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TWO CRITICAL EXPONENTS FOR FINITE REVERSIBLE NEAREST PARTICLE SYSTEMS

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Finite nearest particle systems are certain one parameter families of continuous time Markov chains A_t whose state space is the collection of all finite subsets of the integers. Points are added to or taken away from A_t at rates which have a particular form. The empty set is absorbing for these chains. In the reversible case, the parameter λ is normalized so that extinction at the empty set is certain if and only if $\lambda \le 1$. Let $\sigma(\lambda)$ be the probability of nonextinction starting from a singleton. In a recent paper, Griffeath and Liggett obtained the bounds $\lambda^{-1}(\lambda - 1) \le \sigma(\lambda) \le |\log \lambda^{-1}(\lambda - 1)|^{-1}$ for $\lambda > 1$, and raised the question of determining the correct asymptotics of $\sigma(\lambda)$ as $\lambda \downarrow 1$. In the present paper, this question is largely answered by showing under a moment assumption that for $\lambda > 1$, $\sigma(\lambda)$ is bounded above by a constant multiple of $\lambda - 1$. In the critical case $\lambda = 1$, a similar improvement is made on the known bounds on the asymptotics as $n \to \infty$ of the probability that A_t is of cardinality at least n sometime before extinction. Similar results have been conjectured, but remain open problems in nonreversible situations—for example, for the basic one-dimensional contact process.

1. Introduction and statement of results. Nearest particle systems were introduced and first studied by Spitzer in [9]. Various aspects of their construction and behavior have been treated in [1]–[7]. In order to describe the class of processes to be considered here, let $f(n)$ be a strictly positive probability density on $\{1, 2, \cdots\}$ which satisfies

$$\frac{f(n)}{f(n+1)} \downarrow 1 \quad \text{as} \quad n \uparrow \infty. \tag{1.1}$$

Let $\mathscr{S}$ be the collection of all finite subsets of the set of integers, where subsets which are translates of one another are considered to be the same. The finite nearest particle system corresponding to f with parameter $\lambda > 0$ is the continuous time Markov chain A_t on $\mathscr{S}$ with transition rates given by

$$A \to A\backslash\{x\} \text{ at rate 1 for each } x \in A, \quad \text{and}$$
$$A \to A \cup \{x\} \text{ at rate } \lambda \frac{f(k)f(\ell)}{f(k+\ell)} \text{ for each } x \notin A,$$

where k and ℓ are the distances from x to be the nearest points in A to the left and right respectively. If k or ℓ is infinite, the birth rate is $\lambda f(\ell)$ or $\lambda f(k)$ respectively, which is the natural choice in view of (1.1). The empty set is absorbing. In the foregoing description of the process, we have built in the reversibility and attractiveness assumptions which will be needed here.

In [4], it was shown that absorption at the empty set occurs with certainty if and only if $\lambda \le 1$. Furthermore, the following asymptotics for various quantities near and at $\lambda = 1$ were obtained. We will use * to denote the singleton element of $\mathscr{S}$ and τ to be the time of extinction. For $A \in \mathscr{S}$, $|A|$ will denote the cardinality of A.

(a) The subcritical case $\lambda < 1$:

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$$E^*(\tau) = (1-\lambda)^{-1}, \quad E^*\left\{\int_0^\infty |A_t|\, dt\right\} = (1-\lambda)^{-2}$$

(b) The critical case $\lambda = 1$:

$$\frac{C_1}{n} \le P^*[|A_t| = n \text{ for some } t] \le \frac{C_2}{\log n}$$

for some constants C_1 and C_2.

(c) The supercritical case $\lambda > 1$:

$$\frac{\lambda - 1}{\lambda} \le P^*(\tau = \infty) \le \left|\log \frac{\lambda - 1}{\lambda}\right|^{-1}.$$

A natural problem which we raised in [4] is to determine the correct asymptotics for these quantities in the critical and supercritical cases. This paper is devoted to the proof of the following theorem, which largely solves this problem.

THEOREM 1.2. *In addition to* (1.1), *assume that*

(1.3) $$\textstyle\sum_{n=1}^\infty n^2 f(n) < \infty.$$

(a) *If* $\lambda = 1$, *then for some constant* C,

$$P^*[|A_t| = n \textit{ for some } t] \le \frac{C}{n}$$

for all $n \ge 1$.

(b) *There is a constant* C *so that*

$$P^*(\tau = \infty) \le C(\lambda - 1)$$

for all $\lambda \ge 1$.

This information is of particular interest because results of this type are notoriously difficult to obtain in many important situations. Consider the basic contact process, for instance, which is a nonreversible attractive nearest particle system. In that case, it is not even known whether or not the critical process is absorbed at $\varnothing$ with certainty, let alone results of the above type.

The proof of Theorem 1.2 uses techniques from [4] to deduce the above bounds from similar bounds for Markov chains on the simpler state space $\{0, 1, 2, \cdots\}$. These Markov chains are asymptotically random walks far from 0. The bulk of the proof involves making comparisons between these Markov chains and the random walks which approximate them, since the needed bounds are well known for the random walks. The main difficulty, as might be expected, comes from the necessity of keeping the comparisons uniform in λ near the critical $\lambda = 1$. The comparisons between the nearest particle systems and the Markov chains on $\{0, 1, 2, \cdots\}$ are carried out in Section 2, while the comparisons with the random walks are worked out in Section 3. Theorem 1.2 results from combining Theorems 2.4 and 3.14.

Two natural questions which remain open are: (a) what happens when (1.3) fails? and (b) do the following limits exist:

$$\lim_{n\to\infty} nP^*[|A_t| = n \text{ for some } t] \text{ for } \lambda = 1$$

and

$$\lim_{\lambda\downarrow 1} \frac{P^*(\tau = \infty)}{\lambda - 1}?$$

2. The first comparison. The first comparison uses a slight extension of Theorem 2.10 of [4], which itself is an improved version of the Nash-Williams recurrence criterion for reversible Markov chains. In order to state this extension, let S be a countable set and let $\alpha(i,j) = \alpha(j,i)$ be a symmetric nonnegative matrix indexed by $i, j \in S$ with the property that

$$0 < \alpha(i) = \sum_j \alpha(i,j) < \infty$$

for each $i \in S$. Consider the discrete time Markov chain X_n on S with transition probabilities

$$p(i,j) = \frac{\alpha(i,j)}{\alpha(i)}$$

and assume that this chain is irreducible. Fix a reference point $0 \in S$ and write $S = \bigcup_{k=0}^{\infty} \Lambda_k$ where $\Lambda_0 = \{0\}$ and the Λ_k's are disjoint. Assume that

$$\sum_{i\in\Lambda_k} \alpha(i) < \infty$$

for each k and that

$$P^0(X_n \in \cup_{k=m}^{\infty} \Lambda_k \text{ for some } n) = 1$$

for each $m \geq 1$. Now put $\tilde{S} = \{0, 1, 2, \cdots\}$,

$$\tilde{\alpha}(k, \ell) = \sum_{i\in\Lambda_k, j\in\Lambda_\ell} \alpha(i,j), \quad \tilde{\alpha}(k) = \sum_\ell \tilde{\alpha}(k, \ell), \quad \tilde{p}(k, \ell) = \frac{\tilde{\alpha}(k, \ell)}{\tilde{\alpha}(k)},$$

and let $\tilde{X}_n$ be the corresponding Markov chain on $\tilde{S}$. For each chain, let τ_0 be the hitting time of state 0.

THEOREM 2.1. *Under the above assumptions,*

$$P^0(X_n \in \cup_{k=m}^{\infty} \Lambda_k \text{ for some } n < \tau_0) \leq P^0(\tilde{X}_n \in \{m, m+1, \cdots\} \text{ for some } n < \tau_0).$$

The proof of Theorem 2.1 is identical with that of Theorem 2.10 of [4]. The difference in the statements of the two theorems is simply that in Theorem 2.10 of [4], it is assumed that $\tilde{\alpha}_{k,\ell} = 0$ if $|k - \ell| > 1$, so that $\tilde{X}_n$ is a birth and death chain, and the right hand side of the above inequality can be computed explicitly.

We will now apply Theorem 2.1 with $S = \mathscr{S}$, $\alpha(A, B) = 0$ unless $|A\Delta B| = 1$, $\alpha(*, \varnothing) = \alpha(\varnothing, *) = 1$, and whenever $n \geq 1$, $0 \leq j \leq n$, and $A = \{x_0, x_1, \cdots, x_n\}$ with $x_0 < x_1 < \cdots < x_n$,

$$\alpha(A, A\backslash\{x_j\}) = \alpha(A\backslash\{x_j\}, A) = \begin{cases} 2\lambda^n \prod_{i=1}^n f(x_i - x_{i-1}) & \text{if } j = 0 \text{ or } j = n \text{ and } A\backslash\{x_0\} = A\backslash\{x_n\} \\ \lambda^n \prod_{i=1}^n f(x_i - x_{i-1}) & \text{otherwise.} \end{cases}$$

A moment's thought shows that the chain with transition probabilities $\alpha(A, B)/\alpha(A)$ is the embedded discrete time chain for the nearest particle system up until the time it is absorbed at $\varnothing$. It is here of course that we use the fact that the nearest particle systems we are considering are reversible.

Put $\Lambda_0 = \{\varnothing\}$, $\Lambda_1 = \{*\}$, and for $k \geq 2$,

$$\Lambda_k = \{A \in \mathscr{S}: \max_{x\in A} x - \min_{x\in A} x = k - 1\}.$$

For $k \geq 1$ and $A \in \Lambda_k$, write $A = \{x_0, \cdots, x_n\}$ where $x_0 < x_1 < \cdots < x_n$ and $x_n - x_0 + 1 = k$. Then if $k \geq 2$,

$$\sum_{j=1}^{k-1} \sum_{B\in\Lambda_j} \alpha(A, B) = 2\lambda^n \prod_{i=1}^n f(x_i - x_{i-1}). \tag{2.2}$$

On the other hand, if $1 \leq k < \ell$,

$$\sum_{B\in\Lambda_\ell} \alpha(A, B) = 2\lambda^{n+1} f(\ell - k) \prod_{i=1}^n f(x_i - x_{i-1}),$$

so that

$$\tilde{\alpha}(k, \ell) = \sum_{A\in\Lambda_k} \sum_{B\in\Lambda_\ell} \alpha(A, B) = u_\lambda(k) f(\ell - k)$$

for some $u_\lambda(k)$. Of course $u_\lambda(1) = 2\lambda$, and if $k \geq 2$ it follows from (2.2) and the symmetry of $\tilde{\alpha}(\cdot, \cdot)$ that $u_\lambda(k)$ can be determined recursively by

$$\begin{aligned} u_\lambda(k) &= \lambda \sum_{A\in\Lambda_k, B\in\cup_{j=1}^{k-1}\Lambda_j} \alpha(A, B) = \lambda \sum_{j=1}^{k-1} \tilde{\alpha}(k, j) \\ &= \lambda \sum_{j=1}^{k-1} \tilde{\alpha}(j, k) = \lambda \sum_{j=1}^{k-1} u_\lambda(j) f(k - j). \end{aligned} \tag{2.3}$$

Also, $\tilde{\alpha}(0, 1) = \tilde{\alpha}(1, 0) = 1$ and $\tilde{\alpha}(0, k) = \tilde{\alpha}(k, 0) = 0$ for $k \geq 2$. Therefore $\tilde{\alpha}(0) = 1$,

$$\tilde{\alpha}(1) = 1 + \sum_{\ell=2}^{\infty} \tilde{\alpha}(1, \ell) = 1 + 2\lambda,$$

and for $k \geq 2$,

$$\begin{aligned} \tilde{\alpha}(k) &= \sum_{\ell=1}^{k-1} \tilde{\alpha}(k, \ell) + \sum_{\ell=k+1}^{\infty} \tilde{\alpha}(k, \ell) \\ &= \sum_{\ell=1}^{k-1} u_\lambda(\ell) f(k - \ell) + \sum_{\ell=k+1}^{\infty} u_\lambda(k) f(\ell - k) \\ &= u_\lambda(k)\left(\frac{1}{\lambda} + 1\right) \end{aligned}$$

by (2.3). So, the comparison chain in Theorem 2.1 has the following transition probabilities:

$$q_\lambda(0, 1) = 1, \quad q_\lambda(1, 0) = \frac{1}{1 + 2\lambda}$$

$$q_\lambda(1, \ell) = \frac{2\lambda}{1 + 2\lambda} f(\ell - 1) \quad \text{for} \quad \ell \geq 2$$

and for $k \geq 2$

$$q_\lambda(k, \ell) = \frac{\tilde{\alpha}(k, \ell)}{\tilde{\alpha}(k)} = \begin{cases} \dfrac{\lambda}{\lambda + 1} \dfrac{u_\lambda(\ell) f(k - \ell)}{u_\lambda(k)} & 1 \leq \ell < k \\ \\ \dfrac{\lambda}{\lambda + 1} f(\ell - k) & \ell > k. \end{cases}$$

In this context, Theorem 2.1 becomes

THEOREM 2.4. *Let A_t be the nearest particle system corresponding to the density $f(k)$ and the parameter $\lambda > 0$. Then*

$$P^*[A_t \in \cup_{k=n}^{\infty} \Lambda_k \quad \text{for some} \quad t] \leq P^1[X_m^\lambda \geq n \quad \text{for some} \quad m < \tau_0]$$

where X_m^λ is the chain with transition probabilities $q_\lambda(k, \ell)$.

REMARKS. (a) While the transition probabilities $q_\lambda(k, \ell)$ were obtained by purely formal manipulations, there is an interpretation of them which motivated the manipulations and hence should be described. To do so, take $\lambda \geq 1$ and define $s_\lambda \in (0, 1]$ by

$$\lambda \sum_{n=1}^{\infty} f(n) s_\lambda^n = 1.$$

Put

$$f_\lambda(n) = \lambda f(n) s_\lambda^n, \tag{2.5}$$

which is a probability density. Then (2.3) can be written as

$$u_\lambda(k) s_\lambda^k = \sum_{j=1}^{k-1} u_\lambda(j) f_\lambda(k - j) s_\lambda^j, \tag{2.6}$$

so that $s_\lambda^k u_\lambda(k)$ is a constant multiple of the renewal function corresponding to $f_\lambda(n)$. Call a transition $A \to B$ for the chain A_t an interior transition if A and B are in the same Λ_k, and otherwise call it a boundary transition. If the rates for the interior transitions are formally set to ∞, then one can think of the boundary process as evolving in such a way that the interior of the configuration is always in its stationary distribution. But if the leftmost and rightmost elements of A_t are frozen at $u < v$ respectively, then the stationary distribution for the interior of the configuration is obtained by conditioning the renewal measure corresponding to $f_\lambda(k)$ on the event that u and v are occupied. These heuristics lead to the above transition probabilities $q_\lambda(k, \ell)$ for the evolution of the diameter of A_t when the interior rates are set to ∞.

(b) In [4], the weaker logarithmic bounds for the escape probabilities which are mentioned in the introduction were obtained by using Theorem 2.1 with $\Lambda_k = \{A \in \mathscr{S}: |A| = k\}$. The comparison chain in this case is just the birth and death chain on $\{0, 1, 2, \cdots\}$ which moves from n to $n+1$ at rate $\lambda(n+1)$ and from n to $n-1$ at rate n. Thus our present choice for Λ_k based on the diameter of A rather than on its cardinality leads to a comparison chain which is substantially more difficult to analyze. We are rewarded, however, by obtaining substantially better results.

3. The second comparison. We assume in this section that $\lambda \geq 1$ and $\sum_{n=1}^\infty n^2 f(n) < \infty$. The escape probabilities of interest for the chain with transition probabilities $q_\lambda(k, \ell)$ will be bounded above by comparing them with those for the chain with transition probabilities $p_\lambda(k, \ell)$ defined by

$$p_\lambda(k, \ell) = \begin{cases} \dfrac{1}{\lambda+1} f_\lambda(k-\ell) & \text{if } 1 \leq \ell < k \\ \dfrac{\lambda}{\lambda+1} f(\ell - k) & \text{if } \ell > k \\ \dfrac{1}{\lambda+1} \sum_{j=k}^\infty f_\lambda(j) & \text{if } \ell = 0 \end{cases}$$

for $k \geq 1$ and $p_\lambda(0, 1) = 1$. This is a natural chain to consider for two reasons:

(a) This chain is a random walk on the nonnegative integers until the first time it hits 0.

(b) It follows from the renewal theorem and the remark following Theorem 2.4 that $\lim_{n\to\infty} u_\lambda(n) s_\lambda^n$ exists and is positive, so that

$$\lim_{k\to\infty}[q_\lambda(k, k+\ell) - p_\lambda(k, k+\ell)] = 0$$

for all integers ℓ.

Throughout this section, X_m^λ and Y_m^λ will denote the Markov chains on $\{0, 1, 2, \cdots\}$ with transition probabilities $q_\lambda(k, \ell)$ and $p_\lambda(k, \ell)$ respectively. Let τ_0 be the hitting time of 0 for either chain. Make the following definitions:

$$v_{\lambda,n}(k) = P^k[Y_m^\lambda > n \quad \text{for some} \quad m < \tau_0] \quad \text{if} \quad 1 \leq k \leq n$$

$$w_{\lambda,n}(k) = P^k[X_m^\lambda > n \quad \text{for some} \quad m < \tau_0] \quad \text{if} \quad 1 \leq k \leq n$$

$$v_{\lambda,n}(0) = w_{\lambda,n}(0) = 0, \qquad v_{\lambda,n}(k) = w_{\lambda,n}(k) = 1 \quad \text{if} \quad k > n$$

$$v_\lambda(k) = \lim_{n\to\infty} v_{\lambda,n}(k), \qquad w_\lambda(k) = \lim_{n\to\infty} w_{\lambda,n}(k)$$

$$G_\lambda(k, \ell) = E^k[\text{number of } m \leq \tau_0 \text{ so that } Y_m^\lambda = \ell] \text{ if } k, \ell \geq 1$$

$$P_{\lambda,n} g(k) = E^k g(Y_1^\lambda) \quad \text{if} \quad 1 \leq k \leq n$$

$$Q_{\lambda,n} g(k) = E^k g(X_1^\lambda) \quad \text{if} \quad 1 \leq k \leq n$$

$$Q_{\lambda,n} g(k) = P_{\lambda,n} g(k) = g(k) \quad \text{of} \quad k = 0 \quad \text{or} \quad k > n.$$

$$u_\lambda(k, \ell) = \sum_{j=1}^\infty G_\lambda(k, j)[q_\lambda(j, \ell) - p_\lambda(j, \ell)] \quad \text{if} \quad k, \ell \geq 1.$$

LEMMA 3.1. (a) $s_\lambda^k u_\lambda(k)$ *is nonincreasing in k for each* λ.

(b) $p_\lambda(j, \ell) \le q_\lambda(j, \ell)$ *for* $j, \ell \ge 1$.

(c) $u_\lambda(k, \ell) \ge 0$ *for* $k, \ell \ge 1$.

PROOF. Parts (b) and (c) follow immediately from part (a) and the definitions of $u_\lambda(k, \ell)$, $p_\lambda(j, \ell)$ and $q_\lambda(j, \ell)$. The proof of part (a) involves a coupling argument. Define $f_\lambda(n)$ as in (2.5) and put

$$F_\lambda(n) = \sum_{k=n}^{\infty} f_\lambda(k).$$

Assumption (1.1) implies that $f_\lambda(n)/F_\lambda(n) \downarrow$ as $n \uparrow \infty$. To see this, it suffices to check that

$$f_\lambda(n) \sum_{k=n+1}^{\infty} f_\lambda(k) \ge f_\lambda(n+1) \sum_{k=n}^{\infty} f_\lambda(k),$$

which follows from

$$f_\lambda(n) f_\lambda(k) \ge f_\lambda(n+1) f_\lambda(k-1)$$

for $k \ge n+1$. This last inequality is essentially (1.1). For $k \ge 0$, let (η_k, ζ_k) be random variables with possible values 0 and 1 and distributions given by: $\eta_0 = \zeta_0 = \zeta_1 = 1$ and for $n \ge 1$

$$P[(\eta_n, \zeta_n) = (1, 1) \mid \eta_j, \zeta_j, 1 \le j < n] = \frac{f_\lambda(\ell)}{F_\lambda(\ell)}$$

$$P[(\eta_n, \zeta_n) = (0, 1) \mid \eta_j, \zeta_j, 1 \le j < n] = \frac{f_\lambda(k)}{F_\lambda(k)} - \frac{f_\lambda(\ell)}{F_\lambda(\ell)}$$

$$P[(\eta_n, \zeta_n) = (0, 0) \mid \eta_j, \zeta_j, 1 \le j < n] = 1 - \frac{f_\lambda(k)}{F_\lambda(k)}$$

where $\ell = \ell(\eta_0, \cdots, \eta_{n-1}) = n - \max\{j: 0 \le j < n, \eta_j = 1\}$ and $k = k(\zeta_0, \cdots, \zeta_{n-1}) = n - \max\{j: 0 \le j < n, \zeta_j = 1\}$. Note that at each stage $\eta_n \le \zeta_n$, so that $k \le \ell$ and hence the above probabilities are nonnegative by the monotonicity of $f_\lambda(n)/F_\lambda(n)$. Of course,

$$P[\eta_n = 1 \mid \eta_j, \zeta_j, 1 \le j < n] = \frac{f_\lambda(\ell)}{F_\lambda(\ell)},$$

which is measurable with respect to $\{\eta_0, \cdots, \eta_{n-1}\}$, and

$$P[\zeta_n = 1 \mid \eta_j, \zeta_j, 1 \le j < n] = \frac{f_\lambda(k)}{F_\lambda(k)},$$

which is measurable with respect to $\{\zeta_0, \cdots, \zeta_{n-1}\}$. Therefore

$$P[\eta_n = 1 \mid \eta_j, 1 \le j < n] = \frac{f_\lambda(\ell)}{F_\lambda(\ell)}$$

and

$$P[\zeta_n = 1 \mid \zeta_j, 1 \le j < n] = \frac{f_\lambda(k)}{F_\lambda(k)},$$

so that the distributions of $\{\eta_n, n \ge 0\}$ and $\{\zeta_n, n \ge 1\}$ are those of the renewal process corresponding to $f_\lambda(\cdot)$ conditioned respectively on being 1 at 0 and being 1 at 1. Since $\eta_n \le \zeta_n$ a.s., $P[\eta_n = 1] \le P[\zeta_n = 1]$ for all $n \ge 1$. The result follows now from the fact that $s_\lambda^k u_\lambda(k)$ is a constant multiple of the renewal function corresponding to $f_\lambda(\cdot)$.

LEMMA 3.2. $w_{\lambda,n}(k) - v_{\lambda,n}(k) \le \sum_{\ell=1}^{\infty} u_\lambda(k, \ell) w_{\lambda,n}(\ell)$ *for* $k \ge 1$.

PROOF. By the harmonicity of $w_{\lambda,n}$ for the X_m^λ chain stopped when it exits $\{1, \cdots, n\}$,

$$w_{\lambda,n} - P_{\lambda,n} w_{\lambda,n} = (Q_{\lambda,n} - P_{\lambda,n}) w_{\lambda,n}.$$

Applying $P_{\lambda,n}$ to this relation repeatedly and summing gives

$$w_{\lambda,n} - P^m_{\lambda,n} w_{\lambda,n} = \textstyle\sum_{j=0}^{m-1} P^j_{\lambda,n}(Q_{\lambda,n} - P_{\lambda,n})w_{\lambda,n}.$$

Since $(Q_{\lambda,n} - P_{\lambda,n})w_{\lambda,n} \geq 0$ by (b) of Lemma 3.1 and

$$v_{\lambda,n}(k) = \lim_{m\to\infty}(P^m_{\lambda,n}w_{\lambda,n})(k),$$

$$\begin{aligned} w_{\lambda,n}(k) - v_{\lambda,n}(k) &= \textstyle\sum_{j=0}^{\infty}[P^j_{\lambda,n}(Q_{\lambda,n} - P_{\lambda,n})w_{\lambda,n}](k) \\ &\leq \textstyle\sum_{\ell=1}^{\infty} u_\lambda(k, \ell)w_{\lambda,n}(\ell) \end{aligned}$$

for $k \geq 1$.

LEMMA 3.3. *Put $G_\lambda(k) = G_\lambda(k, k)$ for $k \geq 1$. Then*

(a) *$G_\lambda(k, \ell) \leq G_\lambda(k)$ for $k, \ell \geq 1$.*
(b) *$G_\lambda(k) \leq G_\lambda(k + 1)$ for $k \geq 1$.*
(c) *$G_\lambda(k + \ell) \leq G_\lambda(k) + G_\lambda(\ell)$ for $k, \ell \geq 1$.*
(d) *$v_\lambda(k) \leq v_\lambda(k + 1)$ for $k \geq 0$.*
(e) *$v_\lambda(k + \ell) \leq v_\lambda(k) + v_\lambda(\ell)$ for $k, \ell \geq 0$.*

PROOF. Consider the random walk Z^λ_m on the integers with transition probabilities

$$\bar{p}_\lambda(k, \ell) = \begin{cases} \dfrac{1}{\lambda + 1} f_\lambda(k - \ell) & \text{if} \quad k > \ell \\[2ex] \dfrac{\lambda}{\lambda + 1} f(\ell - k) & \text{if} \quad \ell > k. \end{cases}$$

Note that $\bar{p}_\lambda(k, \ell)s^\ell_\lambda = \bar{p}_\lambda(\ell, k)s^k_\lambda$ for all k, ℓ, so that $[s_\lambda]^{Z^\lambda_m}$ is a martingale. Therefore for $k, \ell \geq 1$, $P^\ell[Z^\lambda_m$ hits k before hitting $\{\cdots, -1, 0\}] \leq s^{\ell-k}_\lambda$. Also $G_\lambda(k, \ell)s^\ell_\lambda = G_\lambda(\ell, k)s^k_\lambda$ for $k, \ell \geq 1$, so that

$$\begin{aligned} G_\lambda(k, \ell) &= s^{k-\ell}_\lambda G_\lambda(\ell, k) \\ &= s^{k-\ell}_\lambda P^\ell[Z^\lambda_m \text{ hits } k \text{ before hitting } \{\cdots, -1, 0\}]G_\lambda(k, k) \\ &\leq G_\lambda(k,k) \end{aligned}$$

giving part (a). For assertion (b), couple together two copies of the Markov chain Y^λ_m, one starting at k and one at $k + \ell$, by letting them use the same increments until the leftmost process hits 0. Up until that time, the leftmost process has hit k the same number of times that the rightmost process has hit $k + \ell$, and of course the rightmost process may visit $k + \ell$ again after that time before hitting 0. This proves (b). At the time the leftmost process hits 0, the rightmost process is in $\{0, 1, \cdots, \ell\}$. Thus

$$G_\lambda(k + \ell) - G_\lambda(k) \leq \max_{1\leq j\leq \ell} G_\lambda(j, k + \ell) \leq G_\lambda(\ell)$$

by parts (a) and (b), so (c) is proved. The proofs of (d) and (e) are similar.

LEMMA 3.4. (a) *If $1 \leq \lambda_1 \leq \lambda_2$, the density f_{λ_1} is stochastically larger than the density f_{λ_2}.*

(b) *If $1 \leq \lambda_1 \leq \lambda_2$ and $Y^{\lambda_1}_0 \leq Y^{\lambda_2}_0$, then $Y^{\lambda_1}_m$ and $Y^{\lambda_2}_m$ can be coupled together so that $Y^{\lambda_1}_m \leq Y^{\lambda_2}_m$ for all m.*

PROOF. Part (b) is an immediate consequence of part (a) and the definition of the transition probabilities for the chain Y^λ_m. For part (a), put

$$\varphi(s) = \textstyle\sum_{n=1}^{\infty} f(n)s^n \quad \text{for} \quad 0 \leq s \leq 1$$

and note that (2.5) can be written as

$$f_\lambda(n) = \frac{f(n)s_\lambda^n}{\varphi(s_\lambda)}.$$

Also, since $\varphi(s)$ is increasing in s, $1 \le \lambda_1 \le \lambda_2$ implies that $1 \ge s_{\lambda_1} \ge s_{\lambda_2}$. Therefore it suffices to show that for any $N \ge 1$,

$$\frac{\sum_{n=1}^N f(n)s^n}{\varphi(s)}$$

is a nonincreasing function of s. But this is easily checked by differentiation.

LEMMA 3.5. $\sum_{\ell=1}^\infty u_\lambda(k,\ell)\ell \le \frac{G_\lambda(k)}{4}[1 + \sum_{\ell=1}^\infty \ell^2 f(\ell)]$.

PROOF.

$$\begin{aligned}\sum_{\ell=1}^\infty u_\lambda(k,\ell)\ell &= \sum_{j,\ell=1}^\infty G_\lambda(k,j)[q_\lambda(j,\ell) - p_\lambda(j,\ell)]\ell \\ &\le G_\lambda(k)\sum_{j,\ell=1}^\infty [q_\lambda(j,\ell) - p_\lambda(j,\ell)]\ell\end{aligned}$$

by (b) of Lemma 3.1 and (a) of Lemma 3.3. Now,

$$q_\lambda(1,\ell) - p_\lambda(1,\ell) = \frac{\lambda}{(1+\lambda)(1+2\lambda)} f(\ell - 1) \quad \text{for} \quad \ell \ge 2,$$

$$q_\lambda(j,\ell) = p_\lambda(j,\ell) \quad \text{for} \quad 2 \le j < \ell$$

and for $1 \le \ell < j$,

$$q_\lambda(j,\ell) - p_\lambda(j,\ell) = \frac{1}{\lambda+1} f_\lambda(j-\ell)\left[\frac{u_\lambda(\ell)s_\lambda^\ell}{u_\lambda(j)s_\lambda^j} - 1\right].$$

Therefore

$$\begin{aligned}\sum_{\ell=1}^\infty u_\lambda(k,\ell)\ell \le \frac{G_\lambda(k)}{\lambda+1}\Bigg[&\frac{\lambda}{1+2\lambda}\sum_{\ell=1}^\infty (\ell+1)f(\ell) \\ &+ \sum_{j=2}^\infty (j-1)\sum_{\ell=1}^{j-1} f_\lambda(j-\ell)\left(\frac{u_\lambda(\ell)s_\lambda^\ell}{u_\lambda(j)s_\lambda^j} - 1\right)\Bigg].\end{aligned}$$

By (2.6),

$$\sum_{\ell=1}^{j-1} f_\lambda(j-\ell)\left[\frac{u_\lambda(\ell)s_\lambda^\ell}{u_\lambda(j)s_\lambda^j} - 1\right] = 1 - \sum_{\ell=1}^{j-1} f_\lambda(j-\ell) = \sum_{\ell=j}^\infty f_\lambda(\ell) \le \sum_{\ell=j}^\infty f(\ell)$$

by (a) of Lemma 3.4. Therefore

$$\sum_{\ell=1}^\infty u_\lambda(k,\ell)\ell \le \frac{G_\lambda(k)}{2}\left[\frac{1}{2}\sum_{\ell=1}^\infty (\ell+1)f(\ell) + \frac{1}{2}\sum_{\ell=1}^\infty \ell(\ell-1)f(\ell)\right],$$

which completes the proof.

LEMMA 3.6. $\limsup_{k\to\infty} \frac{G_1(k)}{k} \le \frac{2}{\sum_{n=1}^\infty n^2 f(n)}$.

PROOF. Let Z_m be the symmetric random walk on the integers with transition probabilities

$$\bar{p}(k,\ell) = \tfrac{1}{2} f(|k-\ell|)$$

for $k \ne \ell$, and let $a(k)$ be the (recurrent) potential kernel for this random walk. Then

$$\begin{aligned}G_1(k) &= \{P^k[Z_m \text{ hits } \{\cdots, -1, 0\} \text{ before returning to } k]\}^{-1} \\ &\le \{P^k[Z_m \text{ hits } \{0\} \text{ before returning to } k]\}^{-1} = 2a(k)\end{aligned}$$

by Theorem 2 of Section 30 of [8]. By Proposition 4 of Section 28 of [8],

$$\lim_{k\to\infty}\frac{a(k)}{k}=\left[\sum_{n=-\infty}^{\infty}n^2\bar{p}(0,n)\right]^{-1}=\left[\sum_{n=1}^{\infty}n^2f(n)\right]^{-1}, \tag{3.7}$$

from which the result follows.

LEMMA 3.8. $\lim_{\lambda\downarrow 1}G_\lambda(k,\ell)=G_1(k,\ell)$ *for* $k,\ell\geq 1$.

PROOF. Note that for $k,\ell\geq 1$,

$$G_\lambda(k,\ell)=\sum_{m=0}^{\infty}P^k[Y_m^\lambda=\ell,\tau_0>m].$$

It is clear that for each m, ℓ and k,

$$P^k[Y_m^\lambda=\ell,\tau_0>m]$$

is continuous in λ. On the other hand,

$$\sum_{m=N}^{\infty}P^k[Y_m^\lambda=\ell,\tau_0>m]\leq P^k[Y_m^\lambda\neq 0\quad\text{for all}\quad m\leq N]G_\lambda(\ell)$$

and

$$\lim_{N\to\infty}\lim_{\lambda\downarrow 1}P^k[Y_m^\lambda\neq 0\quad\text{for all}\quad m\leq N]=\lim_{N\to\infty}P^k[Y_m^1\neq 0\quad\text{for all}\quad m\leq N]=0$$

since Y_m^1 is recurrent. Since $P^k[Y_m^\lambda\neq 0$ for all $m\leq N]$ is decreasing as $N\uparrow\infty$ and $\lambda\downarrow 1$ by (b) of Lemma 3.4, it follows that

$$\lim_{N\uparrow\infty,\lambda\downarrow 1}P^k[Y_m^\lambda\neq 0\quad\text{for all}\quad m\leq N]=0.$$

The result then follows from the fact that for each $\ell\geq 1$, $G_\lambda(\ell)$ is bounded on bounded λ sets.

LEMMA 3.9.

$$\limsup_{\lambda\downarrow 1,k\uparrow\infty}\frac{1}{k}\sum_{\ell=1}^{\infty}u_\lambda(k,\ell)\ell<1.$$

PROOF. Fix $N\geq 1$. By parts (b) and (c) of Lemma 3.3,

$$G_\lambda(k)\leq\frac{k+N}{N}G_\lambda(N)$$

for all $k\geq 1$. By Lemma 3.5,

$$\frac{1}{k}\sum_{\ell=1}^{\infty}u_\lambda(k,\ell)\ell\leq\frac{k+N}{4kN}G_\lambda(N)\left[1+\sum_{\ell=1}^{\infty}\ell^2f(\ell)\right]. \tag{3.10}$$

Since $(1/4)[1+\sum_{n=1}^{\infty}n^2f(n)]<\frac{1}{2}\sum_{n=1}^{\infty}n^2f(n)$, Lemmas 3.6 and 3.8 guarantee that N can be chosen so large that for all λ sufficiently close to 1,

$$\frac{G_\lambda(N)}{N}<\frac{4}{1+\sum_{\ell=1}^{\infty}\ell^2f(\ell)}.$$

The required result follows from this and (3.10).

LEMMA 3.11. $\sum_{\ell=1}^{\infty}u_\lambda(k,\ell)\leq 1-\dfrac{1}{1+2\lambda}G_\lambda(k,1)$.

PROOF.

$$\begin{aligned}\textstyle\sum_{\ell=1}^{\infty} u_\lambda(k,\ell) &= \textstyle\sum_{\ell,j=1}^{\infty} G_\lambda(k,j)[q_\lambda(j,\ell) - p_\lambda(j,\ell)]\\ &= \textstyle\sum_{j=1}^{\infty} G_\lambda(k,j)[p_\lambda(j,0) - q_\lambda(j,0)]\\ &= P^k[Y_m^\lambda = 0 \quad \text{for some} \quad m] - \frac{1}{1+2\lambda} G_\lambda(k,1)\\ &\le 1 - \frac{1}{1+2\lambda} G_\lambda(k,1).\end{aligned}$$

Let $U_\lambda h$ be defined by

$$U_\lambda h(k) = \textstyle\sum_{\ell=1}^{\infty} u_\lambda(k,\ell) h(\ell) \quad \text{for} \quad k \ge 1$$

and let U_λ^n be the nth power of U_λ.

LEMMA 3.12. *Let $\bar{h}(\ell) = \ell$. There exists a constant C so that for all $\lambda \ge 1$ sufficiently close to* 1,

$$\textstyle\sum_{n=0}^{\infty} U_\lambda^n \bar{h} \le C\bar{h}.$$

PROOF. By Lemma 3.9, there is an $N \ge 1$ and a $\gamma \in (0, 1)$ so that for all $\lambda \ge 1$ sufficiently close to 1 and for all $k \ge N$,

$$U_\lambda \bar{h}(k) \le \gamma \bar{h}(k).$$

By Lemmas 3.8 and 3.11, $U_\lambda 1(k) \le 1$ for all k and the γ above can be chosen so that

$$U_\lambda 1(k) \le \gamma$$

for all $k \le N$ and all $\lambda \ge 1$ sufficiently close to 1. Choose $\bar{\gamma} \in (\gamma, 1)$ so that

$$U_\lambda \bar{h}(k) \le \frac{(\bar{\gamma} - \gamma)^2}{1 - \bar{\gamma}} N$$

for all $k \le N$ and all $\lambda \ge 1$ sufficiently close to 1. Then put

$$\bar{C} = \frac{\bar{\gamma} - \gamma}{1 - \bar{\gamma}} N.$$

For $1 \le k \le N$ and all $\lambda \ge 1$ sufficiently close to 1,

$$U_\lambda(\bar{C} + \bar{h}) \le \bar{C}_\gamma + \frac{(\bar{\gamma} - \gamma)^2}{1 - \bar{\gamma}} N = \bar{\gamma}\bar{C} \le \bar{\gamma}(\bar{C} + \bar{h}).$$

For $k \ge N$ and all $\lambda \ge 1$ sufficiently close to 1,

$$U_\lambda(\bar{C} + \bar{h}) \le \bar{C} + \gamma\bar{h} \le \bar{\gamma}(\bar{C} + \bar{h}).$$

Thus $U_\lambda(\bar{C} + \bar{h}) \le \bar{\gamma}(\bar{C} + \bar{h})$ for all $k \ge 1$. Interating this gives

$$\textstyle\sum_{n=0}^{\infty} U_\lambda^n \bar{h} \le \sum_{n=0}^{\infty} U_\lambda^n(\bar{C} + \bar{h}) \le (\bar{C} + \bar{h})(1 - \bar{\gamma})^{-1} \le C\bar{h}$$

where $C = (\bar{C} + 1)(1 - \bar{\gamma})^{-1}$.

LEMMA 3.13. (a) $\sup_{\lambda > 1, k \ge 1} \dfrac{v_\lambda(k)}{k(\lambda - 1)} < \infty$.

(b) $$\sup_n \sup_{1 \le k \le n} \frac{n v_{1,n}(k)}{k} < \infty.$$

PROOF. Consider the random walk Z_m^λ on the integers defined in the proof of Lemma 3.3. By Theorem 2 of Section 25 of [8],

$$\sum_{k\le 0} P^k[Z_m^\lambda > 0 \quad \text{for all} \quad m \ge 1] = E^0 Z_1^\lambda.$$

By part (d) of Lemma 3.3,

$$\sum_{k\le 0} P^k[Z_m^\lambda > 0 \quad \text{for all} \quad m \ge 1] = \frac{\lambda}{\lambda+1} \sum_{k\le 0} \sum_{\ell=1}^{\infty} f(\ell-k) v_\lambda(\ell)$$
$$\ge \frac{\lambda}{\lambda+1} v_\lambda(1) \sum_{n=1}^{\infty} n f(n).$$

Of course

$$E^0 Z_1^\lambda = \frac{\lambda}{\lambda+1} \sum_{\ell=1}^{\infty} \ell f(\ell) - \frac{1}{\lambda+1} \sum_{\ell=1}^{\infty} \ell f_\lambda(\ell),$$

so

$$v_\lambda(1) \le 1 - \frac{1}{\lambda} \frac{\sum_{\ell=1}^{\infty} \ell f_\lambda(\ell)}{\sum_{\ell=1}^{\infty} \ell f(\ell)},$$

or equivalently

$$v_\lambda(1) \le 1 - \frac{s_\lambda \varphi'(s_\lambda)}{\varphi'(1)}$$

where φ is as in the proof of Lemma 3.4. Since

$$\lim_{s\uparrow 1} \frac{\varphi'(1) - s\varphi'(s)}{1-\varphi(s)} = \frac{\varphi''(1) + \varphi'(1)}{\varphi'(1)},$$

it follows that

$$\limsup_{\lambda\downarrow 1} \frac{v_\lambda(1)}{\lambda - 1} \le \frac{\sum_{n=1}^{\infty} n^2 f(n)}{[\sum_{n=1}^{\infty} n f(n)]^2} < \infty.$$

To complete the proof of part (a), use part (e) of Lemma 3.3. For part (b), consider the random walk Z_m defined in the proof of Lemma 3.6 and let $a(k) \ge 0$ be its potential kernel. Then

$$a(k) = \sum_{\ell=-\infty}^{\infty} \bar{p}(k,\ell) a(\ell) \quad \text{for} \quad k \ne 0.$$

Let τ be the hitting time for this random walk of the complement of $\{1, \cdots, n\}$. Then by the martingale stopping theorem,

$$a(k) \ge E^k a(Z_\tau) \quad \text{for} \quad 1 \le k \le n.$$

Therefore, $a(k) \ge P^k(Z_\tau > n) \inf_{m>n} a(m)$, and hence

$$v_{1,n}(k) = P^k(Z_\tau > n) \le \frac{a(k)}{\inf_{m>n} a(m)}.$$

Statement (b) now follows from (3.7)

THEOREM 3.14. (a) $\sup_{\lambda>1, k\ge 1} \dfrac{w_\lambda(k)}{k(\lambda-1)} < \infty.$

(b) $$\sup_n \sup_{1\le k\le n} \left[\frac{n w_{1,n}(k)}{k}\right] < \infty.$$

PROOF. By Lemma 3.2,

$$w_{\lambda,n} \le v_{\lambda,n} + U_\lambda w_{\lambda,n}.$$

Since $0 \le w_{\lambda,n}(k) \le 1$, Lemma 3.12 implies that $\lim_{\ell\to\infty} U_\lambda^\ell w_{\lambda,n} = 0$. Therefore

$$w_{\lambda,n} \le \sum_{\ell=0}^{\infty} U_\lambda^\ell v_{\lambda,n}.$$

Using Lemma 3.12 again, statements (a) and (b) of the Theorem follow from (a) and (b) of Lemma 3.13 respectively.

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