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THE BINARY CONTACT PATH PROCESS

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We study some $\{0, 1, \cdots\}^{Z^d}$ valued Markov interactions η_t called contact path processes. These are similar to branching random walks, in that the normalized size process starting from a singleton is a martingale which converges to a limit M_∞. In contrast to branching, however, M_∞ depends on the spatial dynamics of the path process. The main result is an exact evaluation of the variance of M_∞, achieved by means of the Feynman-Kac formula. The basic contact process of Harris may be viewed as a projection of η_t; as a corollary to the main result we obtain bounds on the contact process critical value $\lambda_c^{(d)}$ in dimension $d \geq 3$.

1. The results. For motivation, let us first consider a continuous time binary spatial branching process $(\eta_t(x))$ on the d-dimensional integer lattice Z^d. In this process each individual at site $x \in Z^d$ waits an independent intensity $1 + \delta$ exponential holding time, and then either dies with probability $\delta/(1 + \delta)$ or stays at x and produces an offspring at some $y \in Z^d$, $y \neq x$, chosen with probability $p(y - x)/(1 + \delta)$. Here $\delta \geq 0$ is a parameter and p is a fixed probability density on Z^d with $p(0) = 0$. Denote by $\eta_t^0(x)$ the number of individuals at site x at time t when the process starts with a single individual at the origin at time 0 (0 will denote the configuration $1_{\{x=0\}}$ throughout the paper.) Thus $\eta_t^0 = (\eta_t^0(x))_{x\in Z^d}$ is a Markov process on $\{0, 1, \cdots\}^{Z^d}$; the possible transitions at site x at time t are

$$\begin{aligned} \eta_t^0(x) &\to \eta_t^0(x) + 1 \quad \text{at rate} \quad \textstyle\sum_y p(x - y)\eta_t^0(y), \\ &\to \eta_t^0(x) - 1 \quad \text{at rate} \quad \delta\, \eta_t^0(x). \end{aligned} \tag{1.1}$$

(In Sections 2 and 3 we consider only chains which live on $S_0 = \{\eta \text{ with finite support}\}$. In Section 4 we will discuss processes with infinite configurations.)

Let X_t be the continuous random walk on Z^d with mean 1 holding times and displacement density p, and write $p(t, x) = P(X_t = x \mid X_0 = 0)$. Then letting

$$m_x(t) = E[\eta_t^0(x)],$$

it is not hard to check that m satisfies

$$\begin{aligned} \frac{dm_x(t)}{dt} &= -\delta m_x(t) + \textstyle\sum_y p(x - y)m_y(t) \\ m_x(0) &= 1_{\{x=0\}}, \end{aligned} \tag{1.2}$$

and so

$$m_x(t) = e^{(1-\delta)t}p(t, x), \quad x \in Z^d,\ t \geq 0. \tag{1.3}$$

Next, write

$$|\eta_t^0| = \textstyle\sum_x \eta_t^0(x), \quad m(t) = E[|\eta_t^0|].$$

From (1.3),

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$$m(t) = e^{(1-\delta)t}. \tag{1.4}$$

In fact, from (1.1) we see that

$$\begin{aligned} |\eta_t^0| &\to |\eta_t^0| + 1 \quad \text{at rate} \quad |\eta_t^0| \\ &\to |\eta_t^0| - 1 \quad \text{at rate} \quad \delta|\eta_t^0|, \end{aligned}$$

so that $|\eta_t^0|$ is simply a continuous time binary branching process. Thus

$$M_t = e^{-(1-\delta)t}|\eta_t^0|$$

is a non-negative mean 1 martingale, and

$$M_t \to M_\infty < \infty \quad \text{a.s.} \quad \text{as } t \to \infty \tag{1.5}$$

for some random M_∞. Of course from branching theory (e.g. [1]) we know that a *critical phenomenon* occurs. If $\delta < \delta_c = 1$, then $\sup_t E[M_t^k] < \infty$ for each k and hence $M_\infty \neq 0$ has all moments finite. In particular one can compute

$$\text{Var}(M_\infty) = \frac{1+\delta}{1-\delta}, \quad \delta < 1. \tag{1.6}$$

If $\delta \le \delta_c$, then $\rho_\eta(\delta) = P(|\eta_t^0| > 0 \text{ for all } t) = 0$, and hence $M_\infty = 0$ a.s. Also, for *any* δ,

$$P(M_\infty > 0) = \rho_\eta(\delta). \tag{1.7}$$

In this paper we study a similar but more complex Markov process η_t^0 on $\{0, 1, \cdots\}^{Z^d}$, which we call the (binary) *contact path process*. Instead of (1.1), the dynamics at x at time t are:

$$\begin{aligned} \eta_t^0(x) &\to \eta_t^0(x) + \eta_t^0(y) \quad \text{at rate} \quad p(x-y) \quad (y \in Z^d), \\ &\to 0 \qquad\qquad\qquad\quad \text{at rate} \quad \delta. \end{aligned} \tag{1.1$'$}$$

With $m_x(t)$ and $m(t)$ defined as above, a little thought reveals that $m_x(t)$ satisfies precisely the same differential equations (1.2), so that (1.3) and (1.4) follow. (In expectation the increments of the two processes agree, but the variation in (1.1′) is more dramatic than in (1.1).) Moreover, for each x such that $\eta_t^0(x) > 0$,

$$\begin{aligned} |\eta_t^0| &\to |\eta_t^0| + \eta_t^0(x) \quad \text{at rate} \quad 1 \\ &\to |\eta_t^0| - \eta_t^0(x) \quad \text{at rate} \quad \delta, \end{aligned}$$

which implies that, as before, M_t is a non-negative mean 1 martingale (a formal proof will be given later), and so (1.5) holds by martingale convergence. In contrast to the branching example, however, $|\eta_t^0|$ now depends nontrivially on the spatial structure of $(\eta_t^0(x))$.

One reason for our interest in contact path processes, as the name implies, is their intimate connection with Harris' contact processes [4], probably the simplest Markov systems on $\{0, 1\}^{Z^d}$ which exhibit critical phenomena. The dynamics of the most widely studied example, the *basic contact process* ξ_t *with parameter* λ, are as follows: at site x at time t,

$$\begin{aligned} 0 \to 1 \quad &\text{at rate} \quad \lambda \textstyle\sum_{y:|y-x|=1} \xi_t(y) \quad \text{if} \quad \xi_t(x) = 0, \\ 1 \to 0 \quad &\text{at rate} \qquad\qquad 1 \qquad\qquad\quad \text{if} \quad \xi_t(x) = 1. \end{aligned} \tag{1.8}$$

(Throughout the paper we will use the *box norm* $|(x_1, \cdots, x_d)| = \sum_{i=1}^d |x_i|$ on Z^d.)

See [3] for a survey of known results concerning the process (1.8), and for a great many references. To explain the connection between (1.1′) and (1.8), let us recall Harris' useful graphical representation [6]. Start with the space-time diagram $Z^d \times [0, \infty)$. For each pair of distinct $x, y \in Z^d$ draw (oriented) arrows from (x, τ_{xy}^n) to (y, τ_{xy}^n) $(n = 1, 2, \cdots)$, where the $\tau_{xy}^n - \tau_{xy}^{n-1}$ are i.i.d. exponential with mean $1/p(y-x)(\tau_{xy}^0 = 0)$. In addition, for each $x \in Z^d$, put down "D"'s at (x, τ_x^n) $(n = 1, 2, \cdots)$, where the $\tau_x^n - \tau_x^{n-1}$ are i.i.d. exponential with mean δ. The resulting random scheme $\mathscr{P}$ is called the *percolation substructure*. By a *path up from* (x_0, t_0) *to* (x_n, t_n) *in* $\mathscr{P}$ we mean a sequence of space-time points

$$(x_0, t_0), (x_0, t_1), (x_1, t_1), \cdots, (x_{n-1}, t_n), (x_n, t_n),$$

with increasing time coordinates t_k, such that there is an arrow from (x_{k-1}, t_k) to (x_k, t_k) for each k, and no D appears on any segment $\{(x_k, u); t_k \le u \le t_{k+1}\}$. See [2], [3] and [6] for more details; the first two references provide helpful pictures. Now the contact path process can be represented as

$$\eta_t^0(x) = \text{the number of distinct paths up from } (0, 0) \text{ to } (x, t) \text{ in } \mathscr{P}.$$

Moreover, if we consider the "projection" ζ_t^0 given by

$$\zeta_t^0(x) = 1_{\{\eta_t^0(x)>0\}}, \tag{1.9}$$

then ζ_t^0 is a contact process with flip rates at x at time t:

$$\begin{array}{llll} 0 \to 1 & \text{at rate} & \sum_y p(x-y)\zeta_t^0(y) & \text{if } \zeta_t^0(x) = 0 \\ 1 \to 0 & \text{at rate} & \delta & \text{if } \zeta_t^0(x) = 1. \end{array} \tag{1.10}$$

(See [6] for a proof.) Comparing (1.7) and (1.10) we see that if $p(y) = (2d)^{-1}1_{\{|y|=1\}}$ and $\lambda = (2d\delta)^{-1}$, then

$$\xi_t^0 = \zeta_{\delta^{-1}t}^0 \quad \text{(in distribution as processes)}. \tag{1.11}$$

Thus information about the contact path process can yield information about the basic contact process via (1.9) and (1.11). For instance, consider the critical values

$$\delta_c = \inf\{\delta \ge 0 : \rho_\eta(\delta) = P(|\eta_t^0| > 0\ \forall t) = 0\},$$

$$\delta_* = \inf\{\delta \ge 0 : M_\infty = 0\},$$

$$\lambda_c = \sup\{\lambda \ge 0 : \rho_\xi(\lambda) = P(|\xi_t^0| > 0\ \forall t) = 0\}.$$

λ_c is the critical value for the basic contact process. By (1.11), (1.9) and (1.4), $\lambda_c \ge 1/2d$. Actually, Harris [4] has proved that $\lambda_c \ge 1/(2d-1)$. On the other hand, (1.11), (1.9) and the trivial inequality $\delta_* \le \delta_c$ give

$$\lambda_c \le 1/2d\delta_*, \tag{1.12}$$

so lower bounds on δ_* give upper bounds for λ_c.

The path process η_t^0 is also intriguing in its own right. In contrast to the branching model, the nature of the limit law M_∞ depends fundamentally on p as well as δ. Let S_n be the discrete symmetrized random walk with displacement density $\tilde{p}(y) = (1/2)[p(y) + p(-y)]$. It turns out that if S_n is transient and

$$\gamma = P(S_n \ne 0\ \forall n \ge 1 \mid S_0 = 0) > 1/2, \tag{1.13}$$

then for all sufficiently small $\delta > 0$, $M_\infty \ne 0$. We will prove this in Section 2 by computing $E[M_t^2]$ exactly for each $t < \infty$ with the aid of the Feynman-Kac formula. Assuming (1.13), we will show that for small δ the second moments of M_t are uniformly bounded in t, so that M_∞ has mean 1. In fact one can integrate M_t^2 to the limit to obtain $E[M_\infty^2]$. The main result of the paper is

THEOREM 1. *If* $\gamma > 1/2$, *then* $\delta_* \ge 2\gamma - 1$. *For* $\delta < 2\gamma - 1$,

$$E[M_\infty] = 1, \quad \operatorname{Var}(M_\infty) = \frac{1+\delta}{2\gamma - (1+\delta)}.$$

Note that the variance agrees with (1.6) in the limit as $\gamma \to 1$. As explained in the preceding paragraph, it follows that $\delta_c \ge 2\gamma - 1$, and so for the basic contact process ξ_t in three or more dimensions, we get a new upper bound on the critical value from (1.12) and Theorem 1.

COROLLARY. *Let $d \geq 3$. Then*

$$\lambda_c^{(d)} \leq [2d(2\gamma_d - 1)]^{-1}, \tag{1.14}$$

and for $\lambda > [2d(2\gamma_d - 1)]^{-1}$,

$$\rho_\xi(\lambda) \geq 1 - \left(\frac{1}{2\gamma_d}\right)\left(1 + \frac{1}{2d\lambda}\right).$$

(The bound for ρ uses the Schwarz inequality:

$$\rho \geq P(M_\infty > 0) \geq (E[M_\infty])^2/E[M_\infty^2].)$$

In [7], Holley and Liggett obtained the bounds

$$\lambda_c^{(d)} \leq \gamma_d^{-1} - 1 \quad (d \geq 3) \tag{1.15}$$

by comparing the basic contact process with a certain "generalized smoothing process." If $d = 3$, then (1.15) gives $\lambda_c^{(3)} \leq .517$, which is slightly better than (1.14) ($\lambda_c \leq .523$). For $d = 4$ it is easy to check that (1.14) is stronger than (1.15), and (1.14) is also asymptotically better as $d \to \infty$:

$$[2d(2\gamma_d - 1)]^{-1} \sim \frac{1}{2d} + \frac{1}{2d^2} + O\left(\frac{1}{d^3}\right),$$

whereas,

$$\gamma_d^{-1} - 1 \sim \frac{1}{2d} + \frac{3}{4d^2} + O\left(\frac{1}{d^3}\right).$$

Computer calculations by Chris Thron give convincing evidence that our bound (1.14) improves (1.15) for any $d \geq 4$. Actually, this paper originated in an attempt to understand a beautiful computation of Kesten (unpublished) which gives a lower bound for the critical value of oriented percolation in four or more dimensions as the return probability of a $(d-1)$ dimensional random walk. Our Theorem 1 is an analogue of Kesten's result, though the proof is rather different.

The similarity between (1.14) and (1.15) is striking, and suggests that there may be other features shared by η_t^0 and the generalized smoothing processes of [7]. As we shall see, the limit behavior of M_t parallels the theory developed by Holley and Liggett in [7]; many of the techniques from [7] and an earlier paper of Liggett and Spitzer [8] apply equally well to the normalized contact path process.

What happens when S_n is recurrent? For $d = 1$ or 2, if p has bounded support, it is shown in Section 3 that the limit variable M_∞ is identically zero for *any* $\delta \geq 0$. The technique is the same as in [7]: one shows that $E[M_t^{1/2}] \to 0$. The precise result is

THEOREM 2. *Suppose $d = 1$ or 2, with $\sum |y| p(y) < \infty$ if $d = 1$, or $\sum |y|^2 p(y) < \infty$ if $d = 2$. Then $M_\infty = 0$ for any $\delta \geq 0$, so $\delta_* = 0$.*

It seems most likely that $M = 0$ whenever S_n is recurrent.

In Section 4 we discuss briefly the contact path processes η_t^η with arbitrary initial states $\eta \in [0, \infty)^{Z^d}$. Using the percolation substructure $\mathscr{P}$, simply define

$$\eta_t^\eta(x) = \textstyle\sum_{y \in Z^d} \eta(y)\, \eta_t^y(x), \tag{1.16}$$

where

$$\eta_t^y(x) = \text{the number of distinct paths from } (y, 0) \text{ to } (x, t) \text{ in } \mathscr{P}.$$

Note that $\eta_t^\eta(x) = \infty$ is possible in (1.16) if η is unbounded and p has infinite support. Now suppose we start the contact path process in state $\mathbf{1}$ = "all 1's on Z^d", and consider the normalized process

$$\bar{\eta}_t^1 = e^{-(1-\delta)t}\eta_t^1.$$

Then there is a $[0, \infty)$ valued random field $\bar{\eta}_\infty$ on Z^d such that $\bar{\eta}_t^1 \to \bar{\eta}_\infty$ as $t \to \infty$, where $\to$ means weak convergence of finite dimensional distributions. Under the hypotheses of Theorem 2, $\bar{\eta}_\infty = \mathbf{0}$ = the "all 0's" field. But under the hypotheses of Theorem 1, the law ν of $\bar{\eta}_\infty$ is a nontrivial invariant measure for the normalized process. In other words, let $\bar{\eta}_\infty$ be ν-distributed and independent of $\mathscr{P}$, and define

$$\eta_t^\nu = \eta_t^\eta \quad \text{on} \quad \{\bar{\eta}^\infty = \eta\};$$

then

$$P(\bar{\eta}_t^\nu \in \cdot) = \nu \quad \text{for each} \quad t \geq 0. \tag{1.17}$$

Moreover, just as in [7] and [8], one can compute the covariances of ν explicitly. In summary, the result is as follows.

THEOREM 3. *For each density p and $\delta \geq 0$ there is a limiting* $[0, \infty)$ *valued field $\bar{\eta}_\infty$ on Z^d such that $\bar{\eta}_t \to \bar{\eta}_\infty$ as $t \to \infty$. The law ν of $\bar{\eta}_\infty$ is invariant in the sense of* (1.17). *If $d = 1$ and $\sum |y| p(y) < \infty$, or if $d = 2$ and $\sum |y|^2 p(y) < \infty$, then $\bar{\eta}_\infty = \mathbf{0}$ for any $\delta \geq 0$. On the other hand, if $\gamma > 1/2$ and $\delta \leq 2\gamma - 1$, then $\bar{\eta}_\infty \neq \mathbf{0}$ has density one and covariances*

$$\operatorname{Cov}(\bar{\eta}_\infty(x), \bar{\eta}_\infty(y)) = \frac{(1+\delta)\pi(y-x)}{2\gamma - (1+\delta)},$$

where $\pi(x) = P(\exists n \geq 0 : S_n = 0 \mid S_0 = x)$ $(\pi(0) = 1)$.

The main tool in the proof of Theorem 3 is a duality equation, like the ones in [7] and [8], which connects $\bar{\eta}_t^1$ and M_t. Presumably one could develop an ergodic theory of normalized contact path processes following the lead of [7] and [8]. For example, one should be able to show that if μ is translation invariant and ergodic with density one, then $\bar{\eta}_t^\mu \to \bar{\eta}_\infty$ as $t \to \infty$.

We conclude this introduction by mentioning the two most intriguing open questions raised by our results. First, what can be said concerning the convergence of M_t to 0 in the recurrent case? Perhaps for the simplest example, $p(1) = 1$ and $\delta = 0$, one can get more detailed results. Second, what is the nature of M_∞ if $\gamma \leq 1/2$ or $0 < 2\gamma - 1 \leq \delta < \delta_c$? The most likely scenario seems to be that M_∞ have infinite second moment in these cases, and that the moments between the second and the first erode continuously as $\delta \uparrow \delta_c$.

2. The proof of Theorem 1. Our main task is to compute $\operatorname{Var}(M_t)$. We accomplish this by deriving an explicit formula for

$$u_x(t) = E[\textstyle\sum_y \eta_t^0(y)\eta_t^0(x+y)].$$

Note that $E[|\eta_t^0|^2] = \sum_x u_x(t)$, so

$$\operatorname{Var}(M_t) = (e^{-2(1-\delta)t} \textstyle\sum_x u_x(t)) - 1.$$

The path processes η_t^η with dynamics (1.1′) have (formal) generator

$$Gf(\eta) = \textstyle\sum_x [\delta[f({}^x\eta) - f(\eta)] + \sum_y p(x-y)[f({}^{yx}\eta) - f(\eta)]], \tag{2.1}$$

where ${}^x\eta$ and ${}^{yx}\eta$ are the modifications of η:

$${}^x\eta(z) = 0,\ z = x, \qquad {}^{yx}\eta(z) = \eta(x) + \eta(y),\ z = x,$$
$$= \eta(z),\ z \neq x; \qquad = \eta(z), \qquad z \neq x.$$

Adopting the usual semigroup notation, write $T_t f(0) = E[f(\eta_t^0)]$. To calculate $u_x(t)$ we will take $f_x(\eta) = \sum_y \eta(y)\eta(x+y)$, verify the forward equations

$$u_x'(t) = \frac{dT_t f_x(0)}{dt} = T_t G f_x(0), \tag{2.2}$$

and solve (2.2) with boundary conditions $u_x(0) = 1_{\{x=0\}}$. Since the path processes have unbounded jump rates, the verification of (2.2) necessarily involves approximation by "tame" Markov chains $\eta_{N,t}^{\eta}$ $(N = 1, 2, \cdots)$. Thus, write $\Lambda_N = \{x \in Z^d: |x| \le N\}$ and define $\eta_{N,T}^{\eta}$ as in (1.16) except that only arrows with heads and tails in $\Lambda_N \times [0, \infty)$ and only D's in $\Lambda_N \times [0, \infty)$ are used. Then $\eta_{N,t}^{\eta}$ is a Markov chain with bounded jump rates; its generator G_N looks like (2.1) with the sums restricted to $x, y \in \Lambda_N$. Therefore, if $f(\eta)$ is bounded and $T_{N,t} f(0) = E[f(\eta_{N,t}^0)]$,

$$\frac{dT_{N,t} f(0)}{dt} = T_{N,t} G_N f(0). \tag{2.3}$$

Also, if $g(t, \eta) = e^{-ct} f(\eta)$ ($c \in R$, f bounded), then

$$g(t, \eta_{N,t}^{\eta}) - \int_0^t (G_N - c) g(s, \eta_{N,s}^{\eta})\, ds \text{ is a martingale.} \tag{2.4}$$

Observe that the nth jump of η_t^0 has rate at most $(1+\delta)n$. Letting τ_n be the time of the nth jump, it follows that $\tau_n \to \infty$ a.s., and hence that for each $t < \infty$, $\eta \in S_0$,

$$P(\exists N_0 < \infty : \eta_s^{\eta} = \eta_{N,s}^{\eta}\ \forall s \le t\ \forall N \ge N_0) = 1. \tag{2.5}$$

We will use (2.3), (2.5) and dominated convergence to establish (2.2).

But first, let us use (2.4) to check that M_t is a martingale. Let $f(\eta) = |\eta|$, $\eta_N(x) = \eta(x) \wedge N$, $f_N(\eta) = f(\eta_N)$. Using (2.5), it is easy to see that as $N \to \infty$,

$$f_N(\eta_{N,t}^0) \to f(\eta_t^0), \quad \text{a.s.}$$

and

$$\int_0^t G_N f_N(\eta_{N,s}^0)\, ds \to \int_0^t G f(\eta_s^0)\, ds \quad \text{a.s.}$$

Write $g(t, \eta) = e^{-ct}|\eta|$. Since (2.4) holds for $g_N(t, \eta) = e^{-ct} f_N(\eta)$, to show that

$$M_t^{(c)} = g(t, \eta_t^0) - \int_0^t (G - c) g(s, \eta_s^0)\, ds \quad \text{is a martingale,}$$

we need only check that the approximating martingales converge in mean. To apply monotone and dominated convergence, it suffices to note that

$$f_N(\eta_{N,t}^0) \uparrow f(\eta_t^0) \quad \text{as} \quad N \to \infty, \tag{2.6}$$

and check that

$$\sup_N E[f_N(\eta_{N,t}^0)] < \infty. \tag{2.7}$$

A straightforward calculation gives

$$|G_N f_N(\eta)| \le (1+\delta) f_N(\eta), \tag{2.8}$$

so by (2.3) the supremum in (2.7) is at most $e^{(1+\delta)}t$. Applying (2.8) again, the approximating martingales are evidently dominated. Now observe that $Gf(\eta) = (1-\delta)|\eta|$, so that $(G - (1-\delta))(e^{-(1-\delta)}|\eta|) \equiv 0$. We conclude that $M_t = M_t^{(1-\delta)}$ is a martingale as claimed.

The argument leading to (2.2) is similar in spirit. First one shows that

$$f_x(\eta_t^0) - \int_0^t G f_x(\eta_s^0)\, ds \quad \text{is a martingale} \tag{2.9}$$

by using the tame approximants $f_N(\eta) = f_x(\eta_N)$ in (2.4) (with $c = 0$) and letting $N \to \infty$. To see that the limit is integrable, set $h(\eta) = |\eta|^2$ and $h_N(\eta) = h(\eta_N)$. Also, let G^0 be the generator G with $\delta = 0$, T_t^0 the corresponding semigroup, G_N^0 and $T_{N,t}^0$ the tame versions. Then for all N,

$$G_N f_N(\eta) \le G_N^0 f_N(\eta) \le G_N^0 h_N(\eta), \tag{2.10}$$

and

$$\begin{aligned} G_N^0 h_N(\eta) &= \textstyle\sum_{x,y\in\Lambda_N} p(x-y)[h_N({}^{yx}\eta) - h_N(\eta)] \\ &\le \textstyle\sum_{y\in\Lambda_N} (\eta(y)\wedge N)(3\sum_{z\in\Lambda_N}\eta(z)\wedge N) \\ &= 3h_N(\eta). \end{aligned} \tag{2.11}$$

By monotonicity, (2.3) and Fatou,

$$T_t f_x(\eta) \le T_t^0 f_x(\eta) \le T_t^0 h(\eta) \le e^{3t}|\eta|^2. \tag{2.12}$$

The same estimates as in (2.10) and (2.11) for the limit generator give

$$Gf_x(\eta) \le 3h(\eta).$$

Also,

$$Gf_x(\eta) \ge -\delta \textstyle\sum_x [h(\eta) - h({}^x\eta)] \ge -2\delta h(\eta).$$

Thus,

$$|Gf_x(\eta)| \le (3 \vee 2\delta)h(\eta), \tag{2.13}$$

and hence

$$|T_t Gf_x(0)| \le (3 \vee 2\delta)e^{3t}. \tag{2.14}$$

Estimates (2.12) and (2.14) give (2.9). Taking expectations we get

$$u_x(t) = \int_0^t T_s Gf_x(0)\, ds.$$

The equations (2.2) follow once we check that $T_s Gf_x(0)$ is continuous in s. By the semigroup property it suffices to check continuity of $T_s Gf_x(\eta)$ for arbitrary $\eta \in S_0$ at $s = 0$. Since $\eta_s^\eta \to \eta$ a.s. as $s \to 0$ and by (2.13) and monotonicity

$$|Gf_x(\eta_s^\eta)| \le (3 \vee 2\delta)h(\eta_s^\eta) \le (3 \vee 2\delta)h(\tilde\eta_1^\eta) \quad \forall s \le 1,$$

where $\tilde\eta$ is the path process with $\delta = 0$, we have

$$E[|Gf_x(\eta_s^\eta) - Gf_x(\eta)|] \to 0 \quad \text{as} \quad s \downarrow 0$$

by (2.12) and dominated convergence. This establishes (2.2).

Let us now proceed to compute $u_x(t)$. To begin, we set $f_{x,y}(\eta) = \eta(x)\eta(y)$ ($x, y \in Z^d$) and calculate $Gf_{y,x+y}(\eta)$. For $x \ne 0$ we get

$$\begin{aligned} Gf_{y,x+y}(\eta) &= \Sigma_z \{\delta[{}^z\eta(y){}^z\eta(x+y) - \eta(y)\eta(x+y)] \\ &\qquad + \textstyle\sum_w p(w)[{}^{z-w,z}\eta(y){}^{z-w,z}\eta(x+y) - \eta(y)\eta(x+y)]\} \\ &= -2\delta\eta(y)\eta(x+y) + \textstyle\sum_w p(w)\{\eta(y-w)\eta(x+y) + \eta(y)\eta(x+y-w)\}. \end{aligned}$$

For $x = 0$,

$$\begin{aligned} Gf_{y,y}(\eta) &= \textstyle\sum_z \{\delta[{}^z\eta^2(y) - \eta^2(y)] + \sum_w p(w)[{}^{z-w,z}\eta^2(y) - \eta^2(y)]\} \\ &= -\delta\eta^2(y) + \textstyle\sum_w p(w)\{2\eta(y)\eta(y-w) + \eta^2(y-w)\}. \end{aligned}$$

Summing on y we find that

$$Gf_x = -2\delta f_x + \sum_y p(y)[f_{x-y} + f_{x+y}], \quad x \neq 0,$$
$$= (1-\delta)f_0 + \sum_y p(y)[f_{-y} + f_y], \quad x = 0.$$

(In the last line we have used the symmetry of f_{xy}). Therefore, according to (2.2), the $u_x(t)$ satisfy

$$(2.15) \qquad \frac{du_x(t)}{dt} = -2\delta u_x(t) + \sum_y (p(y) + p(-y))u_{x+y} \quad x \neq 0,$$
$$= (1-\delta)u_0(t) + \sum_y (p(y) + p(-y))u_y \quad x = 0,$$

with boundary condition $u_x(0) = 1_{\{x=0\}}$. To solve (2.15), write $v_x(t) = e^{-2(1-\delta)t}u_x(t)$. Then $v_x(0) = 1_{\{x=0\}}$ and

$$(2.16) \qquad \frac{dv_x(t)}{dt} = k(x)v_x(t) + Av_x(t),$$

where $k(x) = (1+\delta)1_{\{x=0\}}$ and

$$Af_x = \sum_y (p(y) + p(-y))[f_{x+y} - f_x], \quad x \in Z^d,$$

is the generator of a continuous time random walk $\tilde{X}_t$ with displacement density $\tilde{p}(y)$ and mean ½ exponential holding times. Now recall the Feynman-Kac formula for Markov chains: if k is bounded and the chain $\tilde{X}_t$ has generator A with bounded jump rates, then the unique solution of (2.16) uniformly bounded on $\{(x,s): x \in Z^d, 0 \le s \le t\}$ is:

$$v_x(t) = E_x\left[\exp\left\{\int_0^t k(\tilde{X}_s)\,ds\right\} v_{\tilde{X}_t}(0)\right]$$

(P_x starts at x). (A proof may be fashioned after the one in [10], for example.) In our case, using (2.12),

$$v_x(s) \le E[M_s^2]$$
$$\le e^{-2(1-\delta)s}T_t^0 h(0) \le \exp\{[3 \vee (2\delta+1)]t\} \quad \forall s \le t,$$

so the formula applies. Also, $\tilde{X}_t$ is reversible in the sense that the P_0 law of $x - (\tilde{X}_t - \tilde{X}_{t-s})$, $0 \le s \le t$, equals the P_x law of $\tilde{X}_s$, $0 \le s \le t$. Thus,

$$v_x(t) = E_0\left[\exp\left\{\int_0^t k(x - \tilde{X}_t + \tilde{X}_{t-s})\,ds\right\} v_{x-\tilde{X}_t}(0)\right]$$
$$= E_0\left[\exp\left\{(1+\delta)\int_0^t 1_{\{\tilde{X}_{t-s}=0\}}\,ds\right\} 1_{\{\tilde{X}_t=x\}}\right]$$
$$= E_0\left[\exp\left\{(1+\delta)\int_0^t 1_{\{\tilde{X}_s=0\}}\,ds\right\}, \tilde{X}_t \doteq x\right].$$

Summing on x, we find that

$$(2.17) \qquad v(t) := E[M_t^2] = \sum_x v_x(t)$$
$$= E_0\left[\exp\left\{(1+\delta)\int_0^t 1_{\{\tilde{X}_s=0\}}\,ds\right\}\right].$$

It is easy to evaluate $v(\infty) = \lim_{t\to\infty} v(t)$. In fact,

$$v(\infty) = E_0[\exp\{(1+\delta)\sum_{j=0}^L T_j\}],$$

where

$$T_j = \text{total time of } j\text{th sojourn at } 0 \text{ (initial sojourn} = 0\text{th)},$$

$$L = \text{total number of returns to } 0.$$

Certainly $v(\infty) = \infty$ if $\tilde{X}_t$ is recurrent. The T_j are i.i.d. exponential with mean ½. L is independent of the T_j with geometric density

$$P(L = k) = (1 - \gamma)^k \gamma, \quad k = 0, 1, \cdots$$

in the transient case, since S_n is the imbedded chain for $\tilde{X}_t$. Therefore, assuming $\gamma > 0$,

$$\begin{aligned} v(\infty) &= E_0[\textstyle\prod_{j=0}^{L} e^{(1+\delta)T_j}] \\ &= \textstyle\sum_{k=0}^{\infty} E_0[\prod_{j=0}^{k} e^{(1+\delta)T_j} 1_{\{L=k\}}] \\ &= \textstyle\sum_{k=0}^{\infty} \prod_{j=0}^{k} E_0[e^{(1+\delta)T_j}] P(L = k). \end{aligned}$$

Assume also that $\delta < 1$, so that

$$E_0[e^{(1+\delta)T_j}] = \int_0^\infty e^{(1+\delta)t} 2e^{-2t}\, dt = \frac{2}{1-\delta} < \infty.$$

Then we get

$$\begin{aligned} v(\infty) &= \textstyle\sum_{k=0}^{\infty} \left(\dfrac{2}{1-\delta}\right)^{k+1} (1-\gamma)^k \gamma \\ &= \frac{2\gamma}{2\gamma - (1+\delta)} < \infty \quad \text{provided} \quad 0 \le \delta < 2\gamma - 1. \end{aligned}$$

Under the hypotheses of Theorem 1 we have therefore shown that

$$E[M_\infty] = 1 \quad \text{and} \quad \lim_{t\to\infty} \operatorname{Var}(M_t) = \frac{1+\delta}{2\gamma - (1+\delta)}.$$

It now follows from general martingale theory that the M_t^2 are uniformly integrable. (Presumably for any $\delta < 2\gamma - 1$ there is an $\varepsilon > 0$ such that

$$\sup_t E[M_t^{2+\varepsilon}] < \infty.$$

This would also justify the integration of M_t^2 to the limit, but seems difficult to prove.) A third tack is to use a beautiful argument due to Liggett and Spitzer [8] to compute $\operatorname{Var}(M_\infty)$. Since their approach involves the infinite contact path processes, this final step will be presented in Section 4.

A final remark. The rigorous derivation of (1.2) and (1.3) for the contact path process proceeds along the same lines as the demonstration of (2.2) and (2.17) just completed. To get (1.2) one approximates by tame chains, for (1.3) one uses Feynman-Kac with $k = 0$. We will need (1.3) in the next section; details of the proof are left to the reader.

3. The proof of Theorem 2. Following the lead of Holley and Liggett [7], let us see what we learn by applying the forward equation

(3.1) $$\frac{dT_t f(0)}{dt} = T_t G f(0)$$

to the function $f(\eta) = |\eta|^{1/2}$. The verification of (3.1) for this choice of f is carried out just as before. For the needed integrability simply note that

$$f(\eta_N) - f({}^x\eta_N) \leq \eta_N(x), \quad f({}^{yx}\eta_N) - f(\eta_N) \leq \eta_N(y),$$

and hence that (2.8) holds for our f. Actually, we want to estimate $r(t) = E[M_t^{1/2}]$. By (3.1),

$$r'(t) = e^{-(1/2)(1-\delta)t} T_t(G - \tfrac{1}{2}(1-\delta))f(0). \tag{3.2}$$

After some algebraic manipulations, we find that whenever $|\eta| > 0$,

$$\left(G - \frac{1}{2}(1-\delta)\right)f(\eta) = -\sqrt{|\eta|}\left(\sum_x \delta\left[1 - \frac{1}{2}\frac{\eta(x)}{|\eta|} - \sqrt{1 - \frac{\eta(x)}{|\eta|}}\right] + \sum_{x,y} p(x-y)\left[1 + \frac{1}{2}\frac{\eta(y)}{|\eta|} - \sqrt{1 + \frac{\eta(y)}{|\eta|}}\right]\right). \tag{3.3}$$

Equation (3.3) contains a good deal of information. First, note that the bracketed terms are positive, so that $(G - \tfrac{1}{2}(1-\delta))f$ is negative. Thus $r(t) \downarrow r(\infty)$ as $r \to \infty$. Also, a variant of (3.3) can be used to show that $r_\delta(t)$ is nonincreasing in δ for each $t < \infty$. (See the Appendix for an outline of this argument.) It follows that $r_\delta(\infty)$ is nonincreasing in δ. Now since $E[M_t] \equiv 1$, the $M_t^{1/2}$ are uniformly integrable, and so $M_\infty = 0$ a.s. if and only if $r(\infty) = 0$. These considerations show that δ_* is an honest critical value, i.e. if $\delta > \delta_*$ then $M_\infty = 0$. In fact, we can prove that $P(M_\infty > 0)$ is nonincreasing in δ. Since $\rho_\eta(\delta) = \rho_\xi(\delta)$ is nonincreasing (see e.g. [3]), it suffices to establish the following analogue of (1.7). Thanks to Maury Bramson and Rick Durrett for the simple proof. We will only outline the argument since it is rather tangential to the main direction of the paper.

Proposition. *If $P(M_\infty > 0) > 0$, then $P(M_\infty > 0) = \rho_\eta$.*

Sketch of proof. Since $P(\inf_t M_t = 0, M_\infty > 0) = 0$, if $P(M_\infty > 0) > 0$ then there is an $\varepsilon > 0$ such that

$$P(\inf_t M_t > \varepsilon) = p > 0.$$

Let $\tau_1 = \inf\{t : |\eta_t^0| \leq e^{(1-\delta)t}\varepsilon\}$; $P(\tau_1 = \infty) = p$. On $\{\tau_1 < \infty, |\eta_{\tau_1}^0| > 0\}$, choose a site from $\eta_{\tau_1}^0$, say x_1, and consider the process $\eta_t^{(1)}$ given by

$$\eta_t^{(1)}(x) = \text{the number of paths up from } (x_1, \tau_1) \text{ to } (x, \tau_1 + t).$$

Let $\tau_2 = \inf\{t : |\eta_t^{(1)}| < e^{(1-\delta)t}\varepsilon\}$; $P(\tau_2 = \infty \mid \tau_1 < \infty) = p$. And so on. If $|\eta_t^0|$ stays positive, then after a geometrically distributed number N of trials we get $|\eta_t^{(N)}| > e^{(1-\delta)t}\varepsilon$ for all t. By monotonicity, on $\{|\eta_t^0| \text{ never } 0\}$ we have for all $t \geq \tau_N$,

$$M_t \geq e^{-(1-\delta)\tau_N}[e^{-(1-\delta)(t-\tau_N)}|\eta_{t-\tau_N}^{(N)}|] \geq e^{-(1-\delta)\tau_N}\varepsilon > 0.$$

Hence $\rho_\eta = P(|\eta_t^0| \text{ never } 0) \leq P(M_\infty > 0)$. The opposite inequality is trivial.

If S_n is transient, it seems most likely that $\delta_* = \delta_c$, and that (1.7) holds for all δ. When S_n is recurrent, (1.7) fails for small δ, at least under the hypotheses of Theorem 2, which we now prove. The argument is almost identical to the one for Lemma (4.3) of [7]. Because $r_\cdot(t)$ is monotone decreasing for each t, it suffices to demonstrate that $r_0(\infty) = 0$. A little algebra yields

$$1 + \tfrac{1}{2}a - \sqrt{1+a} \geq a^2/12, \quad a \in [0, 1],$$

so from (3.3), for $\delta = 0$,

$$\left(G - \frac{1}{2}\right)f(\eta) \leq -\frac{1}{12}\sqrt{|\eta|}\left(\sum_x \left[\frac{\eta(x)}{|\eta|}\right]^2\right). \tag{3.4}$$

To estimate the right side, Holley and Liggett use a clever truncation. Let $\mu = \sum yp(y)$, $B_t = \mu t + \{x \in Z^d : |x| \le \varphi(t)\}$, φ to be chosen later. By the Schwarz inequality,

$$\sum_x \left[\frac{\eta(x)}{|\eta|}\right]^2 \ge |B_t|^{-1}\left[1 - \sum_{x \notin B_t} \frac{\eta(x)}{|\eta|}\right]^2 \tag{3.5}$$
$$\ge |B_t|^{-1}\left[1 - 2\sqrt{\sum_{x \notin B_t} \frac{\eta(x)}{|\eta|}}\right].$$

From (3.2), (3.4), (3.5) and Schwarz again one gets

$$r'(t) \le -(12|B_t|)^{-1} r(t) + (6|B_t|)^{-1}\left(\sum_{x \notin B_t} e^{-t} m_t(x)\right)^{1/2}. \tag{3.6}$$

Under the hypotheses of the theorem, using (1.3) and Chebyshev's inequality, there is a $C_0 < \infty$ such that the sum on the right is at most $C_0 t \varphi^{-d}(t)$, $(d = 1, 2)$. Combining (3.6) with this bound one arrives at the first order differential inequality:

$$r'(t) \le -\frac{r(t)}{12|B_t|} + \frac{C_0}{6} \frac{t^{1/2}}{|B_t|\varphi^{d/2}(t)}. \tag{3.7}$$

A judicious choice of φ is

$$\varphi(t) = (1 \vee t \ln t)^{1/d}, \quad |B_t| \sim 2dt \ln t.$$

Solving (3.7) for this φ, one finds that for some $C_1 < \infty$, $c > 0$,

$$r(t) \le C_1(1 \vee \ln t)^{-c} \to 0 \quad \text{as} \quad t \to \infty.$$

Thus $r(\infty) = 0$, and the proof is finished.

4. The proof of Theorem 3. The strategy is due to Liggett and Spitzer [8]; the main ingredients are duality, invariance, and mixing.

I. *Duality.* The idea here is "time/effect reversal;" there are many approaches (e.g. in [2], [3], [4], [5], [6], [7], [8], [9]). We will need two duality equations. First, for any $\eta_1, \eta_2 \in [0, \infty)^{Z^d}$, $t \ge 0$,

$$\sum_x \eta_1(x)\eta_t^{\eta_2}(x) =_d \sum_x \eta_2(x)\eta_t^{\eta_1}(x). \tag{4.1}$$

(Recall (1.16). $=_d$ means equal in distribution.) This sort of equation was introduced by Spitzer [9]. Second, for each $t \ge 0$ in the graphical representation,

$$(\eta_t^1(x))_{x \in Z^d} =_d (|\eta_t^x|)_{x \in Z^d}. \tag{4.2}$$

To prove (4.1) and (4.2) we use the approach of Harris [6]. He observed that on the percolation substructure $\mathscr{P}_t = \mathscr{P}$ restricted to $Z^d \times [0, t]$ one can consider a dual substructure $\hat{\mathscr{P}}_t$ obtained by reversing time and reversing the directions of all arrows. The contact substructure is *self-dual*, meaning that $\hat{\mathscr{P}}_t$ equals $\mathscr{P}_t$ in law. Thus

$\hat{\eta}_s^y(x)$ = the number of (arrow reversed) paths
down from (y, t) to $(x, t - s)$

can be used to define dual processes $(\hat{\eta}_s^{\eta})_{0 \le s \le t}$. In this coupling we clearly have

$$\eta_t^x(y) = \hat{\eta}_t^y(x). \tag{4.3}$$

See [2] or [6] for more details. Using (4.3), it is easy to prove (4.1):

$$\sum_x \eta_1(x)\eta_t^{\eta_2}(x) =_d \sum_x \eta_1(x)\hat{\eta}_t^{\eta_2}(x) = \sum_x \sum_y \eta_1(x)\eta_2(y)\hat{\eta}_t^y(x)$$
$$= \sum_x \sum_y \eta_1(x)\,\eta_2(y)\eta_t^x(y) = \sum_y \eta_2(y)\eta_t^{\eta_1}(y).$$

We can also consider the joint distributions in the coupling:

$$(\eta_t^1(x)) =_d (\hat{\eta}_t^1(x)) = (\sum_y \hat{\eta}_t^y(x)) = (\sum_y \eta_t^x(y)) = (|\eta_t^x|),$$

proving (4.2). It is now a simple matter to show convergence of $\bar{\eta}_t^1$. Write

$$M_t(x) = e^{-(1-\delta)t} |\eta_t^x|,$$

the process η_t^x all defined on $\mathscr{P}$. Then by martingale convergence there is a limit field:

$$P(\lim_{t\to\infty}(M_t(x))_{x\in Z^d} = (M_\infty(x))_{x\in Z^d}) = 1.$$

Put $\bar{\eta}_\infty = (M_\infty(x))$, and let ν be the law of $\bar{\eta}_\infty$. By (4.2), $\bar{\eta}_t^1 =_d (M_t(x))$ for each t. Hence $\bar{\eta}_t^1 \to \bar{\eta}_\infty$ as $t \to \infty$. Under the hypotheses of Theorem 2, $M_\infty(x) \equiv_d M_\infty = 0$, so $\bar{\eta}_\infty = \mathbf{0}$.

II. *Invariance.* Of course for each fixed t

$$P(\bar{\eta}_{s+1}^1 \in \cdot) \to \bar{\eta}_\infty \quad \text{as} \quad s \to \infty.$$

By the Markov property this probability equals $P(\bar{\eta}_t^{\nu_s} \in \cdot)$, where $\nu_s = P(\bar{\eta}_s^1 \in \cdot)$ and $\bar{\eta}_t^{\nu_s}$ is constructed in the obvious way. The same sort of truncation used to prove Lemma (3.7) of [7] shows that for each $t \geq 0$,

$$P(\bar{\eta}_t^{\nu_s} \in \cdot) \to P(\bar{\eta}_t^{\nu} \in \cdot) \quad \text{as} \quad s \to \infty.$$

The finite dimensional distributions determine the limit, so (1.17) follows.

III. *Mixing.* Now assume that the hypotheses of Theorem 1 hold, so that

$$E[\bar{\eta}_\infty(x)] \equiv E[M_\infty] = 1,$$

and

$$E[M_\infty^2] \leq \lim_{t\to\infty} E[M_t^2] = v(\infty) = \frac{2\gamma}{2\gamma - (1+\delta)} < \infty.$$

Let $\bar{\eta}_t^x = e^{-(1-\delta)t}\eta_t^x$. First we compute

$$v_x^z(t) = E[\sum_y \bar{\eta}_t^0(y)\bar{\eta}_t^z(x+y)]$$

and

$$v^z(t) = \sum_x v_x^z(t) = E[|\bar{\eta}_t^0| \cdot |\bar{\eta}_t^z|].$$

To do so, one considers the bivariate process (η_t^0, η_t^z) on $\mathscr{P}$ and determines the forward equation corresponding to $f(\eta_1, \eta_2) = \sum_y \eta_1(y)\eta_2(x+y)$. Mimicking the main computation of Section 2 we find that $v^z_\cdot(\cdot)$ satisfies (2.16), now with boundary conditions $v_x^z(0) = 1_{\{x=z\}}$. It follows that

$$v_x^z(t) = E_z\left[\exp\left\{(1+\delta)\int_0^t 1_{\{\tilde{X}_s=0\}}\,ds\right\}, \tilde{X}_t = x\right], \tag{4.4}$$

and hence

$$v_z(t) = E_z\left[\exp\left\{(1+\delta)\int_0^t 1_{\{\tilde{X}_s=0\}}\,ds\right\}\right].$$

As $|z| \to \infty$,

$$v^z(t) \leq v^z(\infty) = 1 + \pi(z)(v(\infty) - 1),$$

so by (4.2) and Fatou,

$$\limsup_{|z|\to\infty} E[\bar{\eta}_\infty(0)\bar{\eta}_\infty(z)] \leq 1.$$

Since $\bar\eta_\infty$ has density one we get the mixing property,

$$\lim_{|z|\to\infty} E[\bar\eta_\infty(0)\bar\eta_\infty(z)] = 1. \tag{4.5}$$

Now for fixed x, use duality and translation invariance to compute:

$$\begin{aligned} E[\bar\eta_t^\nu(0)\bar\eta_t^\nu(x)] &= E[(\textstyle\sum_y \bar\eta_\infty(y)\bar\eta_t^y(0))(\sum_z \bar\eta_\infty(z+y)\bar\eta_t^{z+y}(x))] \\ &= \textstyle\sum_{y,z} E[\bar\eta_\infty(y)\bar\eta_\infty(z+y)]E[\bar\eta_t^0(y)\bar\eta_t^x(z+y)] \\ &= \textstyle\sum_z E[\bar\eta_\infty(0)\bar\eta_\infty(z)]\sum_y E[\bar\eta_t^0(y)\bar\eta_t^x(z+y)] \\ &= \textstyle\sum_z E[\bar\eta_\infty(0)\bar\eta_\infty(z)]v_z^x(t). \end{aligned} \tag{4.6}$$

From (4.4) we see that for any $N < \infty$,

$$\lim_{t\to\infty} \textstyle\sum_{|z|>N} v_z^x(t) = v^x(\infty). \tag{4.7}$$

Combine (4.5), (4.6) and (4.7) to get

$$\lim_{t\to\infty} E[\bar\eta_t^\nu(0)\bar\eta_t^\nu(x)] = v^x(\infty).$$

But by the invariance of ν,

$$E[\bar\eta_t^\nu(0)\bar\eta_t^\nu(x)] = E[\bar\eta_\infty(0)\bar\eta_\infty(x)] \quad \text{for all} \quad t.$$

A little algebra gives the desired covariances. In particular, $\mathrm{Var}(\eta_\infty(0)) = \mathrm{Var}(M_\infty)$, so we have completed the proofs of Theorems 1 and 3.

APPENDIX

We sketch the proof that $r.(t)$ is nonincreasing. Writing $f(\eta) = |\eta|^{1/2}$, for $\delta < \bar\delta$ we have

$$\begin{aligned} r_{\bar\delta}(t) - r_\delta(t) &= e^{-(1/2)(1-\bar\delta)s}T_s^{\bar\delta}e^{-(1/2)(1-\delta)(t-s)}T_{t-s}^{\delta}f(0)\,|_{s=0}^{t} \\ &= \int_0^t \frac{d}{ds}[e^{-(1/2)(1-\bar\delta)s}T_s^{\bar\delta}e^{-(1/2)(1-\delta)(t-s)}T_{t-s}^{\delta}f(0)]\,ds. \end{aligned}$$

One can show, as in [8], that $T_tGf(\eta) = GT_tf(\eta)$, so the right side above can be rewritten as

$$\int_0^t e^{-(1/2)((1-\bar\delta)s+(1-\delta)(t-s))}T_{t-s}^{\delta}\left[G^{\bar\delta} - G^{\delta} + \frac{1}{2}(\bar\delta - \delta)\right]T_s^{\bar\delta}f(0)\,ds. \tag{$*$}$$

Let ω_t be the path process with parameter $\bar\delta$ starting from $\mathbf{1}$. By duality, if $W = \sum_z \omega_s(z)\eta(z)$, $T_s^{\bar\delta}f(\eta) = E[W^{1/2}]$. Hence

$$\begin{aligned} &\left[G^{\bar\delta} - G^{\delta} + \frac{1}{2}(\bar\delta - \delta)\right]T_s^{\bar\delta}f(\eta) \\ &\qquad = -(\bar\delta - \delta)E\left[\sqrt{W}\left\{\textstyle\sum_x \left(1 - \frac{1}{2}\frac{\omega_s(x)\eta(x)}{W} - \sqrt{1 - \frac{\omega_s(x)\eta(x)}{W}}\right)\right\}\right] \end{aligned}$$

(interpret the sum as 0 if $W = 0$). All the summands are nonnegative. It follows that the integral $(*)$ is nonpositive as desired.

REMARK. Tom Liggett has a more elegant approach. He notes that the normalized path process has generator

$$\Omega f(\eta) = Gf(\eta) - (1-\delta)\textstyle\sum_x \eta(x)\frac{\partial f}{\partial\eta(x)}(\eta),$$

that the normalized semigroup $\bar{T}_t$ maps

$$\mathscr{C} = \left\{ f : \frac{\partial^2 f}{\partial\eta(x)\partial\eta(y)} \le 0, \forall x, y \right\}$$

into itself, and that $(\Omega^{\dot{\bar{\delta}}} - \Omega^{\delta}) f \le 0$ for all $f \in \mathscr{C}$.

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