The Annals of Probability
1983, Vol. 11, No. 3, 827–832

ON THE CESÀRO MEANS OF ORTHOGONAL SEQUENCES OF RANDOM VARIABLES

BY FERENC MÓRICZ

Szeged University

Let $\{\xi_k: k \geq 0\}$ be an orthogonal sequence of random variables with finite second moments $E\xi_k^2 = \sigma_k^2$. It is well-known that if $\sum_{k=0}^\infty \sigma_k^2(k+1)^{-2}[\log(k+2)]^2 < \infty$, then the first arithmetic means $\tau_n^0 := (n+1)^{-1}\sum_{k=0}^n \xi_k \to 0$ a.s. $(n \to \infty)$. Now we prove that the means $\tau_n^1 := (n+1)^{-1}\sum_{k=0}^n (1 - k(n+1)^{-1})\xi_k \to 0$ a.s. $(n \to \infty)$ merely under the condition $\sum_{k=0}^\infty \sigma_k^2(k+1)^{-2} < \infty$. We define the means τ_n^α for every real α, too and prove that under the latter condition $\tau_n^\alpha \to 0$ a.s. $(n \to \infty)$ provided $\alpha > 0$.

1. Cesàro means of numerical sequences. Let α be a real number, let $\{u_k: k \geq 0\}$ be a sequence of real numbers, and define

$$t_n^\alpha = \frac{1}{(n+1)A_n^\alpha}\sum_{k=0}^n A_{n-k}^\alpha u_k \qquad (n = 0, 1, \cdots), \tag{1}$$

where $A_0^\alpha = 1$ and

$$A_n^\alpha = \binom{\alpha+n}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \qquad (n = 1, 2, \cdots).$$

In particular, if $\alpha = -1, -2, \cdots$, then A_n^α is zero for large enough n.

Introducing the notation

$$s_n^\alpha = \sum_{k=0}^n A_{n-k}^\alpha u_k,$$

we can write

$$t_n^\alpha = \frac{s_n^\alpha}{(n+1)A_n^\alpha} \qquad (n = 0, 1, \cdots). \tag{2}$$

In particular,

$$s_n^0 = \sum_{k=0}^n u_k \quad \text{and} \quad s_n^{-1} = u_n.$$

We remind the following well-known identities (see, e.g. [4, pages 76–77]):

$$\sum_{n=0}^\infty A_n^\alpha z^n = (1-z)^{-\alpha-1}$$

and

$$\sum_{n=0}^\infty s_n^\alpha z^n = (1-z)^{-\alpha-1}\sum_{n=0}^\infty u_n z^n,$$

where z is a real or complex parameter, $|z| < 1$. Hence one can deduce that

$$A_n^{\alpha+\beta+1} = \sum_{k=0}^n A_k^\alpha A_{n-k}^\beta \tag{3}$$

and

$$s_n^{\alpha+\beta+1} = \sum_{k=0}^n s_k^\alpha A_{n-k}^\beta \tag{4}$$

for all α and β. Furthermore, for $\alpha \neq -1, -2, \cdots$

$$A_n^\alpha = \frac{n^\alpha}{\Gamma(\alpha+1)}(1 + o(1)) \quad (n \to \infty). \tag{5}$$

Received September 1982.
AMS 1980 *subject classification*. Primary 60F15; secondary 60G46.
Key words and phrases. Orthogonal random variables, Cesàro means of sequences, strong law of large numbers.

2. Main results. Let $\{\xi_k: k \geq 0\}$ be an orthogonal sequence of random variables, i.e.

$$E\xi_k\xi_\ell = 0 \quad (k \neq \ell;\ k, \ell = 0, 1, \cdots)$$

with finite second moments

$$E\xi_k^2 = \sigma_k^2 \quad (k = 0, 1, \cdots). \tag{6}$$

According to (1), we set

$$\tau_n^\alpha = \frac{1}{(n+1)A_n^\alpha} \sum_{k=0}^n A_{n-k}^\alpha \xi_k \quad (n = 0, 1, \cdots).$$

In the special cases $\alpha = 0$, 1 and 2 one gets in turn that

$$\tau_n^0 = \frac{1}{n+1} \sum_{k=0}^n \xi_k, \quad \tau_n^1 = \frac{1}{n+1} \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) \xi_k,$$

$$\tau_n^2 = \frac{1}{n+1} \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right)\left(1 - \frac{k}{n+2}\right) \xi_k.$$

A consequence of the famous Rademacher-Menšov theorem is formulated in the following theorem (see, e.g. [2, pages 86–87]).

THEOREM A. *If*

$$\sum_{k=0}^\infty \frac{\sigma_k^2}{(k+1)^2} [\log(k+2)]^2 < \infty, \tag{7}$$

then

$$\lim_{n\to\infty} \tau_n^0 = 0 \quad \text{a.s.}$$

In this paper the logarithms are of base 2.

It is also pointed out that the sufficient condition (7) is the best possible in the following sense.

THEOREM B. (Tandori [3]). *If $\{\sigma_k: k \geq 0\}$ is a sequence of positive numbers, for which $\sigma_k/(k+1)$ is nonincreasing and*

$$\sum_{k=0}^\infty \frac{\sigma_k^2}{(k+1)^2} [\log(k+2)]^2 = \infty,$$

then there exists an orthogonal sequence $\{\xi_k: k \geq 0\}$ of random variables such that (6) *is satisfied and*

$$\limsup_{n\to\infty} |\tau_n^0| = \infty \quad \text{a.s.}$$

Now, the main result of the present paper is that the a.s. convergence behaviour of τ_n^1 is much more favourable in comparison with that of τ_n^0. This is shown by the following

THEOREM 1. *If*

$$\sum_{k=0}^\infty \frac{\sigma_k^2}{(k+1)^2} < \infty, \tag{8}$$

then

$$\lim_{n\to\infty} \tau_n^1 = 0 \quad \text{a.s.} \tag{9}$$

Another interesting fact is that, under condition (8), the a.s. convergence of the means τ_n^α to 0 coincide for different $\alpha > 0$.

THEOREM 2. *If condition* (8) *is satisfied, then for every* $\alpha > 0$

(10) $$\lim_{n\to\infty} \tau_n^\alpha = 0 \quad \text{a.s.}$$

3. Proof of Theorem 1. It will be done in three steps.

(i) First we prove that

(11) $$\lim_{n\to\infty} \tau_{2^n}^0 = 0 \quad \text{a.s.}$$

In fact, by orthogonality,

$$E[\tau_n^0]^2 = \frac{1}{(n+1)^2} \sum_{k=0}^n \sigma_k^2.$$

Thus

$$\sum_{n=0}^\infty E[\tau_{2^n}^0]^2 = \sum_{n=0}^\infty \frac{1}{(2^n+1)^2} \sum_{k=0}^{2^n} \sigma_k^2 = \sum_{k=0}^\infty \sigma_k^2 \sum_{n:2^n\ge k} \frac{1}{(2^n+1)^2}$$
$$= O(1) \sum_{k=0}^\infty \frac{\sigma_k^2}{(k+1)^2}.$$

By (8), B. Levi's theorem implies (11).

(ii) Our next step is to prove that

(12) $$\lim_{n\to\infty} (\tau_{2^n}^0 - \tau_{2^n}^1) = 0 \quad \text{a.s.}$$

Since

$$\tau_n^0 - \tau_n^1 = \frac{1}{(n+1)^2} \sum_{k=1}^n k\xi_k,$$

a simple calculation gives that

$$\sum_{n=0}^\infty E[\tau_{2^n}^0 - \tau_{2^n}^1]^2 = \sum_{n=0}^\infty \frac{1}{(2^n+1)^4} \sum_{k=1}^{2^n} k^2\sigma_k^2$$
$$= \sum_{k=1}^\infty k^2\sigma_k^2 \sum_{n:2^n\ge k} \frac{1}{(2^n+1)^4} = O(1) \sum_{k=1}^\infty \frac{\sigma_k^2}{(k+1)^2}.$$

Again by (8), B. Levi's theorem implies (12).

(iii) Finally, we prove that

(13) $$\lim_{n\to\infty} \max_{2^n<m\le 2^{n+1}} |\tau_m^1 - \tau_{2^n}^1| = 0 \quad \text{a.s.}$$

To this effect, we use the following estimation:

$$M_n := \max_{2^n<m\le 2^{n+1}} |\tau_m^1 - \tau_{2^n}^1| \le \sum_{j=2^n+1}^{2^{n+1}} |\tau_j^1 - \tau_{j-1}^1|$$
$$\le 2^{n/2} \{\sum_{j=2^n+1}^{2^{n+1}} [\tau_j^1 - \tau_{j-1}^1]^2\}^{1/2} \quad (n = 1, 2, \cdots),$$

where we applied the Cauchy inequality. Since

$$\tau_j^1 - \tau_{j-1}^1 = \sum_{k=0}^j \left(\frac{k(2j+1)}{j^2(j+1)^2} - \frac{1}{j(j+1)}\right) \xi_k \quad (j \ge 1),$$

a simple calculation provides that

$$EM_n^2 \le 2^n \sum_{j=2^n+1}^{2^{n+1}} E[\tau_j^1 - \tau_{j-1}^1]^2$$
$$\le 2^n \sum_{j=2^n+1}^{2^{n+1}} \sum_{k=0}^j \left(\frac{k^2(2j+1)^2}{j^4(j+1)^4} + \frac{1}{j^2(j+1)^2}\right) \sigma_k^2$$
$$\le 5.2^n \sum_{j=2^n+1}^{2^{n+1}} \sum_{k=0}^j \frac{\sigma_k^2}{j^2(j+1)^2} \le \frac{5}{(2^n+1)^2} \sum_{k=0}^{2^{n+1}} \sigma_k^2.$$

Thus,

$$\sum_{n=1}^{\infty} EM_n^2 \leq 5 \sum_{n=1}^{\infty} \frac{1}{(2^n+1)^2} \sum_{k=0}^{2^{n+1}} \sigma_k^2$$

$$= 5 \sum_{k=0}^{\infty} \sigma_k^2 \sum_{n:2^{n+1}\geq k} \frac{1}{(2^n+1)^2} = O(1) \sum_{k=0}^{\infty} \frac{\sigma_k^2}{(k+1)^2},$$

whence B. Levi's theorem implies (13).

Now, putting (11), (12) and (13) together, we obtain statement (9).

4. Two auxiliary results for numerical sequences. In the proof of Theorem 2 we need the following two lemmas.

LEMMA 1. *If for an* $\alpha > -1$

$$\lim_{n\to\infty} t_n^{\alpha} = 0, \tag{14}$$

then for every $\varepsilon > 0$

$$\lim_{n\to\infty} t_n^{\alpha+\varepsilon} = 0. \tag{15}$$

PROOF. It can be essentially found in [4, pages 77–78]. For the sake of completeness, we present the modified proof here. By (2) and (4),

$$t_n^{\alpha+\varepsilon} = \frac{1}{(n+1)A_n^{\alpha+\varepsilon}} \sum_{k=0}^{n} s_k^{\alpha} A_{n-k}^{\varepsilon-1}$$

$$= \frac{1}{(n+1)A_n^{\alpha+\varepsilon}} \sum_{k=0}^{n} t_k^{\alpha}(k+1) A_k^{\alpha} A_{n-k}^{\varepsilon-1} =: \sum_{k=0}^{n} a_{nk} t_k^{\alpha},$$

where

$$a_{nk} = \frac{(k+1)A_k^{\alpha} A_{n-k}^{\varepsilon-1}}{(n+1)A_n^{\alpha+\varepsilon}} \quad (k = 0, 1, \cdots, n;\ n = 0, 1, \cdots).$$

In other words, the $t_n^{\alpha+\varepsilon}$ are linear means of the t_k^{α}. Using (4) and (5) one can verify that

$$\lim_{n\to\infty} a_{nk} = 0 \quad (k = 0, 1, \cdots)$$

and

$$\sum_{k=0}^{n} a_{nk} = O(1) \quad (n = 0, 1, \cdots).$$

Since $a_{nk} \geq 0$, these two conditions are enough to ensure the implication (14) $\Rightarrow$ (15).

LEMMA 2. *If for an* $\alpha > -\frac{1}{2}$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} [t_k^{\alpha}]^2 = 0, \tag{16}$$

then for every $\varepsilon > 0$

$$\lim_{n\to\infty} t_n^{\alpha+1/2+\varepsilon} = 0. \tag{17}$$

PROOF. By (4) and (2),

$$s_n^{\alpha+1/2+\varepsilon} = \sum_{k=0}^{n} s_k^{\alpha} A_{n-k}^{-1/2+\varepsilon} = \sum_{k=0}^{n} t_k^{\alpha}(k+1) A_k^{\alpha} A_{n-k}^{-1/2+\varepsilon}.$$

Applying the Cauchy inequality,

$$|s_n^{\alpha+1/2+\varepsilon}| \leq \{\sum_{k=0}^{n} [t_k^{\alpha}]^2 \sum_{k=0}^{n} (k+1)^2 [A_k^{\alpha} A_{n-k}^{-1/2+\varepsilon}]^2\}^{1/2},$$

then (3) and (5), one can get that

$$\sum_{k=0}^{n} (k+1)^2[A_k^\alpha A_{n-k}^{-1/2+\varepsilon}]^2 = O(n^{2(\alpha+1+\varepsilon)})$$

(cf. [1, page 189]). Hence, by (16),

$$s_n^{\alpha+1/2+\varepsilon} = O(n^{\alpha+3/2+\varepsilon}).$$

Taking this and again (5) into account, we find that

$$t_n^{\alpha+1/2+\varepsilon} = \frac{s_n^{\alpha+1/2+\varepsilon}}{(n+1)A_n^{\alpha+1/2+\varepsilon}} = o(1),$$

in accordance with (17).

5. Proof of Theorem 2. It will be based on Lemmas 1 and 2, and the following Lemmas 3 and 4.

LEMMA 3. *If condition* (8) *is satisfied, then for every* $\alpha > 1/2$

$$\delta_n^\alpha := \frac{1}{n+1} \sum_{j=0}^{n} [\tau_j^\alpha - \tau_j^{\alpha-1}]^2 \to 0 \quad \text{a.s.} \quad (n \to \infty). \tag{18}$$

We note that for $j = 0$ the summand in (18) is zero, since $\tau_0^\alpha = \xi_0$ for all α.

PROOF. In great lines it follows the proof of a corresponding result pertaining to orthogonal series (see, e.g. [1, pages 186–187]).

We begin with the representation

$$\tau_j^\alpha - \tau_j^{\alpha-1} = \frac{1}{(j+1)A_j^\alpha A_j^{\alpha-1}} \sum_{k=0}^{j} (A_{j-k}^\alpha A_j^{\alpha-1} - A_{j-k}^{\alpha-1} A_j^\alpha)\xi_k$$
$$= -\frac{1}{(j+1)A_j^\alpha A_j^{\alpha-1}} \sum_{k=1}^{j} \frac{k}{\alpha} A_{j-k}^{\alpha-1} A_j^{\alpha-1} \xi_k.$$

Hence

$$E[\tau_j^\alpha - \tau_j^{\alpha-1}]^2 = \frac{1}{\alpha^2 (j+1)^2 (A_j^\alpha)^2} \sum_{k=1}^{j} k^2 \sigma_k^2 (A_{j-k}^{\alpha-1})^2$$

and

$$E\delta_{2^n}^\alpha = \frac{1}{\alpha^2(2^n+1)} \sum_{j=1}^{2^n} \frac{1}{(j+1)^2 (A_j^\alpha)^2} \sum_{k=1}^{j} k^2 \sigma_k^2 (A_{j-k}^{\alpha-1})^2$$
$$= \frac{1}{\alpha^2(2^n+1)} \sum_{k=1}^{2^n} k^2 \sigma_k^2 \sum_{j=k}^{2^n} \frac{1}{(j+1)^2} \left(\frac{A_{j-k}^{\alpha-1}}{A_j^\alpha}\right)^2.$$

Now, using (4) and (5), one can obtain that

$$\sum_{j=k}^{\infty} \frac{1}{(j+1)^2} \left(\frac{A_{j-k}^{\alpha-1}}{A_j^\alpha}\right)^2 = O\left(\frac{1}{k^3}\right) \quad \left(k = 1, 2, \cdots; \alpha > \frac{1}{2}\right).$$

Thus, by (8),

$$\sum_{n=1}^{\infty} E\delta_{2^n}^\alpha = O(1) \sum_{n=1}^{\infty} \frac{1}{2^n+1} \sum_{k=1}^{2^n} \frac{\sigma_k^2}{k}$$
$$= O(1) \sum_{k=1}^{\infty} \frac{\sigma_k^2}{k} \sum_{n: 2^n \geq k} \frac{1}{2^n+1} = O(1) \sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty.$$

This implies, via B. Levi's theorem,

$$\lim_{n\to\infty} \delta_{2^n}^\alpha = 0 \quad \text{a.s.}$$

For general m, say $2^n < m \leq 2^{n+1}$, we have

$$0 \leq \delta_m^\alpha \leq 2\delta_{2^{n+1}}^\alpha,$$

and the proof of statement (18) is complete.

LEMMA 4. *If condition* (8) *is satisfied and* (10) *is also satisfied for some* $\alpha > 1/2$, *then*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^n [\tau_j^{\alpha-1}]^2 = 0 \quad \text{a.s.} \tag{19}$$

PROOF. It is clear that

$$\sum_{j=0}^n [\tau_j^{\alpha-1}]^2 \leq 2 \sum_{j=0}^n [\tau_j^{\alpha-1} - \tau_j^\alpha]^2 + \sum_{j=0}^n [\tau_j^\alpha]^2.$$

By virtue of Lemma 3, the first sum on the right-hand side is $o(n)$, while the second sum is also $o(n)$ by assumption.

PROOF OF THEOREM 2. First Theorem 1 shows that statement (10) holds true for $\alpha = 1$. By Lemma 1, statement (10) holds true also for every $\alpha \geq 1$.

Applying Lemma 4, we obtain (19) for $\alpha = 1$, whence Lemma 2 implies the fulfilment of (10) for $\alpha = 1/2 + \varepsilon$ with any $\varepsilon > 0$. Repeating this argument once more, we get (10) for $\alpha = 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the proof of Theorem 2 is complete.

REFERENCES

[1] KACZMARZ, S. und STEINHAUS, H. (1935). *Theorie der Orthogonalreihen.* Warszawa-Lwów.
[2] RÉVÉSZ, P. (1967). *The Laws of Large Numbers.* Akadémiai Kiadó, Budapest.
[3] TANDORI, K. (1972). Bemerkungen zum Gesetz der grossen Zahlen. *Periodica Math. Hungar.* **2** 33–39.
[4] ZYGMUND, A. (1959). *Trigonometric Series.* University Press, Cambridge.

SZEGED UNIVERSITY
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED
HUNGARY