$ and $x > 0$, let

$$T_f = \min\{n \mid S_n > f(n)\}, \qquad f^{(x)}(t) = f([xt])x^{-1/2}, \qquad t \ge 0.$$

In particular we have $S^{(n)}(t) = S_{[nt]}n^{-1/2}$.

(iii) If L and M are topological spaces, $C(L, M)$ denotes the space of continuous functions from L to M with the compact-open topology.

LEMMA 6. *If $t \in (0, 1)$, then*

$$\lim_{n\to\infty}\sup_{-\infty\le x\le\infty}| P_x([nt], n, \infty) - P_x(\rho(c_1, c_2) > \log t^{-1}) | = 0$$

and for each $M > 0$,

$$\lim_{n\to\infty}\sup_{-\infty\le x\le\infty, |y|\le M}| P_x([nt], n, y) - P_x(U(\log t^{-1}) \le y, \rho(c_1, c_2) > \log t^{-1}) | = 0.$$

PROOF.

$$\begin{aligned} P_x([nt], n, y) = P(&x[nt]^{1/2}n^{-1/2} + S_{n-[nt]}n^{-1/2} \le y,\\ &x[nt]^{1/2}n^{-1/2} + S_j n^{-1/2} \in [f_1(j + [nt])n^{-1/2},\\ &f_2(j + [nt])n^{-1/2}], j = 0, \cdots, n - [nt]) \end{aligned}$$

and

$$\begin{aligned} &P_x(U(\log t^{-1}) \le y, \quad \rho(c_1, c_2) > \log t^{-1})\\ &\quad = P_x(B(t^{-1} - 1)t^{1/2} \le y, \quad B(v) \in [c_1(v + 1)^{1/2}, \quad c_2(v + 1)^{1/2}]\forall v \le t^{-1} - 1),\\ &\qquad \text{since} \quad U(s) = B(e^s - 1)e^{-s/2},\\ &\quad = P_{x\sqrt{t}}(B(1 - t) \le y, \quad B(v) \in [c_1(v + t)^{1/2}, c_2(v + t)^{1/2}]\forall v \le 1 - t), \end{aligned}$$

the last by scaling. Now the lemma is almost an immediate consequence of Donsker's Theorem (i.e., $S^{(n)} \to_w B$). To obtain the uniformity in (x, y), let $n_k \uparrow \infty$, $x_k \to x \in [-\infty, \infty]$ and $y_k \to y \in \mathbb{R}$. If f: $\mathbb{N} \cup \{0\} \to \mathbb{R}$ define $\bar{f}^{(n)}$: $[0, \infty) \to \mathbb{R}$ by setting $\bar{f}^{(n)}(k/n)\ f^{(n)}(k/n)$ and then interpolating $\bar{f}^{(n)}$ polygonally. If $x'_k = x_k[n_k t]^{1/2}n_k^{-1/2}$, then

$$x'_k + \bar{S}^{(n_k)} \to_w xt^{1/2} + B \quad \text{on} \quad C([0, 1], [-\infty, \infty]).$$

By Skorohod [30, page 10] and Dudley [10, Theorem 3] we may redefine the $\{\bar{S}^{(n_k)}\}$ on a new probability space so that

$$x'_k + \bar{S}^{(n_k)} \to xt^{1/2} + B \quad \text{a.s. on} \quad C([0, 1], [-\infty, \infty]).$$

By the above,

$$P_x(U(\log t^{-1}) \le y, \rho(c_1, c_2) > \log t^{-1}) = P(xt^{1/2} + B \in A),$$

where

$$\begin{aligned} A = \{w \in C([0, 1], [-\infty, \infty]) \,|\, &w(1 - t)\\ &\le y, \quad w(v) \in [c_1(v + t)^{1/2}, \quad c_2(v + t)^{1/2}]\forall v \le 1 - t\}, \end{aligned}$$

and

$$P_{x_k}([n_k t], n_k, y_k) = P(x'_k + \bar{S}^{(n_k)} \in A_k),$$

where

$$\begin{aligned} A_k = \{w \in C([0, 1], [-\infty, \infty]) \,|\, &w((n_k - [n_k t])n_k^{-1})\\ &\le y_k, w(v) \in [g_1^k(v), g_2^k(v)]\forall v \le (n_k - [n_k t])n_k^{-1}\} \end{aligned}$$

and

$$g_i^k(v) = \bar{f}_i^{(n_k)}((n_k v + [n_k t])n_k^{-1}).$$

Since $\lim_{k\to\infty} g_i^k(v) = c_i(v + t)^{1/2}$ uniformly on compacts, by using elementary properties of Brownian motion one can show that

$$\lim_{k\to\infty} I_{A_k}(x'_k + \bar{S}^{(n_k)}) = I_A(xt^{1/2} + B) \quad \text{a.s.}$$

(notice that if $x = -\infty$ and $c_1 = -\infty$, then $g_i^k \equiv -\infty$ by $(\mathrm{H_b})$). Therefore

$$\begin{aligned}\lim_{k\to\infty} P_{x_k}([n_k t], n_k, y_k) &= \lim_{k\to\infty} P(x'_k + \bar{S}^{(n_k)} \in A_k)\\ &= P(xt^{1/2} + B \in A), \quad \text{which, as above,}\\ &= P_x(U(\log t^{-1}) \le y, \rho(c_1, c_2) > \log t^{-1}).\end{aligned}$$

The uniformity of the second limit in the lemma follows immediately.

The proof of the first statement is similar. □

LEMMA 7. (a) *Let* $f\colon \mathbb{N} \cup \{0\} \to \mathbb{R}$ *satisfy* $P(T_f > n) > 0\ \forall n \in \mathbb{N}$. *Then*

$$P(S_n \le y \mid T_f > n) \ge P(S_n \le y)\ y \in \mathbb{R}, n \in \mathbb{N}.$$

(b) *If* $\varepsilon > 0$, *then*

$$\lim_{n\to\infty} \sup_f P(\max_{i\le n} |X_i| n^{-1/2} \ge \varepsilon \mid T_f > n) = 0,$$

where the supremum is over all functions $f\colon \mathbb{N} \cup \{0\} \to [0, \infty]$.

PROOF. (a) We proceed by induction. If $n = 1$, the result is obvious. Assume the result holds for n. Let $y \le f(n+1)$.

$$\begin{aligned}P(S_{n+1} \le y \mid T_f > n+1) &= \int_{-\infty} P(z + X_1 \le y) P(S_n \in dz \mid T_f > n) P(T_f > n)\\ &\quad \times P(T_f > n+1)^{-1}\\ &\ge \int_{-\infty}^{\infty} g(z) F_n(dz),\end{aligned}$$

where $g(z) = P(z + X_1 \le y)$ and $F_n(dz) = P(S_n \in dz \mid T_f > n)$. Note that g is left-continuous, non-increasing, and satisfies $g(-\infty) = 1$ and $g(+\infty) = 0$. If $g^+(z) = g(z^+)$, then, integrating by parts, we get

$$\begin{aligned}P(S_{n+1} \le y \mid T_f > n+1) &\ge \int_{-\infty}^{\infty} F_n(z)\, d(-g^+(z))\\ &\ge \int_{-\infty}^{\infty} P(S_n \le z)\, d(-g^+(z)) \qquad \text{(by induction)}\end{aligned}$$

$$\therefore \quad P(S_{n+1} \le y \mid T_f > n+1) \ge P(S_{n+1} \le y) \qquad \text{(by parts)}.$$

The above result is obvious if $y > f(n+1)$, and the proof of (a) is complete.

(b) We first show there is a universal constant $c > 0$ and a constant N, depending only on the law of X_1, such that whenever $f\colon \mathbb{N} \cup \{0\} \to [0, \infty)$,

$$P(T_f > n \mid T_f > i) \ge c(in^{-1}) \quad \text{for all} \quad N \le i \le n. \tag{7}$$

It clearly suffices to prove (7) for $N \le i \le n/2$. Fix $i \le n/2$.

$$\begin{aligned}P(T_f > n \mid T_f > i) &= \int_{-\infty}^{f(i)} P(y + S_k \le f(i+k) \text{ for } k \le n-i) P(S_i \in dy \mid T_f > i)\\ &\ge P(\max_{k\le n} S_k \le \sqrt{i}) P(S_i \le -\sqrt{i} \mid T_f > i)\end{aligned}$$

$$P(T_f > n \mid T_f > i) \ge P(\max_{k\le n} S_k \le \sqrt{i}) P(S_i \le -\sqrt{i}) \quad \text{(by (a))} \tag{8}$$

We may assume (Skorohod embedding—Breiman [7, page 276]) that there is a Brownian motion, B, and non-negative i.i.d. random variables $\{\Delta_i\}$ such that $E(\Delta_i) = 1$ and if $\tau_n = \sum_{i=1}^n \Delta_i$, then $S_j = B(\tau_j)$. Therefore

$$P(\max_{k\le n} S_k \le \sqrt{i}) \ge P(\max_{t\le \tau_n} B(t) \le \sqrt{i}) \ge P(\tau_n \le n^2 i^{-1}, \max_{t\le n^2 i^{-1}} B(t) \le \sqrt{i})$$

$$\text{(9)} \qquad \ge P(\max_{t\le 1} B(t) \le in^{-1}) - P(\tau_n > n^2 i^{-1})$$

$$\ge k_1(in^{-1}) - d_{n,i}\, in^{-1},$$

where $k_1 = (2/\pi)^{1/2} e^{-1/2}$ and

$$d_{n,i} = E(\tau_n n^{-1} I(\tau_n n^{-1} > ni^{-1})) \le E(\tau_n n^{-1} I(\tau_n n^{-1} > 2)) \equiv d_n$$

(the last, because $i \le n/2$). By the strong law of large numbers, $\tau_n n^{-1} \to 1$ in L^1 and a.s. This implies $\lim_{n\to\infty} d_n = 0$. From (8) and (9) we have

$$P(T_f > n \mid T_f > i) \ge P(S_i \le -\sqrt{i})(k_1 - d_n)\, in^{-1},$$

and (7) is an immediate consequence of the Central Limit Theorem.

Spitzer showed [31, Theorem 3.5] that

$$\text{(10)} \qquad P(T_f > n) \ge P(\max_{j\le n} S_j \le 0) \ge k_2 n^{-1/2}.$$

Therefore,

$$P(\max_{i\le n}|X_i| n^{-1/2} \ge \varepsilon \mid T_f > n) \le \sum_{i=1}^n P(|X_i| \ge \varepsilon n^{1/2}) P(T_f > i-1) P(T_f > n)^{-1}$$

$$\le P(|X_1| \ge \varepsilon n^{1/2}) \Big(\sum_{i=1}^N n^{1/2} k_2^{-1} + \sum_{i=N}^{n-1} c^{-1} n i^{-1}\Big) \quad \text{(by (7) and (10))}$$

$$\le E(X_1^2 \log^+ X_1 I(|X_1| \ge \varepsilon n^{1/2}))\varepsilon^{-2} n^{-1} \log^+(\varepsilon\sqrt{n})^{-1} \cdot (N n^{1/2} k_2^{-1} + c^{-1} n \log n),$$

$$\text{assuming} \quad N \ge 2, \qquad \varepsilon n^{1/2} \ge 1,$$

$$\le K_\varepsilon E(X_1^2 \log^+ X_1 I(|X_1 \ge \varepsilon n^{1/2})).$$

As the above expression does not depend on f and converges to zero as $n \to \infty$, the result follows. □

LEMMA 8. *Suppose* $K > \varepsilon > 0$.

(a) $\limsup_{n\to\infty} \sup_f P(\inf_{t\le 1} S^{(n)}(t) \le -K \mid T_f > n) \le 2\varepsilon^{-1}(K-\varepsilon)^{-1} \exp\{-(K-\varepsilon)^2/2\}$.

(b) $\liminf_{n\to\infty} \inf_f P(\inf_{t\le 1} S^{(n)}(t) \ge -K \mid T_f > n) \ge P(B(t) \in [\varepsilon - K, \varepsilon/2]$ for all $t \le 1)$,

where the sup *and* inf *are taken over all functions* f: $\mathbb{N} \cup \{0\} \to [0, \infty)$.

PROOF. Fix ε, K as above and define

$$U_n = \inf\{t \mid S^{(n)}(t) < -\varepsilon/2\}.$$

Choose f as above and let

$$G_n^f(dy \times dx) = P((U_n, S^{(n)}(U_n)) \in dy \times dz, T_f n^{-1} > U_n)$$

and $\mathscr{F}_s^n = \sigma(S^{(n)}(u) \mid u \le s)$. Then

$$P(\inf_{t\le 1} S^{(n)}(t) \le -K, T_f > n) = E(P(U_n \le 1, \inf_{t\le 1} S^{(n)}(t) \le -K, T_f n^{-1} > 1 \mid \mathscr{F}_{U_n}^n))$$

$$\le \iint I(y \le 1, z \ge -\varepsilon) P(z + S^{(n)}(t) \le f^{(n)}(t+y)$$

$$\text{for all} \quad t \le 1-y, \qquad \inf_{t\le 1-y} z + S^{(n)}(t) \le -K) G_n^f(dy \times dz)$$

$$+ P(U_n \le 1, S^{(n)}(U_n) \le -\varepsilon, T_f > n)$$
$$\le P(\inf_{t\le 1} S^{(n)}(t) \le -K + \varepsilon) P(U_n \le 1, T_f n^{-1} > U_n)$$
$$+ P(U_n \le 1, S^{(n)}(U_n) \le -\varepsilon, T_f > n).$$

We also have

$$P(T_f > n) \ge \iint I(y \le 1)\, P(z + S^{(n)}(t) \le f^{(n)}(t+y) \quad \text{for all} \quad t \le 1-y)\, G_n^f(dy \times dz)$$
$$\ge P(\sup_{t\le 1} S^{(n)}(t) \le \varepsilon/2)\, P(U_n \le 1, T_f n^{-1} > U_n).$$

Combining the above inequalities, one gets

$$\begin{aligned}(11)\quad P(\inf_{t\le 1} S^{(n)}(t) \le -K \mid T_f > n) &\le P(\inf_{t\le 1} S^{(n)}(t) \le -K + \varepsilon) P(\sup_{t\le 1} S^{(n)}(t) \le \varepsilon/2)^{-1} \\ &\quad + P(U_n \le 1, S^{(n)}(U_n) \le -\varepsilon \mid T_f > n) \\ &\le P(\inf_{t\le 1} S^{(n)}(t) \le -K + \varepsilon) P(\sup_{t\le 1} S^{(n)}(t) \le \varepsilon/2)^{-1} \\ &\quad + \sup_f P(\max_{i\le n} |X_i|\, n^{-1/2} \ge \varepsilon/2 \mid T_f > n).\end{aligned}$$

The last term in (11) converges to zero as $n \to \infty$ by Lemma 7, while the first term in (11) converges to

$$P(\inf_{t\le 1} B(t) \le -K + \varepsilon)\, P(\sup_{t\le 1} B(t) \le \varepsilon/2)^{-1}.$$

Using routine estimates for the above expression one obtains

$$\limsup_{n\to\infty} \sup_f P(\inf_{t\le 1} S^{(n)}(t) \le -K \mid T_f > n) \le 2\varepsilon^{-1}(K-\varepsilon)\exp\{-(K-\varepsilon)^2/2\},$$

and (a) is proved.

To prove (b) note that

$$\begin{aligned}&P(\inf_{t\le 1} S^{(n)}(t) \ge -K, T_f > n) \\ &\quad = P(T_f n^{-1} \wedge U_n > 1) + \iint I(y \le 1)\, P(z + S^{(n)}(t) \\ &\qquad \in [-K, f^{(n)}(t+y)] \forall t \le 1-y) G_n^f(dyx\, dz) \\ (12)\quad &\quad \ge P(T_f n^{-1} \wedge U_n > 1) + P(S^{(n)}(t) \in [-K+\varepsilon, \varepsilon/2] \forall t \le 1) \\ &\qquad \times P(T_f n^{-1} > U_n, U_n \le 1, S^{(n)}(U_n) \in [-\varepsilon, -\varepsilon/2]).\end{aligned}$$

If

$$\varepsilon_n^f = P(U_n \le 1, S^{(n)}(U_n) \le -\varepsilon \mid T_f > n),$$

and

$$\varepsilon_n = \sup_f \varepsilon_n^f$$

(the sup is over all f: $N \cup \{0\} \to [0, \infty)$), then

$$P(T_f > n) \le P(T_f n^{-1} \wedge U_n > 1) + P(T_f n^{-1} > U_n, U_n \le 1,$$
$$S^{(n)}(U_n) \in [-\varepsilon, -\varepsilon/2]) + \varepsilon_n^f P(T_f > n),$$

and therefore

$$\begin{aligned}(13)\quad P(T_f > n) \le (1-\varepsilon_n)^{-1}(&P(T_f n^{-1} \wedge U_n > 1) + P(T_f n^{-1} > U_n, \\ &U_n \le 1, S^{(n)}(U_n) \in [-\varepsilon, -\varepsilon/2])).\end{aligned}$$

Combining (12) and (13) we obtain

$$P(\inf_{t\le 1} S^{(n)}(t) \ge -K \mid T_f > n) \ge (1-\varepsilon_n)\, P(S^{(n)}(t) \in [-K+\varepsilon, \varepsilon/2]\, \forall t \le 1).$$

The last expression is independent of f, and since $\lim_{n\to\infty}\varepsilon_n = 0$ by Lemma 7, and

$$\lim_{n\to\infty} P(S^{(n)}(t) \in [-K+\varepsilon, \varepsilon/2]\,\forall t\le 1) = P(B(t) \in [-K+\varepsilon, \varepsilon/2]\,\forall t \le 1),$$

the result follows. □

LEMMA 9. $\lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} Q_n([c_2-\varepsilon, \infty]) = \lim_{\varepsilon\to 0^+}\limsup_{n\to\infty} Q_n((-\infty, c_1+\varepsilon]) = 0$.

PROOF. By symmetry it suffices to assume $c_2 < \infty$ and show the first limit is zero. If $m < n$,

$$Q_n(A) = \int_{-\infty}^{\infty} P_x(m, n, A)\, Q_m(dx) \left(\int_{-\infty}^{\infty} P_x(m, n, \infty) Q_m(dx)\right)^{-1}$$

and it therefore suffices to show

$$\lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \sup_x P_x([n/2], n, [c_2-\varepsilon, \infty))\, P_x([n/2], n, \infty)^{-1} = 0, \tag{14}$$

where the supremum is over those values of x for which $P_x([n/2], n, \infty) > 0$. A similar convention is used for sup and inf in what follows.

Assume first that $(\mathrm{H_b})$ holds. In order to prove (14) we first establish an auxiliary result. If $0 < s < t < 1$, then

$$\begin{aligned}
&\liminf_{n\to\infty} \inf_x P_x([sn], n, \infty)\, P_x([sn], [tn], \infty)^{-1} \\
&\quad = \liminf_{n\to\infty} \inf_x \int P_y([tn], n, \infty)\, Q_x([sn], [tn], dy) \\
&\quad = \liminf_{n\to\infty} \inf_x \int P_y(\rho(-\infty, c_2) > \log t^{-1})\, Q_x([sn], [tn], dy) \text{ by Lemma 6} \\
&\quad \ge \liminf_{n\to\infty} (\inf_x Q_x([sn], [tn], -1))(\inf_{y\le -1} P_y(\rho(-\infty, c_2) > \log t^{-1})) \\
&\quad = P_{-1}(\rho(-\infty, c_2) > \log t^{-1}) \liminf_{n\to\infty} \inf_{x \le f_2([sn])[sn]^{-1/2}} \\
&\qquad P[S^{([nt]-[ns])}(1) \le (-[tn]^{1/2} - x[sn]^{1/2})([nt]-[ns])^{-1/2} \mid \\
&\qquad S_k \le f_2([ns]+k) - x[ns]^{1/2} \text{ for } k \le [nt]-[ns]] \\
&\quad \ge P_{-1}(\rho(-\infty, c_2) > \log t^{-1}) \liminf_{n\to\infty} P(S^{([nt]-[ns])}(1) \le (-[tn]^{1/2} \\
&\qquad - f_2([sn]))([nt]-[ns])^{-1/2}), \text{ by Lemma 7} \\
&\quad = P_{-1}(\rho(-\infty, c_2) > \log t^{-1})\, P(B(1) \le -(t^{1/2} + c_2 s^{1/2})(t-s)^{-1/2}) \equiv \delta(s,t) > 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \limsup_{n\to\infty} \sup_x P_x([n/2], n, [c_2-\varepsilon, \infty))\, P_x([n/2], n, \infty)^{-1} \\
&\quad = \limsup_{\varepsilon\to 0^+}\limsup_{n\to\infty} \sup_x \int P_y([3n/4], n, [c_2-\varepsilon, \infty))\, Q_x([n/2], [3n/4], dy) \\
&\qquad \times P_x([n/2], [3n/4], \infty)\, P_x([n/2], n, \infty)^{-1} \\
&\quad \le \delta(1/2, 3/4)^{-1} \limsup_{\varepsilon\to 0^+}\limsup_{n\to\infty}\sup_x \int P_y(U(\log(4/3)) \\
&\qquad \ge c_2 - \varepsilon, \rho(-\infty, c_2) > \log 4/3)\, Q_x([n/2], [3n/4], dy), \quad \text{by Lemma 6} \\
&\quad \le \delta(1/2, 3/4)^{-1} \limsup_{\varepsilon\to 0^+} \sup_{y\le c_2} P_y(U(\log 4/3) \in [c_2-\varepsilon, c_2], \rho(-\infty, c_2) > \log 4/3) \\
&\quad \le \delta(1/2, 3/4)^{-1} \limsup_{\varepsilon\to 0^+} \sup_{y\le c_2} P_y(B(1/3)(4/3)^{-1/2} \in [c_2-\varepsilon, c_2]) = 0
\end{aligned}$$

and (14) holds. The same argument shows that if $\lim_{\varepsilon\to 0^+} m(A_\varepsilon \cap (-\infty, c_2]) = 0$ (m denotes Lebesgue measure), then

$$\limsup_{\varepsilon\to 0^+} \limsup_{n\to\infty} \sup_x P_x([n/2], n, A_\varepsilon)\, P_x([n/2], n, \infty)^{-1} = 0. \tag{15}$$

Assume now that (H_a) holds. Clearly (14) is implied by the "symmetrical" condition

$$\limsup_{\varepsilon\to 0^+} \limsup_{n\to\infty} \sup_x P_x([n/2], n, [c_2 - \varepsilon, c_2 + \varepsilon] \cup [c_1 - \varepsilon, c_1 + \varepsilon])\, P_x([n/2], n, \infty)^{-1} = 0.$$

By symmetry we need only prove

$$\limsup_{\varepsilon\to 0^+} \limsup_{n\to\infty} \sup_{x\ge c_0} P_x([n/2], n, I_\varepsilon)\, P_x([n/2], n, \infty)^{-1} = 0, \tag{16}$$

where $c_0 = (c_1 + c_2)/2$ and $I_\varepsilon = [c_1 - \varepsilon, c_1 + \varepsilon] \cup [c_2 - \varepsilon, c_2 + \varepsilon]$. Let $P'_x(m, n, y)$ be defined as $P_x(m, n, y)$ but with f_1 replaced by $-\infty$. Therefore (15) implies

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \sup_x P'_x([n/2], n, I_\varepsilon)\, P'_x([n/2], n, \infty)^{-1} \\ &\ge \limsup_{\varepsilon\to 0^+} \limsup_{n\to\infty} \sup_{x\ge c_0} P_x([n/2], n, I_\varepsilon)\, P_x([n/2], n, \infty)^{-1} \\ &\quad \times (P_x([n/2], n, \infty)\, P'_x([n/2], n, \infty)^{-1}). \end{aligned} \tag{17}$$

We may use Lemma 8(b) to get a lower bound on the last ratio as follows:

$$\begin{aligned} &\liminf_{n\to\infty} \inf_{x\ge c_0} P_x([n/2], n, \infty)\, P'_x([n/2], n, \infty)^{-1} \\ &\quad \ge \liminf_{n\to\infty} \inf_{x\ge c_0} P(S_j \ge f_1(j + [n/2]) - x[n/2]^{1/2}\ \forall j \le n - [n/2] \mid \\ &\qquad S_j \le f_2(j + [n/2]) - x[n/2]^{1/2}\ \forall j \le n - [n/2]) \\ &\quad \ge \liminf_{n\to\infty} \inf_{f_2^{([n/2])}(1) \ge x \ge c_0} P(\inf_{t\le 1} S^{(n-[n/2])}(t) \ge [n/2]^{1/2}(n - [n/2])^{-1/2} \\ &\qquad (f_1^{([n/2])}(1) - c_0) \mid S_j \le f_2(j + [n/2]) - x[n/2]^{1/2}\ \forall j \le n - [n/2]) \\ &\quad \ge P(B(t) \in [-(c_0 - c_1)/4, (c_0 - c_1)/8] \quad \text{for all} \quad t \le 1) \equiv \delta > 0. \end{aligned}$$

We have used Lemma 8(b) in the last line with $K = (c_0 - c_1)/2$ and $\varepsilon = (c_0 - c_1)/4$, since for large n, $f_{x,n}(j) = f_2(j + [n/2]) - x[n/2]^{1/2} \ge 0$ for $x \le f_2^{([n/2])}(1)$ and $[n/2]^{1/2}(n - [n/2])^{-1/2}(f_1^{([n/2])}(1) - c_0) \le (c_1 - c_0)/2$. Substitute the above inequality into (17) to obtain (16) and hence complete the proof. □

We are finally ready for the following.

Proof of Theorem 5. (a) Assume first that (H_b) holds. Lemma 8(a) shows that $\{Q_n\}$ are tight. Clearly every limit point is supported on $(-\infty, c_2]$. Choose a subsequence such that $Q_{n_k} \to Q$ for some Q. By the usual diagonalization procedure we may assume $Q_{[n_k n^{-1}]} \to Q^{(n)}$ for some $Q^{(n)}$, for all $n \in N$. Fix y such that $Q(\{y\}) = 0$. If $n \ge 2$, then

$$\begin{aligned} Q((-\infty, y]) &= \lim_{k\to\infty} Q_{n_k}((-\infty, y]) \\ &= \lim_{k\to\infty} \int_{-\infty}^{\infty} P_x([n_k n^{-1}], n_k, y)\, Q_{[n_k n^{-1}]}(dx) \\ &\qquad \left(\int_{-\infty}^{\infty} P_x([n_k n^{-1}], n_k, \infty)\, Q_{[n_k n^{-1}]}(dx)\right)^{-1} \\ &= \int_{-\infty}^{c_2} P_x(U(\log n) \le y, \rho(-\infty, c_2) > \log n)\, Q^{(n)}(dx) \\ &\qquad \left(\int_{-\infty}^{c_2} P_x(\rho(-\infty, c_2) > \log n)\, Q^{(n)}(dx)\right)^{-1} \quad \text{(Lemma 6)} \end{aligned}$$

$$(18) \qquad \therefore \quad Q((-\infty, y]) = \int_{-\infty}^{c_2} P_x(U(\log n) \le y \,|\, \rho(-\infty, c_2) > \log n) R_n(dx),$$

where R_n is the probability measure on $(-\infty, c_2]$ defined by

$$R_n(B) = \int_B P_x(\rho(-\infty, c_2) > \log n) Q^{(n)}(dx) \Big(\int_{-\infty}^{c_2} P_x(\rho(-\infty, c_2) > \log n) Q^{(n)}(dx) \Big)^{-1}.$$

The strong Markov property implies that $P_x(\rho(-\infty, c_2) > \log n)$ is non-increasing in x and therefore

$$\begin{aligned} \limsup_{\delta\to 0^+} \sup_{n\ge 2} R_n([c_2-\delta, c_2]) &\le \limsup_{\delta\to 0^+} \sup_{n\ge 2} P_{c_2-\delta}(\rho(-\infty, c_2) > \log n) \\ &\qquad Q^{(n)}([c_2-\delta, c_2]) \\ &\qquad (P_{c_2-\delta}(\rho(-\infty, c_2) > \log n) Q^{(n)}((-\infty, c_2-\delta]))^{-1} \\ &= \limsup_{\delta\to 0^+} Q^{(n)}([c_2-\delta, c_2]) Q^{(n)}((-\infty, c_2-\delta))^{-1} \end{aligned}$$

$$(19) \qquad \therefore \quad \limsup_{\delta\to 0^+} \sup_{n\ge 2} R_n([c_2-\delta, c_2]) = 0.$$

The last follows from Lemma 9 and the fact that $Q_{[n_k n^{-1}]} \to Q^{(n)}$.

Fix $N > 0$ such that $(N-1)^2 > 2\lambda_0(-\infty, c_2)$. We claim that

$$(20) \qquad \lim_{n\to\infty} R_n((-\infty, -N(\log n)^{1/2})) = 0.$$

By Lemma 8(a), there is a universal constant $c > 0$ such that

$$\begin{aligned} Q^{(n)}((-\infty, -N(\log n)^{1/2})) &\le \liminf_{k\to\infty} Q_{[n_k n^{-1}]}((-\infty, -N(\log n)^{1/2})) \\ &\le c \exp\{-(N-1)^2 \log n/2\} = cn^{-(N-1)^2/2}. \end{aligned}$$

By Theorem 1.1 of Uchiyama [36], if $x \ge -N(\log n)^{1/2}$ and

$$0 < \delta < (\lambda_1(-\infty, c_2) - \lambda_0(-\infty, c_2)) N^{-2},$$

then there is a $c' > 0$ and an $n_0 \in N$, both independent of x such that

$$\begin{aligned} P_x(\rho(-\infty, c_2) > \log n) &\ge n^{-\lambda_0(-\infty, c_2)} (\theta(-\infty, c_2)\psi_0(-\infty, c_2, x) - c_\delta n^{(\lambda_1(-\infty,c_2)-\lambda_0(-\infty,c_2))} e^{\delta N^2 \log n}) \\ &\ge c' n^{-\lambda_0(-\infty, c_2)} \end{aligned}$$

for $n \ge n_0$. We have used Proposition 2(d). Combine the above estimates to see that for $n \ge n_0$,

$$R_n((-\infty, -N(\log n)^{1/2})) \le cc'^{-1} n^{-((N-1)^2/2) + \lambda_0(-\infty, c_2)} (1 - cn^{-(N-1)^2/2})^{-1}.$$

The choice of N shows that the right side of the above converges to zero as $n \to \infty$ and hence (20) holds.

Let $n \to \infty$ in (18) and use (19), (20) and Proposition 4(a) to conclude that

$$Q((-\infty, y]) = \int_{-\infty}^{y} \psi_0(-\infty, c_2, z) m(dz) \theta(-\infty, c_2)^{-1}$$

for all y such that $Q(\{y\}) = 0$ and hence for all y. As Q is an arbitrary limit point of $\{Q_n\}$, the proof is complete in this case. The theorem holds under (H_c) by symmetry.

Assume (H_a) holds. The proof proceeds almost exactly as above. In fact some simplification occurs since $\{Q_n\}$ and $\{R_n\}$ are trivially tight. The only change is in proving the analogue of (19), namely

$$\lim_{\delta\to 0^+} \limsup_{n\to\infty} R_n([c_1, c_1+\delta] \cup [c_2-\delta, c_2]) = 0,$$

where R_n is defined in the obvious way. If $\varepsilon > 0$ and $I_\delta = [c_1, c_1 + \delta] \cup [c_2 - \delta, c_2]$, then

$$\limsup_{n\to\infty} R_n(I_\delta) \le \limsup_{n\to\infty} Q^{(n)}(I_\delta)\sup_{x\in I_\delta} P_x(\rho(c_1, c_2) > \log n)$$
$$Q^{(n)}([c_1+\varepsilon, c_2-\varepsilon])^{-1}(\inf_{x\in[c_1+\varepsilon,c_2-\varepsilon]} P_x(\rho(c_1, c_2) > \log n))^{-1}$$
$$\le \limsup_{n\to\infty} Q^{(n)}(I_\delta) Q^{(n)}([c_1+\varepsilon, c_2-\varepsilon])^{-1}$$
$$\times (\sup_{x\in I_\delta}\psi_0(c_1, c_2, x))(\inf_{x\in[c_1+\varepsilon,c_2-\varepsilon]}\psi_0(c_1, c_2, x))^{-1} \quad \text{(see (6))}.$$

Use Proposition 2(d) and Lemma 9 (recall that $Q_{[n_k n^{-1}]} \to_w Q^{(n)}$) to conclude that the right side converges to zero as $\delta \to 0^+$ for small enough $\varepsilon > 0$. The rest of the proof proceeds as before.

(b) Note that if $t \in (0, 1)$,

$$\lim_{n\to\infty} P(T > n) P(T > [nt])^{-1} = \lim_{n\to\infty} \int_{-\infty}^{\infty} P_y([nt], n, \infty) Q_{[nt]}(dy)$$

$$= \int_{c_1}^{c_2} P_y(\rho > \log t^{-1})\psi_0(y) m(dy)(\theta)^{-1} \quad \text{(by (a) and Lemma 6)}$$

$$= t^{\lambda_0} \quad \text{(Lemma 3(b))}.$$

It follows that $P(T > n) = n^{-\lambda_0}\pi(n)$, where π is slowly varying. □

We suspect that the condition $E(X_1^2\log^+(X_1)) < \infty$, used only to derive Lemma 7(b), is unnecessary but we have been able to drop this hypothesis only in a few special cases. One such case is $f_1 = 0$, $f_2 = +\infty$. Indeed using Spitzer's result (10) one obtains, instead of (7), the stronger result

$$P(T_f > n \mid T_f > i) \ge c(in^{-1})^{1/2} \quad \text{for all} \quad N \le i \le n$$

for some N. The proof of Lemma 7(b) now goes through without the condition $E(X_1^2\log^+(X_1)) < \infty$. This observation gives us another proof of the following result, originally due to Iglehart [16] if $E(|X_1|^3) < \infty$, and Bolthausen [5] if $E(X_1^2) < \infty$.

COROLLARY 10. *If $\{X_i\}$ are* i.i.d. *random variables with $E(X_1) = 0$ and $E(X_1^2) = 1$, then for $y \ge 0$*

$$\lim_{n\to\infty} P(S_n n^{-1/2} \le y \mid S_i \ge 0\ i = 1, \cdots, n) = \int_0^y z e^{-z^2/2}\, dz.$$

PROOF. By the above remarks we may use Theorem 5 with $f_1 = 0$ and $f_2 = +\infty$. The result follows from that result and the fact that $\psi_0(0, \infty, z)\theta(0, \infty)^{-1} = z/2$ (Proposition 2(c)). □

A functional form of Theorem 5, and in particular of the above Corollary (see Iglehart [16], Bolthausen [5]), will be derived from the above results in a subsequent paper [14].

It is not hard to prove a partial converse of Theorem 5, namely the following.

THEOREM 11. *Let $\{X_i\}$ be* i.i.d. *with mean zero and variance one. Assume $f_1 : N \cup \{0\} \to [-\infty, 0]$ and $f_2 : N \cup \{0\} \to [0, \infty]$ are non-increasing and non-decreasing, respectively, for large n, and define Q_n and T as before. If $Q_n \to_w Q_\infty$ for some Q_∞ and $\limsup_{n\to\infty} P(T > n/2) P(T > n)^{-1} < \infty$, then $\lim_{n\to\infty} f_i(n) n^{-1/2} = c_i \in [-\infty, \infty]$, where $[c_1, c_2]$ is the support of Q_∞.* □

We omit the proof. The basic idea is that, for a "reasonable" sequence of measures, $Q_n \to_w Q_\infty$ should imply support $(Q_n) \to$ support (Q_∞). One expects, moreover, that support $(Q_n) = [f_1(n)n^{-1/2}, f_2(n)n^{-1/2}]$, whence the result. The monotonicity of f_1 and f_2 play a crucial role in the proof.

We close this section with another conditional limit theorem that will prove useful in Section 5 and in subsequent applications [14] to random walk.

PROPOSITION 12. *Assume the hypotheses* (H) *hold with* $|c_i| < \infty$ $(i = 1, 2)$. *Then*

$$\begin{aligned} \lim_{n\to\infty} P(S_T > 0 \mid T > n) &= \psi_0(c_1, c_2)'(c_2)e^{-c_2^2/2}(\lambda_0(c_1, c_2)\theta(c_1, c_2))^{-1} \\ &= 1 + \psi_0(c_1, c_2)'(c_1)e^{-c_1^2/2}(\lambda_0(c_1, c_2)\theta(c_1, c_2))^{-1} \\ &= \int_{c_1}^{c_2} (s(x) - s(c_1))(s(c_2) - s(c_1))^{-1}\psi_0(c_1, c_2, x)m(dx)\theta(c_1, c_2)^{-1} \\ &\equiv \alpha(c_1, c_2), \end{aligned}$$

where $s(x) = \int_0^x e^{y^2/2}\,dy$.

PROOF. Let

$$T_n(x) = \inf\{t \mid x + S^{(n)}(t) \notin [f_1^{(n)}(1 + t), f_2^{(n)}(1 + t)]\}$$

and

$$p(n, x) = P(x + S^{(n)}(T_n(x)) > 0).$$

Since $\lim_{n\to\infty} f_i^{(n)}(1 + t) = c_i(1 + t)^{1/2}$ uniformly on compacts, it is easy to see that if

$$T' = \inf\{t \mid B(t) \notin [c_1(1 + t)^{1/2}, c_2(1 + t)^{1/2}]\},$$

then

$$\lim_{n\to\infty} p(n, x) = P_x(B(T') = c_2(1 + T')^{1/2})$$

(use Donsker's Theorem and Theorem 5.5 of Billingsley [3]). Transforming B into an Ornstein-Uhlenbeck process, one sees that this limiting probability equals

$$P_x(U(\rho) = c_2) = (s(x \wedge c_2) - s(x \wedge c_1))(s(c_2) - s(c_1))^{-1}.$$

Since this limit is continuous in x and $p(n, x)$ is non-decreasing in x, it follows that

(21) $$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |p(n, x) - (s(x \wedge c_2) - s(x \wedge c_1))(s(c_2) - s(c_1))^{-1}| = 0.$$

By conditioning with respect to $\sigma(X_1, \cdots, X_n)$ one gets

$$\begin{aligned} \lim_{n\to\infty} P(S_T > 0 \mid T > n) &= \lim_{n\to\infty} \int_{-\infty}^{\infty} p(n, x)\,Q_n(dx) \\ &= \int_{c_1}^{c_2} (s(x) - s(c_1))(s(c_2) - s(c_1))^{-1}\psi_0(x)m(dx)\theta^{-1} \\ &(\equiv \alpha(c_1, c_2)) \end{aligned}$$

(by (21) and Theorem 5). To obtain the other expressions for α recall that

$$d/dx(\psi_0'(x)e^{-x^2/2}) = 2\lambda_0\psi_0(x)e^{-x^2/2}.$$

Integrate by parts to see that

$$\alpha(c_1, c_2) = (\lambda_0)^{-1} \int_{c_1}^{c_2} (s(x) - s(c_1))(s(c_2) - s(c_1))^{-1} d/dx(\psi_0'(x)e^{-x^2/2})\,dx\,\theta^{-1}$$

$$= (\lambda_0)^{-1}\psi_0'(c_2)e^{-c_2^2/2}\theta^{-1}$$

$$= (\lambda_0\theta)^{-1}\Big(\psi_0'(c_1)e^{-c_1^2/2} + \int_{c_1}^{c_2} 2\lambda_0\psi_0(x)e^{-x^2/2}\,dx\Big)$$

$$= 1 + (\lambda_0\theta)^{-1}\psi_0'(c_1)e^{-c_1^2/2}. \qquad \square$$

3. Joint convergence. Throughout this section, in addition to the Conditions (H) at the beginning of Section 2, we will assume that $\lambda_0 = \lambda_0(c_1, c_2) < 1$. This will be true, for example, if $1 \le \min(|c_1|, |c_2|)$ and $1 < \max(|c_1|, |c_2|)$, or $\max(|c_1|, |c_2|) = \infty$ (see Proposition 2(c).)

Inductively define a sequence of stopping times, $\{T_i\}$, by $T_0 = 0$ and

$$T_{i+1} = \min\{n > T_i \mid S_n - S_{T_i} \notin [f_1(n - T_i), f_2(n - T_i)]\}.$$

By Theorem 5 there is a slowly varying function π such that $P(T_1 > n) = n^{-\lambda_0}\pi(n)\Gamma(1 - \lambda_0)^{-1}$. Therefore T_1 is in the domain of attraction of a stable law of index λ_0. Let $b_n = n^{\lambda_0}\pi(n)^{-1}$. Then $\tau_n(t) = T_{[b_n t]}n^{-1}$ converges weakly to $\tau(t)$ as $n \to \infty$, where $\tau(t)$ is a stable subordinator of index λ_0, scaled so that $E(e^{-u\tau(1)}) = \exp\{-u^{\lambda_0}\}$ $(u \ge 0)$ (see Feller [13, page 424] and Skorohod [29, Theorem 2.7] but note that $\{b_n\}$ is the inverse sequence of Feller's normalizing constants $\{a_n\}$). The weak convergence is in the space $D([0, \infty), \mathbb{R})$ of right-continuous functions with left limits, with the Skorohod J_1 topology (see Stone [32]). Let

$$L_n(t) = \textstyle\sum_{i=0}^{\infty} I(T_i n^{-1} \le t)\, b_n^{-1} = \inf\{u \mid \tau_n(u) > t\}.$$

By Whitt [37, Theorem 7.2] $L_n \to_w L$ on $D([0, \infty), \mathbb{R})$ where L is the continuous inverse of the subordinator τ. Donsker's theorem states that $S^{(n)}$ converges weakly to a Brownian motion B. We will show that $(L_n, S^{(n)})$ converges jointly. Some auxiliary processes will be needed.

NOTATION. Functions from $D([0, \infty), \mathbb{R})$ to $D([0, \infty), \mathbb{R})$ are defined by

$$g^-(d)(t) = \sup\{d(u) \mid d(u) \le t\}, \qquad g^+(d)(t) = \inf\{d(u) \mid d(u) > t\},$$

$$(\sup\varnothing = \inf\varnothing = 0), \qquad i(d)(t) = \inf\{u \mid d(u) > t\}.$$

$$A_n(t) = t - g^-(\tau_n)(t); \quad Y_n(t) = S^{(n)}(t) - S^{(n)}(g^-(\tau_n)(t)). \qquad \square$$

Note that for a non-decreasing function, d, $i(d)$ is the right continuous inverse of d. It is easy to show that if d is strictly increasing with $d(0) = 0$ and $d(\infty) = \infty$, then d is a continuity point of i, g^+, and g^-, e.g., through the nonstandard treatment of the J_1 topology outlined in the appendix.

For convenience we consider the joint law of $(A_n, Y_n, S^{(n)}, L_n, \tau_n)$. It is natural to consider the first four of these processes on a common time domain, and τ_n on a separate copy of $[0, \infty)$. Hence we consider $(A_n, Y_n, S^{(n)}, L_n, \tau_n)$ as a random vector in $D([0, \infty), \mathbb{R}^4) \times D([0, \infty), \mathbb{R})$.

THEOREM 13. *Under the hypotheses* (H), *if* $\lambda_0(c_1, c_2) < 1$, *then* $(A_n, Y_n, S^{(n)}, L_n, \tau_n)$ *converges weakly on* $D([0, \infty), \mathbb{R}^4) \times D([0, \infty), \mathbb{R})$ *to a process* $X = (A, Y, B, L, \tau)$ *whose joint law* (P) *depends only on the pair* (c_1, c_2).

Assume X is defined on the canonical space of paths

$$(\Omega, \mathscr{F}, P) = (D([0, \infty), \mathbb{R}^4) \times D([0, \infty), \mathbb{R}), \text{Borel sets}, P).$$

Let

$$\mathscr{F}_t = \cap_{s>t}\sigma(\{B(u), A(u) \mid u \le s\} \cup \{P\text{-null sets}\}),$$

$$\mathscr{F}'_t = \cap_{s>t}\sigma(\{A(u), Y(u) \mid u \le s\} \cup \{P\text{-null sets}\}).$$

(a) *B is an* $\mathscr{F}_t$*-Brownian motion* ($B(t) - B(s)$ *is independent of* $\mathscr{F}_s$) *and* $\mathscr{F}_t = \mathscr{F}'_t$ *for all* $t \ge 0$.

(b) (A, Y) *is a homogeneous strong Markov process with respect to* $\{\mathcal{F}'_t\}$, *starting at* $(0, 0)$, *with transition probability*

$$P((A(t), Y(t)) \in B_1 \times B_2 \mid (A(0), Y(0)) = (a, y))$$
$$= I_{B_1}(a + t) P(y + B(t) \in B_2, T(a, y) > t)$$
$$+ E\left(\int_0^{(t-T(a,y))^+} I_{B_1}(u)(t - T(a, y) - u)^{\lambda_0 - 1} u^{-\lambda_0}\right.$$
$$\left.\theta^{-1} \int_{c_1}^{c_2} I_{B_2}(zu^{1/2})\psi_0(z) m(dz) du (\Gamma(\lambda_0)\Gamma(1 - \lambda_0))^{-1}\right),$$

where

$$T(a, y) = \inf\{s \mid y + B(s) \notin [c_1(s + a)^{1/2}, c_2(s + a)^{1/2}]\}.$$

(c) τ *is a stable subordinator of index* $\lambda_0(c_1, c_2)$ *scaled so that*

$$E(e^{-u\tau(1)}) = \exp\{-u^{\lambda_0(c_1,c_2)}\}(u \geq 0).$$

(d) L *is the continuous inverse of* τ *and is also the local time* (*in the sense of Maisonneuve* [25]) *of the regular regenerative set,*

$$M = \{t \mid (A(t), Y(t)) = (0, 0)\} = \overline{\{\tau(t) \mid t \geq 0\}}.$$

(e) *For* a.a. ω *and all* $t \geq 0$,

$$Y(t) = B(t) - B(g^-(\tau)(t)) \quad and \quad A(t) = t - g^-(\tau)(t)$$

(22) $$B(\tau(t^-) + u) - B(\tau(t^-)) \in (c_1 u^{1/2}, c_2 u^{1/2}) \quad for \quad 0 < u < \tau(t) - \tau(t^-)$$

$$B(\tau(t)) - B(\tau(t^-)) \in \{c_1(\tau(t) - \tau(t^-))^{1/2}, c_2(\tau(t) - \tau(t^-))^{1/2}\}.$$

PROOF. The tightness of $\{(A_n, Y_n, S^{(n)}, L_n, \tau_n)\}$ in $D([0, \infty), \mathbb{R}^4) \times D([0, \infty), \mathbb{R})$ is established in the appendix using a nonstandard characterization of tightness in $D([0, \infty), \mathbb{R}^k)$ (see Proposition A.3). Note that tightness in $D([0, \infty), \mathbb{R}^k)$ does not follow from the tightness of each component.

Choose a subsequence $\{n_k\}$ such that $(A_{n_k}, Y_{n_k}, S^{(n_k)}, L_{n_k}, \tau_{n_k}) \to (A, Y, B, L, \tau)$ (the latter being defined as the limit on $\{n_k\}$) on $D([0, \infty), \mathbb{R}^4) \times D([0, \infty), \mathbb{R})$. By changing the underlying probability space, we may assume $(A_{n_k}, Y_{n_k}, S^{(n_k)}, L_{n_k}, \tau_{n_k}) \to (A, Y, B, L, \tau)$ a.s. on $D([0, \infty), \mathbb{R}^4) \times D([0, \infty], \mathbb{R})$ (see Skorohod [30, page 10] and Dudley [10, Theorem 3]). Let $\mathcal{F}^k_t = \sigma(S^{(n_k)}(s) \mid s \leq t)$ and define $\mathcal{F}_t$ and $\mathcal{F}'_t$ as described earlier (but on the probability space chosen above). By definition we have for all $t \geq 0$,

(i) $$Y_{n_k}(t) = S^{(n_k)}(t) - S^{(n_k)}(g^-(\tau_{n_k})(t)); \qquad A_{n_k}(t) = t - g^-(\tau_{n_k})(t); \qquad L_{n_k} = i(\tau_{n_k})$$

(ii) $$S^{(n_k)}(t) - S^{(n_k)}(g^-(\tau_{n_k})(t)) \in [f_1^{(n_k)}(A_{n_k}(t)), f_2^{(n_k)}(A_{n_k}(t))]$$

(iii) $$S^{(n_k)}(g^+(\tau_{n_k})(t)) - S^{(n_k)}(g^-(\tau_{n_k})(t)) \notin [f_1^{(n_k)}(g^+(\tau_{n_k})(t) - g^-(\tau_{n_k})(t)),$$
$$f_2^{(n_k)}(g^+(\tau_{n_k})(t) - g^-(\tau_{n_k})(t))].$$

Since τ is a.s. a continuity point of i and $g^{\pm}$ we may let $k \to \infty$ in the above (recall that $\lim_{k\to\infty} f_i^{(n_k)}(u) = c_i u^{1/2}$ uniformly on compacts excluding zero) to see that for a.a. ω, (i), (ii) and (iii) hold in the limit for all t which are continuity points of $(A, Y, g^-(\tau), g^+(\tau))$ and satisfy $A(t) > 0$:

(i)′ $$Y(t) = B(t) - B(g^-(\tau)(t)); \qquad A(t) = t - g^-(\tau)(t); \qquad L = i(\tau)$$

(ii)′ $$B(t) - B(g^-(\tau)(t)) \in [c_1(A(t))^{1/2}, c_2(A(t))^{1/2}]$$

(iii)′ $$B(g^+(\tau)(t)) - B(g^-(\tau)(t)) \notin (c_1(g^+(\tau)(t) - g^-(\tau)(t))^{1/2},$$
$$c_2(g^+(\tau)(t) - g^-(\tau)(t))^{1/2}).$$

By right-continuity (i)$'$, (ii)$'$ and (iii)$'$ hold for all $t \geq 0$ a.s. Note that (e) follows for our limit point (A, Y, B, L, τ) except that we have established (22) only for the closed intervals $[c_1 u^{1/2}, c_2 u^{1/2}]$.

Our strategy now is to show that (A, Y) is a Markov process with the transition probabilities of (b) above. We then will show that B, L and τ are measurable functions of (A, Y) so that the multivariate distribution of the limit (A, Y, B, L, τ) is uniquely determined.

Let $\phi: \mathbb{R} \to \mathbb{R}$ and $\psi: [0, \infty) \to \mathbb{R}$ be bounded continuous functions such that ψ vanishes on $[0, \varepsilon]$ for some $\varepsilon > 0$. If $G_k(t) = E(\phi(Y_{n_k}(t))\psi(A_{n_k}(t)))$, then

$$G_k(t) = \psi(t) E(\phi(S^{(n_k)}(t)) I(T_1 n_k^{-1} > t)) + E(I(T_1 n_k^{-1} \leq t) E(\phi(Y_{n_k}(t))\psi(A_{n_k}(t)) \mid \mathscr{F}^k_{T_1 n_k^{-1}}))$$

$$G_k(t) = \psi(t) E(\phi(\sqrt{t} S^{(tn_k)}(1)) I(T_1 n_k^{-1} > t)) + \int_0^t G_k(t-u) P(T_1 n_k^{-1} \in du).$$

The above renewal equation has the unique solution (see Feller [13, page 185, Theorem 1])

$$G_k(t) = \int_0^t I_k(t-s) R_k(ds), \tag{23}$$

where

$$I_k(u) = \psi(u) E(\phi(u^{1/2} S^{(un_k)}(1)) \mid T_1 > n_k u) b_{n_k} P(T_1 > n_k u)$$

$$R_k([0, t]) = E(\textstyle\sum_{i=0}^{\infty} I(T_i n_k^{-1} \leq t) b_{n_k}^{-1}) = E(L_{n_k}(t)).$$

Note that

$$\begin{aligned} E(L_n(t)^2) &\leq 2 \textstyle\sum\sum_{0 \leq i \leq j} P(T_i \leq nt, T_j \leq nt) b_n^{-2} \\ &\leq 2 \textstyle\sum\sum_{0 \leq i \leq j} P(T_i \leq nt) P(T_j - T_i \leq nt) b_n^{-2} \\ &\leq 2(\textstyle\sum_{i=0}^{\infty} P(T_i \leq nt) b_n^{-1})^2 \\ &\leq 2(\textstyle\sum_{i=0}^{\infty} P(T_1 \leq nt)^i b_n^{-1})^2 \\ &= 2(P(T_1 > nt) b_n)^{-2} \to 2t^{2\lambda_0}\Gamma(1-\lambda_0)^2 \quad \text{as} \quad n \to \infty. \end{aligned}$$

In particular for each $t \geq 0$, $\{L_n(t) \mid n \in \mathbb{N}\}$ is a uniformly integrable family. Since $L_n(t) \to_w L(t)$, Theorem 5.4 of Billingsley [3] implies,

$$\lim_{k\to\infty} R_k([0, t]) = E(L(t)) = t^{\lambda_0} E(L(1)) \equiv R([0, t]). \tag{24}$$

Using the fact that for a slowly varying function π, $\lim_{k\to\infty} \pi(n_k u)\pi(n_k)^{-1} = 1$ uniformly for u in compacts excluding zero, (see Feller [13, page 277, Lemma 2]), one obtains

$$\lim_{k\to\infty} \sup_{\varepsilon \leq u \leq N} | b_{n_k} P(T_1 > n_k u) - u^{-\lambda_0}\Gamma(1-\lambda_0)^{-1} | = 0, \qquad \forall N \in \mathbb{N}. \tag{25}$$

It follows easily from Theorem 5 that

$$\lim_{k\to\infty} \sup_{\varepsilon \leq u \leq N} | E(\phi(u^{1/2} S^{(un_k)}(1)) \mid T > un_k) - \int_{c_1}^{c_2} \phi(u^{1/2} y) Q_\infty(dy) | = 0, \tag{26}$$

where Q_∞ is as in Theorem 5. The fact that the convergence is uniform in u over $[\varepsilon, N]$ is easy to check. Therefore (25), (26), and the fact that $\psi = 0$ on $[0, \varepsilon]$ imply

$$\lim_{k\to\infty} \sup_{0 \leq u \leq N} | I_k(u) - I(u) | = 0, \tag{27}$$

where

$$I(u) = \psi(u) \int_{c_1}^{2} \phi(u^{1/2}y) Q_\infty(dy) u^{-\lambda_0} \Gamma(1-\lambda_0)^{-1}.$$

Let $G(t) = \int_0^t I(t-s)R(ds)$ and choose sequences $k_j \uparrow \infty$ and $t_j \to t$. Then

$$|G_{k_j}(t_j) - G(t_j)| \le \left| \int_0^{t_j} I_{k_j}(t_j - s) R_{k_j}(ds) - \int_0^t I(t-s) R_{k_j}(ds) \right|$$
$$+ \left| \int_0^t I(t-s) R_{k_j}(ds) - \int_0^t I(t-s) R(ds) \right|$$
$$\to 0 \quad \text{as} \quad j \to \infty \quad \text{by (24) and (27).}$$

We have proved

$$\lim_{k\to\infty} \sup_{0\le t\le N} |G_k(t) - G(t)| = 0, \qquad \forall N > 0. \tag{28}$$

If $s < t$, then for a.a. ω

$$E(\phi(Y_{n_k}(t))\psi(A_{n_k}(t)) \mid \mathscr{F}_s^k)(\omega)$$
$$= \psi(t - s + A_{n_k}(s, \omega)) E(I(g^+(\tau_{n_k})(s) > t)\phi(Y_{n_k}(s) + S^{(n_k)}(t) - S^{(n_k)}(s)) \mid \mathscr{F}_s^k)(\omega)$$
$$+ E(E(\phi(Y_{n_k}(t))\psi(A_{n_k}(t)) \mid \mathscr{F}^k_{g^+(\tau_{n_k})(s)}) I(g^+(\tau_{n_k})(s) \le t) \mid \mathscr{F}_s^k)(\omega)$$
$$= \psi(t - s + A_{n_k}(s, \omega)) H_k(A_{n_k}(s, \omega), Y_{n_k}(s, \omega))$$
$$+ E(G_k(t - g^+(\tau_{n_k})(s)) I(g^+(\tau_{n_k})(s) \le t) \mid \mathscr{F}_s^k)(\omega),$$

where

$$H_k(a, y) = E(\phi(y + S^{(n_k)}(t-s)) I(T_k(a, y) > t - s))$$

and

$$T_k(a, y) = \inf\{u \mid y + S^{(n_k)}(u) \notin [f_1^{(n_k)}(u + a), f_2^{(n_k)}(u + a)]\}.$$

Since

$$P(g^+(\tau_{n_k})(s) \le u \mid \mathscr{F}_s^k)(\omega) = P(s + T_k(A_{n_k}(s, \omega), Y_{n_k}(s, \omega)) \le u) \quad \text{a.s.,}$$

it follows from the above that

$$E(\phi(Y_{n_k}(t))\psi(A_{n_k}(t)) \mid \mathscr{F}_s^k)(\omega) = \psi(t - s + A_{n_k}(s, \omega)) H_k(A_{n_k}(s, \omega), Y_{n_k}(s, \omega))$$
$$+ F_k(A_{n_k}(s, \omega), Y_{n_k}(s, \omega)), \tag{29}$$

where

$$F_k(a, y) = E(G_k(t - s - T_k(a, y)) I(T_k(a, y) \le t - s)).$$

Let $0 \le s_1 < \cdots < s_m \le s$ and let α be bounded and continuous on $\mathbb{R}^{2m}$. By (i)$'$ and the fact that τ is a stable subordinator, $s, t, s_1, \cdots, s_m$ are a.s. continuity points of (A, Y). Therefore, if

$$\alpha_k = \alpha(A_{n_k}(s_1), Y_{n_k}(s_1), \cdots, A_{n_k}(s_m), Y_{n_k}(s_m))$$
$$\alpha = \alpha(A(s_1), Y(s_1), \cdots, A(s_m), Y(s_m)),$$

then

$$E(\phi(Y(t))\psi(A(t))\alpha) = \lim_{k\to\infty} E(\phi(Y_{n_k}(t))\psi(A_{n_k}(t))\alpha_k)$$
$$= \lim_{k\to\infty} E((\psi(t - s + A_{n_k}(s)) H_k(A_{n_k}(s), Y_{n_k}(s)) \tag{30}$$
$$+ F_k(A_{n_k}(s), Y_{n_k}(s)))\alpha_k), \quad \text{by (29).}$$

Theorem 5.5 of Billingsley [3] shows that if $(a_k, y_k) \to (a, y)(a > 0, y \in \mathbb{R})$, then

$$(S^{(n_k)}, T_k(a_k, y_k)) \to_w (B, T(a, y)) \quad \text{on} \quad C([0, \infty), \mathbb{R}) \times \mathbb{R}$$

where $T(a, y)$ is as in the theorem, and therefore

$$\lim_{k\to\infty} H_k(a_k, y_k) = E(\phi(y + B(t-s))I(T(a, y) > t - s)) \equiv H(a, y).$$

Also if $(a_k, y_k) \to (a, y)(a > 0)$, then (28) shows that

$$\lim_{k\to\infty} F_k(a_k, y_k) = E(G(t - s - T(a, y))I(T(a, y) \le t - s)) \equiv F(a, y).$$

Apply the dominated convergence theorem in (30) to get

$$E(\phi(Y(t))\psi(A(t))\alpha) = E((\psi(t - s + A(s))H(A(s), Y(s)) + F(A(s), Y(s)))\alpha).$$

Therefore for a.a. ω

$$\begin{aligned} &E(\phi(Y(t))\psi(A(t)) \mid (A(u), Y(u)), u \le s)(\omega) \\ &= \psi(t - s + A(s, \omega))E(\phi(Y(s, \omega) + B(t-s))I(T(A(s, \omega), Y(s, \omega)) > t - s)) \\ &\quad + E\left(\int_0^\infty I(u \le (t - s - T(A(s, \omega), Y(s, \omega)))^+\right) \int_{c_1}^{c_2} \phi(u^{1/2}y)Q_\infty(dy)\psi(u)u^{-\lambda_0} \\ &\quad \times \lambda_0(t - s - T(A(s, \omega), Y(s, \omega)) - u)^{\lambda_0 - 1}\, du\, \Gamma(1 - \lambda_0)^{-1}E(L(1)). \end{aligned}$$

By taking bounded pointwise limits, the above holds for all bounded measurable ϕ and ψ. Setting $\phi = \psi = 1$ gives $E(L(1)) = \Gamma(1 + \lambda_0)^{-1}$. Therefore (A, Y) is a homogeneous Markov process with the transition probabilities described in the theorem. $(A(0), Y(0)) = 0$ a.s. is immediate from (i)$'$. It is easy to check that the above semigroup maps bounded continuous functions into bounded continuous functions and hence $(A(t), Y(t))$ is a strong Markov process with respect to $\{\mathscr{F}_t'\}$ (see Blumenthal and Getoor [4, page 41]). It follows from (i)$'$ and the fact that τ is a subordinator that (0, 0) is regular for itself, for the process (A, Y). Therefore,

$$M = \{t \mid (A(t), Y(t)) = (0, 0)\} = \overline{\{\tau(t) \mid t \ge 0\}}$$

(see (i)$'$) is a regular regenerative random set in the sense of Maisonneuve [25, Definition X.1] and therefore has a continuous local time, C_t. The Lévy measure of τ is $\nu(dx) = \Gamma(1 - \lambda_0)\lambda_0^{-1}x^{-(1+\lambda_0)}\,dx$. If

$$N_\varepsilon(t) = \operatorname{card}\{s \le t \mid \tau(s^-) - \tau(s) \ge \varepsilon\},$$

then

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \lambda_0^2 \Gamma(1 - \lambda_0)^{-1}\varepsilon^{\lambda_0}N_\varepsilon(t) \\ &= \lim_{\varepsilon\to 0^+} \operatorname{card}\{s \le L(t) \mid \tau(s) - \tau(s^-) \ge \varepsilon\} \times \nu([\varepsilon, \infty))^{-1} = L(t) \quad \text{a.s.} \end{aligned} \tag{31}$$

The last follows easily from the law of large numbers. Since $N_\varepsilon(t)$ is the number of excursions away from zero completed by A by time t, Theorem X.4 of Maisonneuve [25] implies that $C = L = f(A)$ for some measurable $f: D([0, \infty), \mathbb{R}) \to D([0, \infty), \mathbb{R})$ depending only on (c_1, c_2). We have chosen our multiplication constant in the definition of C to get equality here.

We now show that B is a measurable function of (A, Y). As $S^{(n_k)}(t) - S^{(n_k)}(s)$ is independent of $\mathscr{F}_s^k$ and $(A_{n_k}, S^{(n_k)}) \to (A, B)$ a.s., it follows easily that $B(t) - B(s)$ is independent of $\mathscr{F}_s$, and hence B is an $\{\mathscr{F}_t\}$-Brownian motion. For $\varepsilon > 0$ define stopping times with respect to $\{\mathscr{F}_t \cap \mathscr{F}_t'\}$ as follows:

$$\begin{aligned} T_0(\varepsilon) &= U_0(\varepsilon) = 0 \\ T_{i+1}(\varepsilon) &= \inf\{t \ge U_i(\varepsilon) \mid A(t) = \varepsilon\} \\ U_{i+1}(\varepsilon) &= \inf\{t \ge T_{i+1}(\varepsilon) \mid A(t) = 0\}. \end{aligned}$$

Let $V_\varepsilon = \cup_{i=1}^{\infty}]]T_i(\varepsilon), U_i(\varepsilon)]]$, an $\mathcal{F}_t$-predictable set, and define

$$M_\varepsilon(t) = \int_0^t I_{V_\varepsilon}(s)\,dB(s).$$

If m denotes Lebesgue measure, then

$$\begin{aligned}E(\sup_{s\le t}(B(s) - M_\varepsilon(s))^2) &\le cE(m(\{s \le t \mid (s, \omega) \notin V_\varepsilon\}))\\ &\le cE(t - \textstyle\sum_{i=1}^{\infty} I(T_i(\varepsilon) \le t)(U_i(\varepsilon) \wedge t - (T_i(\varepsilon) - \varepsilon))\\ &\quad + \varepsilon(N_\varepsilon(t) + 1))\\ &\le cE(m([0, t] - \cup_{\Delta\tau(s)\ge\varepsilon, \tau(s^-)\le t}[\tau(s^-), \tau(s))))\\ &\quad + c\varepsilon + cE(\varepsilon N_\varepsilon(t)).\end{aligned}$$

As $\varepsilon N_\varepsilon(t) \le t$, (31) implies that $\lim_{\varepsilon\to 0^+} E(\varepsilon N_\varepsilon(t)) = 0$. Also since the range of τ is a.s. Lebesgue null, the right-hand side of the above inequality converges to zero. Therefore we may choose a sequence, $\{\varepsilon_n\}$, decreasing to zero and depending only on $\lambda_0(c_1, c_2)$, such that

$$\lim_{n\to\infty} \sup_{s\le n} |B(s) - M_{\varepsilon_n}(s)| = 0 \quad \text{a.s.} \tag{32}$$

Note that if $T_i(\varepsilon_n) \le s$, then

$$g^-(\tau)((U_i(\varepsilon_n) \wedge s)^-) = g^-(\tau)(T_i(\varepsilon_n)) = T_i(\varepsilon_n) - \varepsilon_n$$

and therefore (i)′ implies that

$$Y((U_i(\varepsilon_n) \wedge s)^-) - Y(T_i(\varepsilon_n)) = B(U_i(\varepsilon_n) \wedge s) - B(T_i(\varepsilon_n)).$$

It follows that

$$M_{\varepsilon_n}(s) = \textstyle\sum_{i=1}^{\infty} I(T_i(\varepsilon_n) \le s)(Y(U_i(\varepsilon_n) \wedge s^-) - Y(T_i(\varepsilon_n))) \equiv h_n(A, Y)(s) \tag{33}$$

for some measurable $h_n: D([0, \infty), \mathbb{R}^2) \to D([0, \infty), \mathbb{R})$ By (32) there is a measurable map $h: D([0, \infty), \mathbb{R}^2) \to D([0, \infty), \mathbb{R})$, depending only on (c_1, c_2), such that $B = h(A, Y)$ a.s. Moreover (32) and (33) show that $\mathcal{F}_t \subseteq \mathcal{F}'_t$. Also, (i)′ implies $Y(t) = B(t) - B(t - A(t))$ so that $\mathcal{F}'_t \subseteq \mathcal{F}_t$. The law of the limit point

$$(A, Y, B, L, \tau) = (A, Y, h(A, Y), f(A), i(f(A)))$$

(on $D([0, \infty), \mathbb{R}^4) \times D([0, \infty), \mathbb{R})$) depends only on the law of (A, Y) and hence only on (c_1, c_2). This proves the weak convergence result. The remaining statements of the theorem, except for (22), have already been proved for the limit process, (A, Y, B, L, τ). It is clear, for example, that if (a) holds on some probability space then it holds on the canonical space of paths described in the theorem. For (22), apply the law of the iterated logarithm, at $t = 0$, to the Brownian motions

$$B(t + S(r)) - B(S(r)), \quad r \text{ rational},$$

where $S(r)$ is the $\mathcal{F}_t$-stopping time defined by

$$S(r) = \inf\{t \ge r \mid B(t) - B(t - A(t)) \in \{c_1(A(t))^{1/2}, c_2(A(t))^{1/2}\}\},$$

to see that (ii)′ may be strengthened to

(ii)″ For a.a. ω, $B(t) - B(g^-(\tau)(t)) \in (c_1(A(t))^{1/2}, c_2(A(t))^{1/2})$ whenever $A(t) > 0$.

(22) is now immediate. This completes the proof. □

The strong Markov process (A, Y) is not a Hunt process as it has predictable jumps.

4. A Lévy decomposition for Y. We wish to establish some further properties of the process (A, Y, B, L, τ) constructed in the previous section. Consider first the case $c_1 = 0$, $c_2 = \infty$. Recall that Y_n is constructed by running $S^{(n)}$ until it becomes negative at which time we reset Y_n to zero, and then repeat this procedure. This definition and a short computation leads to

$$Y_n(t) = S^{(n)}(t) - \inf_{s \le t} S^{(n)}(s).$$

Using Theorem 13, we may let $n \to \infty$ to get

(34) $$Y(t) = B(t) - \inf_{s \le t} B(s).$$

A well-known result of Lévy identifies $-\inf_{s \le t} B(s)$ as the local time of the zero set of Y, which in this setting equals the zero set of (A, Y). Therefore we also have

(35) $$\alpha L(t) = \inf_{s \le t} B(s)$$

for some $\alpha > 0$. We wish to extend these results to the more general square root boundaries corresponding to $c_1 \le 0$, $c_2 = \infty$. Clearly the situation here differs in that Y has discontinuous paths if $c_1 < 0$.

NOTATION. Let $J(t) = \sum_s I(\tau(s) \le t)(\tau(s) - \tau(s^-))^{1/2}$.

Since τ is a subordinator of index $\lambda_0(c_1, c_2)$, $J(t)$ is finite for all t if and only if $\lambda_0(c_1, c_2) < \frac{1}{2}$.

LEMMA 14. *If* $\lambda_0(c_1, c_2) < \frac{1}{2}$, *then*

$$A(t)^{1/2} + J(t) = \tfrac{1}{2} \int_0^t A(v)^{-1/2}\, dv.$$

PROOF. The definitions of A and J imply

$$A(t)^{1/2} + J(t) = \tfrac{1}{2} \int_{g^-(\tau)(t)}^t (v - g^-(\tau)(t))^{-1/2}\, dv + \tfrac{1}{2} \sum_{\tau(s) \le t} \int_{\tau(s^-)}^{\tau(s)} (v - \tau(s^-))^{-1/2}\, dv$$

$$= \tfrac{1}{2} \int_0^t A(v)^{-1/2}\, dv. \qquad \square$$

If $c_1 = 0$ in the following theorem, $c_1 J(t)$ and $c_1 \int_0^t A_1(s)^{-1/2}\, ds$ are understood to be zero, even though $\lambda_0(0, \infty) = \frac{1}{2}$.

THEOREM 15. *If* $c_2 = \infty$ *and* $-\infty < c_1 \le 0$, *there is a constant* $\alpha(c_1) > 0$ *such that for* a.a. ω *and all* $t \ge 0$,

$$\alpha(c_1) L(t) = -\inf_{s \le t}\left(B(s) - (c_1/2) \int_0^s A(u)^{-1/2}\, du\right)$$

$$= \tfrac{1}{2} \lim_{\varepsilon \to 0^+} \int_0^t I(Y(u) - c_1 A(u)^{1/2} \in [0, \varepsilon])\, du\, \varepsilon^{-1}$$

$$= c_1 J(t) - B(g^-(\tau)(t))$$

and

$$Y(t) = B(t) - c_1 J(t) - \inf_{s \le t}\left(B(s) - c_1/2 \int_0^t A(s)^{-1/2}\, ds\right)$$

$$= B(t) - c_1 J(t) + \alpha(c_1) L(t).$$

The proof is a straightforward application of the following result of El Karoui and Chaleyat-Maurel [12, Theorem I.1.2 and Proposition I.2.1]. All processes have sample paths in $D([0, \infty), \mathbb{R})$ and are adapted with respect to a filtration $\{\mathcal{F}_t\}$ satisfying the "usual hypotheses", i.e., $\{\mathcal{F}_t\}$ is right-continuous and $\mathcal{F}_0$ contains all the null sets.

THEOREM 16. *Let M_t be a continuous local martingale and let V_t be a continuous process whose sample paths are of bounded variation on bounded intervals. Suppose that Z and K are continuous processes satisfying, for* a.a. ω,

(i) $Z = (M + V) + K$
(ii) $Z \geq 0$
(iii) *K is non-decreasing, $K_0 = 0$ and*

$$\int_0^\infty I(Z(s) > 0)\, dK(s) = 0.$$

Then

(a) $K_t = -\inf_{s\leq t}(M_s + V_s)$
(b) *Let $\tilde{L}_t^\circ$ be the local time at zero of the semimartingale, Z,*

$$\tilde{L}_t^\circ = \lim_{\varepsilon\to 0^+} \int_0^t I(Z_s \in [0, \varepsilon])\, d[M, M]_s \varepsilon^{-1}$$

(*see Yor* [38, *Corollary* 2]). *Then*

$$K_t = -\int_0^t I(Z_s = 0)\, dV_s + \tfrac{1}{2}\tilde{L}_t^\circ.$$

□

PROOF OF THEOREM 15. We apply the above result with $M = B$, $V(t) = -c_1/2 \int_0^t A(s)^{-1/2}\, ds$, $Z(t) = Y(t) - c_1(A(t))^{1/2}$, and $K(t) = c_1 J(t) - B(g^-(\tau)(t))$. Our filtration is $\{\mathcal{F}_t\} = \{\mathcal{F}_t'\}$, introduced in Theorem 13. It must be shown that for a.a. ω and all $t \geq 0$,

$$Y(t) - c_1(A(t))^{1/2} = \left(B(t) - (c_1/2)\int_0^t A(s)^{-1/2}\, ds\right) + (c_1 J(t) - B(g^-(\tau)(t))) \tag{36}$$

$$Y(t) - c_1(A(t))^{1/2} \geq 0 \tag{37}$$

$c_1 J(t) - B(g^-(\tau)(t))$ is continuous, non-decreasing and satisfies

$$\int_0^\infty I(Y(t) - c_1(A(t))^{1/2} > 0)\, d(c_1 J(t) - B(g^-(\tau)(t))) = 0. \tag{38}$$

Note that the continuity of $Y - c_1 A^{1/2}$ follows from (36) and (38). (36) follows from Lemma 14 and Theorem 13(e), and (37) also is a consequence of Theorem 13(e) (see (ii)′ in the proof of Theorem 13). The last part of Theorem 13(e) implies that

$$B(g^-(\tau)(t)) - B(g^-(\tau)(t^-)) = c_1(J(t) - J(t^-)) \tag{39}$$

and therefore $c_1 J(t) - B(g^-(\tau)(t))$ is continuous. By Theorem 13 and the result of Skorohod and Dudley used there, we may assume $(\tau_n, S^{(n)}) \to_{\text{a.s.}} (\tau, B)$ in $D([0, \infty), \mathbb{R})^2$. Therefore $(g^-(\tau_n), S^{(n)}) \to_{\text{a.s.}} (g^-(\tau), B)$ in $D([0, \infty), \mathbb{R})^2$, since g^- is continuous at τ a.s. From this, and the continuity of B, it is easy to see that $S^{(n)}\circ g^-(\tau_n) \to_{\text{a.s.}} B\circ g^-(\tau)$ on $D([0, \infty), \mathbb{R})$. In particular, $B\circ g^-(\tau)$ is non-increasing since this is obviously true of $S^{(n)}\circ g^-(\tau_n)$, and therefore $c_1 J(t) - B(g^-(\tau)(t))$ is non-decreasing, as it is the continuous (non-decreasing) part of $-B\circ g^-(\tau)$ by (39). Fix ω so that the conclusions of Theorem 13(e) hold and suppose that $B\circ g^{-}(\tau)(t+\varepsilon) < B\circ g^{-}(\tau)(t)$ for each $\varepsilon > 0$. Then clearly $g^{-}(\tau)(t+\varepsilon) > g^{-}(\tau)(t)$ for each $\varepsilon > 0$ which implies $g^{-}(\tau)(t) = t$ and hence $(A, Y)(t) = (0, 0)$. In particular it follows that for a.a. ω,

$$\{t \mid c_1 J(t+\varepsilon) - B\circ g^{-}(\tau)(t+\varepsilon) > c_1 J(t) - B\circ g^{-}(\tau)(t) \text{ for all } \varepsilon > 0\} \subset \{t \mid (A, Y)(t) = (0, 0)\}. \tag{40}$$

This completes the proof of (38).

Theorem 16(a) now implies

$$K(t) \equiv c_1 J(t) - B(g^{-}(\tau)(t)) = -\inf_{s\le t}\left(B(s) - (c_1/2)\int_0^s A(u)^{-1/2}\,du\right). \tag{41}$$

Theorem 13(e) shows that

$$\{s \mid Y(s) - c_1(A(s))^{1/2} = 0\} \subset \overline{\{\tau(s) \mid s \ge 0\}} \quad \text{a.s.}$$

and therefore, as the latter is a Lebesgue null set,

$$\int_0^\infty I(Y(s) - c_1(A(s))^{1/2} = 0)A(s)^{-1/2}\,ds = 0 \quad \text{a.s.} \tag{42}$$

Theorem 16(b) and (42) give us,

$$K(t) = ½\,\tilde{L}_t^\circ \equiv ½\lim_{\varepsilon\to 0^+}\int_0^t I(Y(u) - c_1 A(u)^{1/2} \in [0, \varepsilon])\,du\,\varepsilon^{-1}. \tag{43}$$

This shows that $K(t)$ is a continuous additive functional of the strong Markov process (A, Y) which by (40) and (41) increases only on the zero set of (A, Y). Uniqueness of the local time, L, (see Theorem X.2 of Maisonneuve [25]) implies that $K(t) = \alpha(c_1)L(t)$ for some $\alpha(c_1) > 0$, $\alpha(c_1) = 0$ being excluded by (41). Combine this with (41) and (43) to get the first set of equalities of the theorem. Finally, these equations may be used in (36) to complete the proof. □

REMARKS. 1) In the case $c_1 = 0$, our scaling of L gives $\alpha(0) = 2^{-1/4}$ and Theorem 15 gives us

$$2^{-1/4}L(t) = -\inf_{s\le t}B(s) = ½\lim_{\varepsilon\to 0^+}\int_0^t I(Y(u) \in [0, \varepsilon])\,du\,\varepsilon^{-1},$$

where

$$Y(t) = B(t) - \inf_{s\le t}B(s)$$

(i.e., the familiar results of Lévy, described earlier).

2) Theorem 15 shows that for $c_2 = \infty$,

$$B(t) = Y(t) + c_1 J(t) - \alpha(c_1)L(t) = h(A, Y)$$

for some measurable $h: D([0, \infty), \mathbb{R}^2) \to D([0, \infty), \mathbb{R})$. The existence of such an h was established in general, in the proof of Theorem 13.

5. On the small oscillations of a Brownian path. In this section we prove results on the existence of $O(\sqrt{h})$ oscillations in a Brownian path. Recall that $-\lambda_0(c_1, c_2)$ denotes the largest eigenvalue of the Sturm-Liouville equation

$$½(d^2/dx^2 - x\,d/dx)\psi = -\lambda\psi \quad \text{on} \quad (c_1, c_2), \qquad \psi(c_i) = 0 \quad \text{if} \quad |c_i| < \infty$$

(see Proposition 2).

THEOREM 17. *If* $c_1 \le 0 \le c_2$, *then*

$$P(\exists t \ge 0, \Delta > 0 \text{ such that } B(t+h) - B(t) \in [c_1 h^{1/2}, c_2 h^{1/2}] \text{ for all } h \in [0, \Delta]) = \begin{cases} 0 & \text{if } \lambda_0(c_1, c_2) > 1 \\ 1 & \text{if } \lambda_0(c_1, c_2) < 1. \end{cases}$$

PROOF. If $\lambda_0(c_1, c_2) < 1$, the theorem is an immediate consequence of Theorem 13. Indeed choose t such that $\tau(t) > \tau(t^-)$ and note that

$$B(\tau(t^-) + h) - B(\tau(t^-)) \in (c_1 h^{1/2}, c_2 h^{1/2}) \quad \text{for} \quad h \in (0, \tau(t) - \tau(t^-))$$

(by Theorem 13(e)).

Assume $\lambda_0(c_1, c_2) > 1$. (Therefore $|c_1|$ and $|c_2|$ are finite by Proposition 2.) Fix $\Delta > 0$. It suffices to show that for a.a. ω there is no $t \le 1$ such that

$$B(t+h) - B(t) \in [c_1 h^{1/2}, c_2 h^{1/2}] \quad \text{for} \quad h \in [0, \Delta]. \tag{44}$$

Let

$$A(i, n) = \{\omega \mid (44) \text{ holds for some } t \in [(i-1)/n, i/n]\}.$$

Since $P(A(i, n)) = P(A(1, n))$, it suffices to show

$$\lim_{n\to\infty} nP(A(1, n)) = 0. \tag{45}$$

If $\omega \in A(1, n)$ and $t(\omega) \in [0, n^{-1}]$ is a point at which (44) holds, then for $n^{-1} \le \Delta$ and $s \in [t(\omega), t(\omega) + \Delta]$,

$$c_1(s-t)^{1/2} \le B(s) - B(t) \le c_2(s-t)^{1/2}$$

$$B(t) - B(n^{-1}) + c_1 s^{1/2} \le B(s) - B(n^{-1}) \le B(t) - B(n^{-1}) + c_2 s^{1/2}$$

$$-c_2 n^{-1/2} + c_1 s^{1/2} \le B(s) - B(n^{-1}) \le -c_1 n^{-1/2} + c_2 s^{1/2}, \tag{46}$$

the last because $B(n^{-1}) - B(t) \in [c_1 n^{-1/2}, c_2 n^{-1/2}]$. By the continuity of λ_0 (Proposition 2) we may choose $m \in \mathbb{N}$ such that if $c_1^m = c_1 - c_2 m^{-1/2}$ and $c_2^m = c_2 - c_1 m^{-1/2}$, then $\lambda_0(c_1^m, c_2^m) > 1$. If $s \in [m/n, \Delta]$, then using (46) we have

$$c_1^m s^{1/2} \le -c_2 m^{-1/2}(m/n)^{1/2} + c_1 s^{1/2} \le B(s) - B(n^{-1}) \le -c_1 m^{-1/2}(m/n)^{1/2} + c_2 s^{1/2} \le c_2^m s^{1/2}.$$

Therefore

$$\begin{aligned} P(A(1, n)) &\le P(B(s) - B(n^{-1}) \in [c_1^m s^{1/2}, c_2^m s^{1/2}] \quad \text{for all} \quad s \in [m/n, \Delta]) \\ &= E(P(B(s) - B(m/n) + (B(m/n) - B(n^{-1})) \in [c_1^m s^{1/2}, c_2^m s^{1/2}] \\ &\qquad \text{for all} \quad s \in [m/n, \Delta] \mid B(u), u \le m/n)) \end{aligned}$$

$$\begin{aligned} P(A(1, n)) \le \sup_y P(y + B(u) \in [c_1^m(u + m/n)^{1/2}, c_2^m(u + m/n)^{1/2}] \\ \text{for all} \quad u \le \Delta - m/n). \end{aligned} \tag{47}$$

Note that if U denotes an Ornstein-Uhlenbeck process, $\rho(c_1^m, c_2^m)$ is the first time U exits from $[c_1^m, c_2^m]$, $y \in \mathbb{R}$ and $\varepsilon, \delta > 0$, then for some $k > 0$,

$$\begin{aligned} &P(y + B(u) \in [c_1^m(u+\varepsilon)^{1/2}, c_2^m(u+\varepsilon)^{1/2}] \quad \text{for} \quad u \le \delta) \\ &\quad = P(y\varepsilon^{-1/2} + B(u) \in [c_1^m(u+1)^{1/2}, c_2^m(u+1)^{1/2}] \quad \text{for} \quad u \le \delta\varepsilon^{-1}) \quad \text{by scaling} \\ &\quad = P_{y\varepsilon^{-1/2}}(\rho(c_1^m, c_2^m) > \log(\delta\varepsilon^{-1} + 1)) \\ &\quad \le k \exp\{-\lambda_0(c_1^m, c_2^m)\log(\delta\varepsilon^{-1} + 1)\} \quad \text{by} \quad (6) \\ &\quad = k(1 + \delta\varepsilon^{-1})^{-\lambda_0(c_1^m, c_2^m)}, \end{aligned}$$

where k is independent of (y, ε, δ).

Substituting this upper bound into (47) leads to

$$nP(A(1, n)) \leq nk(1 + (\Delta - m/n)(n/m))^{-\lambda_0(c_1''', c_2''')} = k(\Delta/m)^{-\lambda_0(c_1''', c_2''')} n^{1-\lambda_0(c_1''', c_2''')}$$
$$\to 0 \quad \text{as} \quad n \to \infty,$$

the last, since $\lambda_0(c_1'''; c_2''') > 1$. This gives (45) and hence completes the proof. □

Theorem 1(a) is immediate from the above result, the strict monotonicity of $\lambda_0(-c, c)$, and the fact that $\lambda_0(-1, 1) = 1$ (see Proposition 2).

In the course of the above proof we have established the following.

LEMMA 18. *If $-\infty < c_1' \leq 0 \leq c_2' < \infty$, there is a constant $k = k(c_1', c_2')$ such that*

$$P(B(s) - B(n^{-1}) \in [c_1' s^{1/2}, c_2' s^{1/2}] \quad \text{for all} \quad s \in [m/n, \Delta])$$
$$\leq k(\Delta m^{-1} n)^{-\lambda_0(c_1', c_2')} \qquad \forall m, n, \Delta > 0. \qquad \square$$

REMARKS. Let

$$R(c_1, c_2) = \{t \mid B(t + h) - B(t) \in [c_1 h^{1/2}, c_2 h^{1/2}] \quad \text{for all} \quad h \in [0, \Delta], \quad \text{for some} \quad \Delta > 0\},$$

and assume $\lambda_0(c_1, c_2) < 1$. We claim that $R(c_1, c_2)$ is a.s. dense in $[0, \infty)$. The proof of Theorem 17 shows that $P(R(c_1, c_2) \cap [0, \varepsilon] \neq \phi$ for each $\varepsilon > 0) = 1$. The claim follows by applying this result to the countable collection of Brownian motions $B_{(r)}(t) = B(r + t) - B(r)$ for r a rational.

The proof of Theorem 17 could have been given immediately after the proof of Theorem 5. In this case (B, τ) would be a limit point of the tight sequence $\{(S^{(n)}, \tau_n)\}$ and one would only need (22) for each such limit point, a fact which is fairly easy to prove (see the proof of Theorem 13).

Our proof showing that $P(R(c_1, c_2) = \phi) = 1$ if $\lambda_0(c_1, c_2) > 1$ is a refinement of the argument in Dvoretsky [11]. The harder problem is to find times in $R(c_1, c_2)$ and this follows from what little we know about the joint law of (B, τ).

The situation in the critical case, $\lambda_0(c_1, c_2) = 1$, remains unresolved. It would also be of interest to see if the results of Section 2 could be extended to the case $c_2 > c_1 > 0$ as this would lead to an interesting analogue of Theorem 17.

We next consider the existence of times of $O(h^{1/2})$ oscillations in the zero set of a Brownian motion, B. Knight [20] showed that for $k > \frac{1}{2}$ and for a.a. ω there are times t such that

(48) $$B(t) = 0 \quad \text{and} \quad \limsup_{h \to 0^+} |B(t + h)| (2h \log \log h^{-1})^{-1/2} < k.$$

Kahane [19] later showed that (48) could be strengthened to

$$B(t) = 0 \quad \text{and} \quad \limsup_{h \to 0^+} |B(t + h)| h^{-1/2} < \infty.$$

A more precise result is the following:

THEOREM 19. *If $c_1 \leq 0 \leq c_2$, then*

$$P(\exists t \geq 0, \Delta > 0 \textit{ such that } B(t) = 0 \textit{ and } B(t + h) \in [c_1 h^{1/2}, c_2 h^{1/2}] \textit{ for all } h \in [0, \Delta])$$

$$= \begin{cases} 0 & \text{if} \quad \lambda_0(c_1, c_2) > \frac{1}{2} \\ 1 & \text{if} \quad \lambda_0(c_1, c_2) < \frac{1}{2}. \end{cases}$$

The proof illustrates the fact, noted in the above remarks, that only a small portion of Theorem 13 is needed to establish Theorem 17. The argument is similar to the proof of

Theorem 17, but we assume

$$P(X_i = 1) = P(X_i = -1) = \tfrac{1}{2},$$

and instead of

$$T = \min\{n \mid S_n \notin [c_1 n^{1/2}, c_2 n^{1/2}]\},$$

we work with

$$V = \min\{n > T \mid S_n = 0\}.$$

The main step is to prove that if $\lambda_0(c_1, c_2) < \tfrac{1}{2}$, $P(V > n)$ has the same asymptotic behaviour as $P(T > n)$ as $n \to \infty$. Roughly speaking, this is true because the sample path takes *about* as long to return to zero from a boundary poiint as it took to reach that point.

LEMMA 20. *If $\lambda_0(c_1, c_2) < \tfrac{1}{2}$, and $\alpha(c_1, c_2)$ is as in Proposition 12, then*

$$\lim_{n\to\infty} P(V > n) P(T > n)^{-1}$$
$$= 1 + (2/\pi)^{1/2} \int_0^\infty e^{-z^2/2} z^{-2\lambda_0(c_1,c_2)}((1 - \alpha(c_1, c_2))(c_1^2 + z^2)^{\lambda_0(c_1,c_2)}$$
$$+ \alpha(c_1, c_2)(c_1^2 + z^2)^{\lambda_0(c_1,c_2)})\, dz.$$

In particular, V is in the domain of attraction of a stable law of index $\lambda_0(c_1, c_2)$.

PROOF.

$$(49)\quad \begin{aligned} P(V > n) &= P(T > n) + \int_0^n \int_{-\infty}^\infty P(y + S_k \neq 0, \forall k \le n - t) P((T, S_T) \in dt \times dy) \\ &= P(T > n) + \int_0^n \int_{-\infty}^\infty P(S_{n-t} \in [-|y|, |y|)) P((T, S_T) \in dt \times dy), \end{aligned}$$

the last by the reflection principle. Use (49) and the fact that $|S_k - S_{k-1}| = 1$ to see that

$$P(V > n) P(T > n)^{-1}$$

$$(50)\quad \begin{aligned} &\le 1 + \int_0^n P(S_{n-t} \in [c_1 t^{1/2} - 1, -c_1 t^{1/2} + 1)) P(T \in dt, S_T < 0) P(T > n)^{-1} \\ &\quad + \int_0^n P(S_{n-t} \in [-c_2 t^{1/2} - 1, c_2 t^{1/2} + 1)) P(T \in dt, S_T > 0) P(T > n)^{-1} \\ &\le 1 + \int_0^1 P(S^{(n(1-t))}(1) \in I_t^{n,1}) P(T/n \in dt, S_T < 0) P(T > n)^{-1} \\ &\quad + \int_0^1 P(S^{(n(1-t))}(1) \in I_t^{n,2}) P(T/n \in dt, S_T > 0) P(T > n)^{-1}, \end{aligned}$$

where

$$I_t^{n,i} = [-\alpha_i(t) - (n(1-t))^{-1/2}, \alpha_i(t) + (n(1-t))^{-1/2}),$$
$$\alpha_i(t) = |c_i| t^{1/2}(1-t)^{-1/2} \qquad (i = 1, 2).$$

The Berry-Esseen theorem (Feller [13, page 542]) shows that

$$\begin{aligned} &|P(S^{(n(1-t))}(1) \in I_t^{n,i}) - P(|B(1)| \le \alpha_i(t))| \\ &\le |P(S^{(n(1-t))}(1) \in I_t^{n,i}) - P(B(1) \in I_t^{n,i})| + |P(B(1) \in I_t^{n,i}) - P(|B(1)| \le \alpha_i(t))| \\ &\le \min(2, c(n(1-t))^{-1/2}) (c > 0). \end{aligned}$$

Therefore if $\varepsilon \in (0, 1)$,

$$(51) \quad \limsup_{n\to\infty} \int_0^1 | P(S^{(n(1-t))}(1) \in I_t^{n,i}) - P(|B(1)| \le \alpha_i(t))| \, P(T/n \in dt)P(T > n)^{-1}$$

$$\le \limsup_{n\to\infty} c(n\varepsilon)^{-1/2}P(T > n)^{-1} + 2P(n(1-\varepsilon) \le T \le n)P(T > n)^{-1}$$
$$= 2((1-\varepsilon)^{-\lambda_0(c_1,c_2)} - 1), \quad \text{by Theorem 5(b).}$$

Letting ε approach 0 in the above, we see that (51) equals zero. Substitute this result into (50) to obtain (let $\beta_i(z) = z^2(c_i^2 + z^2)^{-1}$)

$$\limsup_{n\to\infty} P(V > n)P(T > n)^{-1}$$

$$\le 1 + \limsup_{n\to\infty}\left(\int_0^1 P(|B(1)| \le \alpha_1(t))P(T/n \in dt, S_T < 0)P(T > n)^{-1}\right.$$

$$\left. + \int_0^1 P(|B(1)| \le \alpha_2(t))P(T/n \in dt, S_T > 0)P(T > n)^{-1}\right)$$

$$= 1 + \limsup_{n\to\infty}\left((2/\pi)^{1/2}\int_0^\infty e^{-z^2/2}P(T > n\beta_1(z), S_T < 0)\, dz P(T > n)^{-1}\right.$$

$$\left. + (2/\pi)^{1/2}\int_0^\infty e^{-z^2/2}P(T > n\beta_2(z), S_T > 0)\, dz P(T > n)^{-1}\right) \quad \text{(Fubini Theorem)}$$

$$\therefore \limsup_{n\to\infty} P(V > n)P(T > n)^{-1} \le 1 + \limsup_{n\to\infty}\left((2/\pi)^{1/2}\right.$$

$$(52) \qquad \cdot \int_0^\infty e^{-z^2/2}P(S_T < 0 \mid T > n\beta_1(z))P(T > n\beta_1(z))P(T > n)^{-1}\, dz$$

$$\left. + (2/\pi)^{1/2}\int_0^\infty e^{-z^2/2}P(S_T > 0 \mid T > n\beta_2(z))P(T > n\beta_2(z))P(T > n)^{-1}\, dz\right).$$

By Proposition 12 and Theorem 5(b), if $z > 0$, then

$$(53) \quad \lim_{n\to\infty} P(S_T < 0 \mid T > n\beta_1(z))P(T > n\beta_1(z))P(T > n)^{-1} = (1 - \alpha(c_1, c_2))\beta_1(z)^{-\lambda_0(c_1,c_2)},$$

$$(54) \quad \lim_{n\to\infty} P(S_T > 0 \mid T > n\beta_2(z))P(T > n\beta_2(z))P(T > n)^{-1} = \alpha(c_1, c_2)\beta_2(z)^{-\lambda_0(c_1,c_2)}.$$

If $P(T > x) = \pi(x)x^{-\lambda_0(c_1,c_2)}$, where π is slowly varying at ∞ (Theorem 5(b)), then by Feller [13, page 282],

$$\pi(x) = a(x)\exp\left(\int_1^x \varepsilon(t)t^{-1}\, dt\right),$$

where $\lim_{x\to\infty} a(x) = c \in (0, \infty)$, $\lim_{t\to 0^+} \varepsilon(t) = 0$. Choose $\varepsilon > 0$ such that $2(\lambda_0(c_1, c_2) + \varepsilon) < 1$, and $N > 0$ such that $|\varepsilon(t)| < \varepsilon$ and $|a(x)a(y)^{-1} - 1| < 1$ whenever $t, x, y > N$. Note that

$$\limsup_{n\to\infty} \int I(n\beta_i(z) \le N)e^{-z^2/2}P(T > n\beta_i(z))P(T > n)^{-1}\, dz$$

$$\le \limsup_{n\to n} c(n - N)^{-1/2}P(T > n)^{-1} = 0,$$

the last, by Theorem 5(b). By combining this result with (52), one gets

$$\limsup_{n\to\infty} P(V>n)P(T>n)^{-1} \le 1 + (2/\pi)^{1/2}\limsup_{n\to\infty}\Bigg(\int_0^\infty I(n\beta_1(z)>N) \cdot e^{-z^2/2}P(S_T<0 \mid T>n\beta_1(z))P(T>n\beta_1(z))P(T>n)^{-1}\,dz$$

(55)

$$+\int_0^\infty I(n\beta_2(z)>N)e^{-z^2/2}P(S_T>0 \mid T>n\beta_2(z))P(T>n\beta_2(z))P(T>n)^{-1}\,dz\Bigg).$$

Note that

$$\begin{aligned} &I(n\beta_i(z)>N)P(T>n\beta_i(z))P(T>n)^{-1}\\ &\quad= \beta_i(z)^{-\lambda_0(c_1,c_2)}a(n\beta_i(z))a(n)^{-1}\exp\left\{-\int_{n\beta_i(z)}^{n}\varepsilon(t)t^{-1}\,dt\right\}I(n\beta_i(z)>N)\\ &\quad\le \beta_i(z)^{-\lambda_0(c_1,c_2)}2\exp\{\varepsilon\log(\beta_i(z)^{-1})\} = 2(\beta_i(z))^{-\lambda_0(c_1,c_2)-\varepsilon}\\ &\quad\le c(I(z>1)+z^{-2(\lambda_0+\varepsilon)}I(z\le 1)) \quad (\text{for some } c>0). \end{aligned}$$

The choice of ε makes the above expression integrable with respect to $e^{-z^2/2}\,dz$. We therefore may use the dominated convergence theorem, (53) and (54) to evaluate the right side of (55) and obtain

$$\limsup_{n\to\infty} P(V>n)P(T>n)^{-1} \le 1 + (2/\pi)^{1/2}\int_0^\infty e^{-z^2/2}((1-\alpha(c_1,c_2))\beta_1(z)^{-\lambda_0(c_1,c_2)} + \alpha(c_1,c_2)\beta_2(z)^{-\lambda_0(c_1,c_2)})\,dz.$$

By making minor changes in the above arguments, one sees that $\liminf_{n\to\infty} P(V>n)$ $P(T>n)^{-1}$ is bounded below by the same expression, and we have evaluated $\lim_{n\to\infty} P(V>n)P(T>n)^{-1}$, as required. The last statement of the lemma is immediate from Theorem 5(b). □

Proof of Theorem 19. Suppose first that $\lambda_0(c_1, c_2) < \frac{1}{2}$. Define a sequence of stopping times $\{V_n\}$ by setting $V_0 = 0$ and

$$V_{i+1} = V_i + V\circ\theta_{V_i}$$

where $S_n(\theta_{V_i}(\omega)) = S_{n+V_i(\omega)}(\omega)$. If $b'_n = P(V>n)^{-1}$, then it follows from the previous lemma that

$$\tau'_n(t) \equiv V_{[b'_n t]}n^{-1} \to_w \tau'(t) \quad \text{on} \quad D(\mathbb{R}),$$

where $\tau'(t)$ is a stable subordinator of index $\lambda_0(c_1, c_2)$. The sequence of measures induced on $D([0,\infty), \mathbb{R})^2$ by $\{(S^{(n)}, \tau'_n) \mid n \in \mathbb{N}\}$ is therefore tight and by applying the result of Skorohod [30, page 10] and Dudley [10] we may assume, by changing the probability space, if necessary, that there is a subsequence such that

$$(S^{(n_k)}, \tau'_{n_k}) \to (B, \tau') \quad \text{a.s. on} \quad D([0,\infty), \mathbb{R})^2,$$

where B is a Brownian motion and τ' is as above.

Suppose that for some $N \in \mathbb{N}$,

(56) $P(\exists\Delta>0,\ t\le N \text{ such that } B(t)=0 \text{ and } B(t+h)\in[c_1h^{1/2}, c_2h^{1/2}]\ \forall h\le\Delta) \equiv p > 0.$

Expression (56) for the Brownian motion $\varepsilon^{-1/2}B(\varepsilon t)$ is

$$P(\exists\Delta > 0,\ t \le \varepsilon N \text{ such that } B(t) = 0 \text{ and } B(t+h) \in [c_1 h^{1/2}, c_2 h^{1/2}]\forall h \le \Delta) = p > 0.$$

$$\therefore\quad P(\forall\varepsilon > 0,\ \exists t \le \varepsilon \text{ and } \Delta > 0 \text{ such that } B(t) = 0 \text{ and } B(t+h) \in [c_1 h^{1/2}, c_2 h^{1/2}]\forall h \le \Delta)$$
$$= p > 0.$$

The zero-one law implies that the above probability must be one, as required. It suffices, therefore, to prove (56). Note first that

$$S^{(n_k)}(\tau'_{n_k}(t)) = 0 \qquad \forall t \ge 0 \quad \text{a.s.}$$

Letting $k \to \infty$ gives

$$B(\tau'(t)) = 0 \qquad \forall t \ge 0 \quad \text{a.s.}$$

It follows that

$$B(g^-(\tau')(t)) = B(g^+(\tau')(t)) = 0 \qquad \forall t \ge 0 \quad \text{a.s.} \tag{57}$$

Now argue as follows: Since the range of τ' is a subset of the zero set of B, if (56) fails and there are boundary crossings after all the 0's of B then τ' has range the zero set of B and hence has index ½. But τ' has index $\lambda_0 < ½$. The details:

Suppose (56) is false. Then by (57)

$$N = \{\omega \mid \exists\Delta > 0,\ t > 0 \text{ such that } B(g^-(\tau')(t) + h) \in [c_1 h^{1/2}, c_2 h^{1/2}]\forall h \le \Delta\}$$

is a null set. Fix $\omega \notin N$ such that $(S^{(n_k)}, \tau'_{n_k}) \to (B, \tau')$ in $D([0, \infty), \mathbb{R})^2$, $\tau'(\cdot, \omega)$ is a continuity point of g^- (see the remarks preceding Theorem 13), and $g^-(\tau')(\cdot)$ is continuous at each rational $r \ge 0$, by omitting a null set. Fix a rational $r \ge 0$. There is a sequence $h_n(r, \omega) \downarrow 0$ such that

$$B(g^-(\tau')(r) + h_n) \notin [c_1 h_n^{1/2}, c_2 h_n^{1/2}].$$

Therefore, since $\lim_{k\to\infty} g^-(\tau'_{n_k})(r) = g^-(\tau')(r)$, there is a $k_n(r, \omega)$ such that $k \ge k_n$ implies

$$S^{(n_k)}(g^-(\tau'_{n_k})(r) + h_n) \notin [c_1 h_n^{1/2}, c_2 h_n^{1/2}]$$

and hence

$$S^{(n_k)}(u) \ne 0\ \forall u \in [g^-(\tau'_{n_k})(r) + h_n, g^+(\tau'_{n_k})(r))$$

(by the definition of τ'_{n_k}). As $S^{(n_k)} \to B$ uniformly on compacts and B has no local extrema in $\{t \mid B(t, \omega) = 0\}$, by omitting another null set, we see that, by letting $k \to \infty$ and $n \to \infty$, in that order,

$$B(u) \ne 0 \qquad \forall u \in (g^-(\tau')(r), g^+(\tau')(r))\ \forall \text{ rational } r \ge 0. \tag{58}$$

(57) and (58) together show that the jumps of τ' coincide exactly with the excursions of B away from zero. Therefore (see Itô and McKean [17, page 43])

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \varepsilon^{1/2} \operatorname{card}\{s \mid \tau'(s) \le t, \Delta\tau'(s) > \varepsilon\} \\ &\quad = \lim_{\varepsilon\to 0^+} \varepsilon^{1/2} \times (\text{no. of excursions of } B \text{ exceeding } \varepsilon \text{ in length and} \\ &\qquad\qquad \text{completed by time } t) \\ &\quad = cL_t^\circ, \end{aligned} \tag{59}$$

where L_t° is local time of B at zero and $c > 0$. On the other hand, the law of large numbers implies that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\lambda_0} \operatorname{card}\{s \mid \tau'(s) \le t, \Delta\tau'(s) > \varepsilon\} = c'\tau'^{-1}(t), \tag{60}$$

for some $c' > 0$, where τ'^{-1} is the continuous functional inverse of τ'. Since $\lambda_0 < ½$, (59) and (60) give us a contradiction. Therefore (56) must be true and the proof is complete for the case $\lambda_0(c_1, c_2) < ½$.

Suppose now that $\lambda_0(c_1, c_2) > \frac{1}{2}$. It suffices to fix $\Delta > 0$ and show that for a.a. ω there is no $t \leq 1$ such that

$$B(t) = 0 \quad \text{and} \quad B(t+h) \in [c_1 h^{1/2}, c_2 h^{1/2}] \quad \text{for} \quad h \in [0, \Delta]. \tag{61}$$

Let

$$A'(i, n) = \{\omega \mid (61) \text{ holds for some } t \in [(i-1)/n, i/n]\}.$$

Choose $m \in N$ such that if $c_1^m = c_1 - c_2 m^{-1/2}$ and $c_2^m = c_2 - c_1 m^{-1/2}$, then $\lambda_0(c_1^m, c_2^m) > \frac{1}{2}$. Argue just as in the proof of Theorem 17 to see that

$$\begin{aligned} A'(i, n) \subset \{\omega \mid & B(t) = 0 \text{ for some } t \in [(i-1)/n, i/n] \quad \text{and} \\ & B(s) - B(in^{-1}) \in [c_1^m (s - (i-1)/n)^{1/2}, c_2^m (s - (i-1)/n)^{1/2}] \\ & \text{for all} \quad s \in [(i + m - 1)/n, (i-1)/n + \Delta]\}. \end{aligned}$$

Therefore there is a constant $c > 0$, depending only on (c_1^m, c_2^m), such that

$$\begin{aligned} & P(\exists t \leq 1 \text{ such that (61) holds}) \\ & \leq \textstyle\sum_{i=1}^n P(A'(i, n)) \\ & \leq P(B(s) - B(n^{-1}) \in [c_1^m s^{1/2}, c_2^m s^{1/2}] \quad \text{for all} \quad s \in [m/n, \Delta]) \\ & \quad \times \textstyle\sum_{i=1}^n P(B(t) = 0 \quad \text{for some} \quad t \in [(i-1)/n, i/n]) \\ & \leq c(\Delta/m)^{-\lambda_0(c_1^m, c_2^m)} n^{-\lambda_0(c_1^m, c_2^m)} \textstyle\sum_{i=1}^n i^{-1/2} \\ & \quad \text{(by Lemma 18 and Lévy [22, Chapter VI, Theorem 44.1])} \\ & \leq 2c(\Delta/m)^{-\lambda_0(c_1^m, c_2^m)} n^{(1/2) - \lambda_0(c_1^m, c_2^m)} \\ & \to 0 \text{ as } n \to \infty, \end{aligned}$$

the last since $\lambda_0(c_1^m, c_2^m) > \frac{1}{2}$. This completes the proof. □

REMARKS. It follows easily from the proof of Lemma 20 that

$$\lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} P(T \leq \varepsilon n \mid U > n) = 0.$$

By using this result and being just a little more careful in the previous proof one can show that for a.a. ω, if $\tau'(t^-) < \tau'(t)$ then there is a $\Delta(t, \omega) > 0$ such that

$$B(\tau'(t^-)) = 0 \quad \text{and} \quad B(\tau'(t^-) + u) \in (c_1 u^{1/2}, c_2 u^{1/2}) \quad \text{for} \quad 0 < u < \Delta(t, \omega).$$

Here (B, τ') is any limit point of $\{(S^{(n)}, \tau'_n) \mid n \in N\}$. Again, the jumps of τ' select an "excursion" of B with the required properties.

The situation in the critical case $\lambda_0(c_1, c_2) = \frac{1}{2}$ remains unresolved, except of course when $(c_1, c_2) = (-\infty, 0)$ or $(0, \infty)$ (in which case there do exist excursions away from zero!)

Finally we are ready to complete the proof of Theorem 1.

PROOF OF THEOREM 1(b). In view of the previous result and the strict monotonicity of $\lambda_0(-c, c)$, it suffices to show that if $c_0 > 0$ satisfies

$$\textstyle\sum_{n=1}^\infty (c_0^2/2)^n ((2n-1)n!)^{-1} = 1, \tag{62}$$

then $\lambda_0(-c_0, c_0) = \frac{1}{2}$. For this we argue as in the proof of Theorem 1 in Breiman [6]. It follows from (6) that $-\lambda_0(-c, c)$ is the largest real pole of

$$\phi(\lambda) = E_0(e^{-\lambda \rho(-c, c)}).$$

From Breiman [6] (but note that his Ornstein-Uhlenbeck process has generator $d^2/dy^2 - y d/dy$) we have, for $\text{Re}(\lambda) \geq 0$,

$$\phi(\lambda)^{-1} = 2^{1-\lambda}\Gamma(\lambda)^{-1}\int_0^\infty e^{-t^2/2}t^{2\lambda-1}\cosh(ct)\,dt$$

(63)
$$= 2^{1-\lambda}\Gamma(\lambda)^{-1}\left(\int_0^\infty e^{-t^2/2}t^{2\lambda-1}\,dt + \int_0^\infty e^{-t^2/2}t^{2\lambda-1}\textstyle\sum_{n=1}^\infty (ct)^{2n}/(2n)!\,dt\right)$$
$$= 1 + 2^{1-\lambda}\Gamma(\lambda)^{-1}\textstyle\sum_{n=1}^\infty c^{2n}((2n)!)^{-1}2^{n+\lambda-1}\Gamma(n+\lambda)$$
$$= 1 + \textstyle\sum_{n=1}^\infty (2c^2)^n((2n)!)^{-1}(n+\lambda-1)\times(n+\lambda-2)\times\cdots\times\lambda.$$

The above expression may be used to analytically continue $\phi(\lambda)^{-1}$ to $Re(\lambda) < 0$. Set $\lambda = -1/2$ and note that

$$\phi(-1/2)^{-1} = 0 \Leftrightarrow \textstyle\sum_{n=1}^\infty c^{2n}(2n-3)(2n-5)\cdots(1)((2n)!)^{-1} = 1$$
$$\Leftrightarrow \textstyle\sum_{n=1}^\infty (c^2/2)^n((2n-1)n!)^{-1} = 1 \Leftrightarrow c = c_0.$$

For any fixed c, (63) shows that $\phi(\lambda)^{-1} = 0$ has at most one solution $\lambda \in (-1, 0)$. Therefore if such a λ exists, it must be $-\lambda_0(-c, c)$, the largest zero of $\phi(\lambda)^{-1}$. In particular, $-\lambda_0(-c_0, c_0) = -1/2$, as required. The estimate $c_0 \simeq 1.3069$ is given by Shepp [28] (as the series in (62) converges rapidly, c_0 is easy to estimate). □

APPENDIX

In this appendix we use some nonstandard techniques to establish the tightness of the sequence of processes considered in Theorem 13. The nonstandard viewpoint simplifies compactness problems in

$$D([0,\infty), \mathbb{R}^k) = \{d : [0,\infty) \to \mathbb{R}^k \mid d \text{ right-continuous with left limits}\},$$

equipped with the Skorohod J_1 topology. A good introduction to nonstandard probability theory may be found in Loeb [24], while more general introductions to nonstandard analysis are Davis [9] and Stroyan and Luxemburg [34].

We work in an ω_1-saturated enlargement of a superstructure containing $\mathbb{R}$. If $x \in {}^*\mathbb{R}^k$, ${}^\circ x$ denotes the standard part of x, when it exists.

DEFINITIONS. A function $F \in {}^*D([0,\infty), \mathbb{R}^k)$ is SDJ if the following conditions hold:

(i) ${}^\circ F(t)$ is finite whenever ${}^\circ t$ is finite.
(ii) ${}^\circ F(t) = {}^\circ F(0)$ whenever ${}^\circ t = 0$.
(iii) For each $t \in [0,\infty)$ there is a $\underline{t} \approx t$ such that
${}^\circ F(u) = {}^\circ F(\underline{t})$ for all $u \in \{u \mid u \geq \underline{t}, {}^\circ u = t\}$, and
${}^\circ F(u) = {}^\circ F(\underline{t}^-)$ for all $u \in \{u \mid u < \underline{t}, {}^\circ u = t\}$.

If instead of (iii) we assume that ${}^\circ F(t_1) = {}^\circ F(t_2)$ whenever ${}^\circ t_1 = {}^\circ t_2 < \infty$, then F is SC (S-continuous).

If $F \in {}^*D([0,\infty), \mathbb{R}^k)$ is SDJ, define a function $st(F) : [0,\infty) \to \mathbb{R}^k$ by

$$st(F)(t) = \lim_{{}^\circ u \to t^+} {}^\circ F(u).$$

(It is understood that ${}^\circ u > t$ in the limit). □

It is trivial to check that $st(F)$ is well-defined for F SDJ.

The nonstandard interpretation of the J_1 topology is contained in the following result which may be found in Hoover and Perkins [15, Theorem 2.6] or Stroyan and Bayod [33].

PROPOSITION A1. *F is* SDJ *if and only if F is nearstandard in ${}^*D([0,\infty), \mathbb{R}^k)$ and in this case $st(F)$ is the standard part of F in $D([0,\infty), \mathbb{R}^k)$.* □

The following proposition is an immediate consequence of the above and the nonstandard characterization of weak convergence and tightness of a sequence of measures (see Anderson and Rashid [2] or Loeb [23]) on a Polish space.

PROPOSITION A2. *Let X_n be a sequence of stochastic processes with sample paths in $D([0, \infty), \mathbb{R}^k)$, defined on a common probability space $(X, \mathcal{G}, Q)$. Let $(\Omega, \mathcal{F}, P) = ({}^*X, L({}^*\mathcal{G}), L({}^*Q))$ be the Loeb space generated by $(X, \mathcal{G}, Q)$. Then the sequence $\{X_n\}$ is tight in law in $D([0, \infty), \mathbb{R}^k)$ if and only if for each infinite η in ${}^*\mathbb{N}$, $P(X_\eta$ is SDJ$) = 1$. The laws of $\{X_n\}$ converge to a law Q on $D([0, \infty), \mathbb{R}^k)$ if, in addition, for each infinite η in ${}^*\mathbb{N}$,*

$$P(st(X_\eta) \in \cdot) = Q(\cdot). \qquad \square$$

The following result was used in the proof of Theorem 13.

PROPOSITION A3. *$\{(A_n, Y_n, S^{(n)}, L_n, \tau_n) \mid n \in \mathbb{N}\}$ is tight in $D([0, \infty), \mathbb{R}^4) \times D([0, \infty), \mathbb{R})$.*

PROOF. Since we have already seen that $\tau_n \to_w \tau$ in $D([0, \infty), \mathbb{R})$, it suffices to show that $\{(A_n, Y_n, S^{(n)}, L_n)\}$ is tight in $D([0, \infty), \mathbb{R}^4)$. Fix $\eta \in {}^*\mathbb{N} - \mathbb{N}$ and let $(\Omega, \mathcal{F}, P)$ denote the Loeb space used in Proposition A2 (clearly we may assume the sequence of processes are defined on a common probability space). Since $S^{(n)} \to_w B$, it follows (Proposition A2) that $S^{(\eta)}$ is SDJ a.s. The continuity of B implies that in fact $S^{(\eta)}$ must be SC a.s. Note that $g^-(\tau_\eta)$ is SDJ by definition. Therefore

$$Y_\eta = S^{(\eta)} - S^{(\eta)} \circ g^-(\tau_\eta) \quad \text{and} \quad A_\eta(t) = t - g^-(\tau_\eta)(t)$$

are both SDJ. If ω is fixed so that $S^{(\eta)}(\cdot, \omega)$ is SC, then ${}^\circ Y_\eta(t) \neq {}^\circ Y_\eta(t^-)$ $({}^\circ t < \infty)$ implies ${}^\circ A_\eta(t) \neq {}^\circ A_\eta(t^-)$. That is, the macroscopic jumps of Y_η coincide precisely with macroscopic jumps of A_η. Therefore, (A_η, Y_η) is SDJ. Recall that L_η is the right-continuous inverse of τ_η and $st(\tau_\eta)$ is a.s. strictly increasing to ∞ (since $\mathrm{st}(\tau_\eta)$ is a stable subordinator by Proposition A2.) It follows that L_η is SC. The facts that $(L_\eta, S^{(\eta)})$ is SC (a.s.) and (A_η, Y_η) is SDJ (a.s.) together imply that $(A_\eta, Y_\eta, S^{(\eta)}, L_\eta)$ is SDJ (a.s.) and hence the tightness of $\{(A_n, Y_n, S^{(n)}, L_n)\}$ follows from Proposition A2. $\square$

Added in proof. After submitting this manuscript, we learnt of work of Burgess Davis ("On Brownian slow points", to appear in *Z. Wahrsh. verw. Gebiete*) that was done independently and at about the same time. Davis gives a different and more direct proof of Theorem 1(a) and also proves a version of this result with $c_1 = 1$ and $c_2 = \infty$. This answers a question raised in the remarks following Lemma 18.

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