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EXPONENTIAL MOMENTS FOR HITTING TIMES OF UNIFORMLY ERGODIC MARKOV PROCESSES[1]

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Let μ be an invariant measure for a Markov process which is assumed μ-uniformly ergodic in the following sense: the corresponding semigroup of operators on $L^2(d\mu)$, say $\{P_t;\ t \geq 0\}$, is such that the time average $(1/T)\int_0^T P_t\,dt$ converges to a rank one projection in the uniform norm of operators. We prove that hitting times of sets having non zero μ-measure possess moment generating functions.

1. Introduction. In a very interesting paper [19], J. B. Walsh proposed a new mathematical model for the problem of neural response. He considered the following equation:

$$\frac{\partial V(t, x)}{\partial t} = \left(\frac{\partial^2}{\partial x^2} - 1\right) V(t, x) + W(t, x) \tag{1.1}$$

for the electrical potential $V(t, x)$ at time t and position x of a neuron which is idealized as a line segment of length $L > 0$. Here $W(t, x)$ denotes a Gaussian white noise in both variables $t \geq 0$ and $x \in [0, L]$ which appears as a limit of Poisson source noises, and the operator $\partial^2/\partial x^2 - 1$ is considered with Neuman boundary condition as a self-adjoint operator, say A, on the Hilbert space $L^2([0, L], dx)$. A generates a strongly continuous semigroup $\{e^{tA};\ t \geq 0\}$ of self-adjoint operators on $L^2([0, L], dx)$. These operators are more than contractive. In fact they satisfy:

$$\| e^{tA} \| \leq e^{-mt} \tag{1.2}$$

for some $m > 0$ independent of $t > 0$, where $\|\cdot\|$ stands for the norm of operators on $L^2([0, L], dx)$. Moreover they possess kernels which we denote by $e^{tA}(x, y)$ for x and y in $[0, L]$ and $t > 0$. If we add the initial condition $V(0, x) = v_0(x)$ for $x \in [0, L]$, equation (1.1) can be solved and the unique solution is given, at least formally, by:

$$V(t, x) = \int_0^L e^{tA}(x, y) v_0(y)\, dy + \int_0^t \int_0^L e^{(t-u)A}(x, y) W(u, y)\, du\, dy \tag{1.3}$$

where the first term can be rewritten as $[e^{tA}v_0](x)$ and is deterministic (it is in fact the solution of (1.1) in the absence of the source noise W) and the second one is a stochastic integral given by the "variation of constants method". The latter can be given a rigorous meaning and the process $V = \{V(t, x);\ t \geq 0,\ x \in [0, L]\}$ appears as a two parameter Gaussian process with the mean $[e^{tA}v_0](x)$ and covariance $\mathrm{Cov}\{V(s, x)V(t, y)\}$ given by the integral kernel of the self-adjoint operator $(2A)^{-1}[e^{(t+s)A} - e^{|t-s|A}]$ computed at $(x, y) \in [0, L] \times [0, L]$.

For each $t \geq 0$, the function $X_t: [0, L] \ni x \mapsto V(t, x)$ is easily seen to be almost surely continuous and a little extra work leads to a stochastic process $\mathbf{X} = \{X_t;\ t \geq 0\}$ with state

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space the Banach space $\mathcal{X} = C([0, L])$ of continuous functions on $[0, L]$ equipped with its Borel σ-field $\mathscr{X}$, and a family $\{\mathbb{P}_{\mathbf{x}}; \mathbf{x} \in \mathcal{X}\}$ of the probability measures on the space of continuous functions from $[0, \infty)$ to $\mathcal{X}$ such that under $\mathbb{P}_x$, the process $\{X_t(x); t \geq 0, x \in [0, L]\}$ has the same law as the solution of (1.1) with initial condition $v_0 = \mathbf{x} \in \mathcal{X}$ (note that we implicitly use the fact that the operators e^{tA} map $\mathcal{X} = C([0, L])$ into itself). The continuous strong Markov process so obtained is a prototype of an Ornstein-Uhlenbeck process in infinite dimensions. Indeed the above construction mimics the finite dimensional one (see [13. Section 8]).

One of the problems addressed by Walsh concerns the firing time of the neuron, namely the first time, say T, that the electrical potential exceeds a given level. T can be interpreted as the first hitting time of some open set in the state space and the questions to be answered concern the almost sure finiteness of T and the existence of moments of T. Walsh proved that for each $n \geq 0$:

$$\mathbb{E}_0\{T^n\} < +\infty, \tag{1.4}$$

where $\mathbb{E}_{\mathbf{x}}$ stands for the expectation with respect to the conditional probability $\mathbb{P}_{\mathbf{x}}$ given that the initial condition is $\mathbf{x} \in \mathcal{X}$.

The aim of this note is to prove an abstract theorem for hitting times of uniformly ergodic stationary Markov processes (see Theorem 1 below) which will imply that the firing time of the neuron in Walsh's model actually possesses exponential moments rather than just power moments as in (1.4). In other words, our result implies the existence of the moment generating function of hitting times (and in particular of the above firing time). It is stated and proved in full generality in Section 2.

Infinite dimensional Ornstein-Uhlenbeck processes have been extensively studied prior to Walsh's work. They appeared a) as infinite dimensional symmetric diffusion processes (see for example [9], [15] or [18]), b) as limiting cases in some problems of infinite systems of particles [6], c) as basic objects to be perturbed in constructive quantum field theory (see for example [5]), or d) in problems on stochastic partial differential equations (see for example [12]), and some of their ergodic and sample path properties are well understood (see [6] and [2]). Nevertheless the existence of exponential moments for hitting times was not investigated despite the existence of some one dimensional results [1].

We will now push further the analysis of Walsh's model in order to shed some light on our intuition and our proof. It is easily seen that the centered Gaussian probability measure μ on $\mathcal{X}$, the covariance of which is given by the integral kernel of the inverse of the operator $-2A$, is invariant for the process. Consequently, the semigroup $\{P_t; t \geq 0\}$ formally defined by:

$$[P_t f](\mathbf{x}) = \mathbb{E}_{\mathbf{x}}\{f(X_t)\} \tag{1.5}$$

for non negative measurable functions f on $\mathcal{X}$, is actually a strongly continuous contraction semigroup of self-adjoint operators on $L^2(\mathcal{X}, d\mu)$, the spectral properties of which are well known thanks to results obtained in the context of quantum field theory (see [17] for an historical perspective). The Hilbert space $L^2(\mathcal{X}, d\mu)$ can be decomposed into a direct sum of orthogonal subspaces:

$$L^2(\mathcal{X}, d\mu) = \oplus_{n=0}^{\infty} H_n \tag{1.6}$$

which reduce the operators P_t (i.e., the P_t's leave invariant the H_n's), so that the spectra of the P_t's can be obtained by superposition of the spectra of their restrictions to the H_n's. The decomposition (1.6) is known to probabilists as the Wiener chaos and to the physicists as the Fock space representation (see [14. Chap. VII] and [17 Chap. I] respectively). Let us assume for a short while that $t > 0$ is fixed. H_0 is the subspace of all constant functions on $\mathcal{X}$. They are left invariant by P_t so that the contribution to the spectrum reduces to the singleton $\{1\}$. H_1 is the closed subspace of $L^2(\mathcal{X}, d\mu)$ spanned by the functions $f: \mathcal{X} \ni \mathbf{x} \to \mathbf{x}'(\mathbf{x})$ where $\mathbf{x}'$ runs through the dual space $\mathcal{X}'$ of $\mathcal{X}$. Moreover, for such a function f, we have:

$$[P_t f](\mathbf{x}) = \mathbb{E}_{\mathbf{x}}\{\mathbf{x}'(X_t)\} = \mathbf{x}'(e^{tA}\mathbf{x})$$

so that P_t coincides on H_1 with e^{tA} if we consider the action of e^{tA} on the space $\mathcal{X}'$ of measures x' on $[0, L]$ given by:

$$[e^{tA}x'](y)\,dy = \left[\int e^{tA}(x, y)\mathbf{x}'(dx)\right] dy.$$

It then follows from (1.2) that the contribution to the spectrum is contained in the interval $[0, e^{-mt}]$. More generally, H_n is the closed subspace of $L^2(\mathcal{X}, d\mu)$ spanned by the symmetric tensors $\mathbf{x}_1'\otimes_s \cdots \otimes_s \mathbf{x}_n'$ where $\mathbf{x}_1', \cdots, \mathbf{x}_n'$ run through $\mathcal{X}'$, and, with the above convention we have:

$$P_t(\mathbf{x}_1'\otimes_s \cdots \otimes_s \mathbf{x}_n') = (e^{tA}\mathbf{x}_1')\otimes_s \cdots \otimes_s(e^{tA}\mathbf{x}_n')$$

which implies (using again (1.2)) that the spectral contribution of the restriction of P_t to H_n is contained in the interval $[0, e^{-nmt}]$. Thus, the whole spectrum of the operator P_t is of the form $S_t \cup \{1\}$, where the set S_t is contained in the interval $[0, e^{-mt}]$. The isolated eigenvalue 1 is expected to be sensitive to perturbations. In fact, our proof shows that killing the process when it hits a set $U \in \mathscr{X}$ such that $\mu(U) > 0$ is enough to push this eigenvalue inside the interval $[0, 1)$ so that the semigroup of the killed process, say $\{P_t^U, t \geq 0\}$, satisfies:

$$\| P_t^U \| \leq e^{-m\mu(U)t}$$

for all $t > 0$. Consequently, its Laplace transform exists for some strictly negative reals $-\alpha$, with $\alpha > 0$, and we have:

$$\int_0^{+\infty} e^{\alpha t}[P_t^U 1](\mathbf{x})\,dt = \int_0^{+\infty} e^{\alpha t}\mathbb{P}_{\mathbf{x}}\{t < T_U\}\,dt = \alpha^{-1}(\mathbb{E}_{\mathbf{x}}\{e^{\alpha T_U}\} - 1) \tag{1.7}$$

where T_U denotes the first hitting time of the set U, namely:

$$T_U = \inf\{t > 0; X_t \in U\}. \tag{1.8}$$

(1.7) proves the desired result.

We will show that the intuition behind the above argument is quite general and applies as well to nonsymmetric non-Gaussian Markov processes, provided the resolvent of the killed process exists for some strictly negative reals. In general this resolvent is not given by the Laplace transform so we will not be able to use (1.7) directly and a more sophisticated argument will be needed.

We would like to emphasize that these facts from operator theory will not be needed in the proof below even though they were a guide line for our intuition.

Note that the existence of exponential moments for hitting times has been known and used for a long time in the study of recurrent diffusion processes in $\mathbb{R}^n$ (see for example [8] and [11]). The proofs rely heavily on the local compactness of the state space and on the strong Feller property of these processes. Unfortunately both properties are restrictive and are not satisfied in general, particularly in the case of infinite dimensional diffusion processes.

To illustrate this last point we would like to recall that we can have:

$$\mathbb{E}_{\mathbf{x}}\{e^{\alpha T_u}\} < +\infty \tag{1.9}$$

for some $\alpha > 0$ uniformly in $\mathbf{x} \in \mathcal{X}$ provided $U \in \mathscr{X}$ is such that $\mu(U) > 0$ and:

$$\lim_{t\to\infty} P_t(\mathbf{x}, U) = \mu(U) \tag{1.10}$$

uniformly in $\mathbf{x} \in \mathcal{X}$. Indeed this last assumption implies that:

$$\mathbb{P}_{\mathbf{x}}\{T_U \geq nt_0\} \leq [1 - \tfrac{1}{2}\mu(U)]^n$$

for some t_0 fixed, all integers n, uniformly in $\mathbf{x} \in \mathcal{X}$, because (1.10) allows us to pick t_0 such that:

$$\mathbb{P}_{\mathbf{x}}\{T_U > t_0\} \leq \mathbb{P}_{\mathbf{x}}\{X_{t_0} \notin U\} \leq 1 - ½\mu(U),$$

uniformly in $\mathbf{x} \in \mathcal{X}$.

Nevertheless the uniformity in the limit (1.10) is usually checked using compactness arguments that are not available in the infinite dimensional setting presented above nor in some cases of non strongly Feller processes (see Remark 2 below). Moreover an elementary computation shows that it is not even true for the one dimensional Ornstein-Uhlenbeck process.

2. Uniformly ergodic Markov processes. Throughout this section $(\mathcal{X}, \mathscr{X})$ will be a fixed separable measurable space.

2.1 *The continuous time case.* Let $P = \{P_t(x, A); t \geq 0, x \in \mathcal{X}, A \in \mathscr{X}\}$ be a transition probability kernel on $(\mathcal{X}, \mathscr{X})$. Note that we implicitly assume that $P_0(x, \cdot)$ is the unit mass at $x \in \mathcal{X}$. A probability measure μ on $(\mathcal{X}, \mathscr{X})$ is said to be invariant for P if:

$$\int_{\mathcal{X}} P_t(x, A)\mu(dx) = \mu(A), \quad t \geq 0, A \in \mathscr{X}.$$

When this is the case, the transition probability kernel defines a semigroup, say $\{P_t; t \geq 0\}$, of bounded operators on the Hilbert space $L^2(\mathcal{X}, d\mu)$ of μ-equivalence classes of square integrable functions on $\mathcal{X}$. Our main assumption concerns the ergodic properties of this semigroup and the following notation will simplify its statement. For each $T > 0$ we set:

$$Q_T = \frac{1}{T}\int_0^T P_t \, dt.$$

DEFINITION. The transition probability kernel $P = \{P_t(x, A); t \geq 0, x \in \mathcal{X}, A \in \mathscr{X}\}$ is said to be μ-uniformly ergodic if μ is an invariant probability measure for P and if there exist a positive function $c(T)$ such that $\lim_{T\to\infty} c(T) = 0$ and:

$$\left\| Q_T f - \int f \, d\mu \right\|_2 \leq c(T) \| f \|_2$$

for all $f \in L^2(X, d\mu)$ where $\| \cdot \|_2$ stands for the norm of the Hilbert space $L^2(\mathcal{X}, d\mu)$.

Note that, when the semigroup defined on $L^2(\mathcal{X}, d\mu)$ by the transition kernel P is strongly continuous, namely

$$\lim_{t\to 0} P_t(x, A) = 1_A(x)$$

in μ-probability for all $A \in \mathcal{X}$, our assumption of uniform ergodicity has been extensively studied in the context of contractive semigroups of operators on general Banach spaces (see for example [3. Chap. VII] and [10]).

In order to make the connection with the discussion of the problem in the introduction we note that it can be restated in the following equivalent form (which is implicit in [10]). 0 is a simple isolated eigenvalue in the spectrum Σ of the infinitesimal generator A of the semigroup $\{P_t; t \geq 0\}$. Our result is

THEOREM 1. *Let $\{X_t;\ t \geq 0\}$ be a measurable stationary Markov process in the standard measurable space $(\mathcal{X}, \mathscr{X})$ and let us assume the existence of a transition probability $P_t(x, dy)$ which is μ-uniformly ergodic for the invariant probability measure μ. Then, there exists $\alpha > 0$ such that the function*:

$$\mathcal{X} \ni x \mapsto \mathbb{E}_{\mathbf{x}}\{e^{\alpha T_U}\} \in [0, \infty]$$

is in $L^2(\mathcal{X}, d\mu)$ *(and hence finite* μ − a.e.) *for all* $U \in \mathscr{X}$ *such that* $\mu(U) > 0$ *and for which the hitting time* T_U *defined by* (1.8) *is a random variable.*

Here $\mathbb{E}_x$ denotes the conditional expectation knowing that the paths start from x at time $t = 0$. Moreover, we would like to emphasize that the proof shows that the measurability of T_U can be bypassed (provided the conclusion is appropriately restated).

PROOF. Enlarging the probability space if necessary, we can assume without any loss of generality the existence of a sequence $\{\xi_n : n \geq 1\}$ of independent identically distributed random variables with uniform distribution on $[0, T]$ (where $T > 0$ will be chosen later on) which is independent of the process $\mathbf{X} = \{X_t;\, t \geq 0\}$. For each integer $n \geq 1$ we set:

$$\tau_n = \xi_1 + \cdots \xi_n.$$

Then $\{X_{\tau_n};\, n \geq 1\}$ is a Markov chain with transition kernel Q_T defined by (2.1) and for which μ is still an invariant probability measure. We first remark that $T_U \leq \tau_N \leq TN$ provided we set:

$$N = \inf\{n \geq 1;\, X_{\tau_n} \in U\}$$

and consequently our proof reduces to proving that

$$\mathcal{X} \ni x \mapsto \mathbb{E}_x\{e^{\alpha N}\}$$

is square integrable for $\alpha > 0$ small enough. At this point we note that N is measurable (because the process X is assumed measurable) whether T_U is measurable or not. So we are left with a problem depending only on the chain we embedded in the continuous time process we started with. Before going further we notice the following crucial property of its transition kernel Q_T. If $f \in L^2(\mathcal{X}, d\mu)$ we have:

$$\| Q_T f \|_2^2 = \left\| \int f\, d\mu + \left(Q_T f - \int f\, d\mu \right) \right\|_2^2$$

$$= \left(\int f\, d\mu \right)^2 + \left\| Q_T f - \int f\, d\mu \right\|_2^2 + 2\left(\int f\, d\mu \right) \int \left[Q_T f - \left(\int f\, d\mu \right) \right] d\mu$$

$$= \left(\int f\, d\mu \right)^2 + \left\| Q_T f - \int f\, d\mu \right\|_2^2$$

(by the invariance of μ)

$$\geq \left(\int f\, d\mu \right)^2 + c(T)^2 \| f \|_2^2.$$

(by our assumption (2.2)).

In particular, if f is of the form $1_U g$ for some $U \in \mathscr{X}$ and $g \in L^2(\mathcal{X}, d\mu)$, we obtain by using Schwarz inequality:

(2.3) $$\| Q_T(1_{U^c} g) \|_2 \leq [\mu(U)^2 + c(T)^2]^{1/2} \| g \|_2.$$

Now, if $\alpha > 0$ and $U \in \mathscr{X}$ is such that $\mu(U) > 0$, for every $x \in \mathcal{X}$ we have:

(2.4) $$\mathbb{E}_x\{e^{\alpha N}\} = \textstyle\sum_{n=1}^{\infty} \mathbb{P}_x\{X_{\tau_1} \notin U, \cdots, X_{\tau_{n-1}} \notin U, X_{\tau_n} \in U\}.$$

But, an easy computation shows that:

$$\mathbb{P}_x\{X_{\tau_1} \notin U, \cdots, X_{\tau_{n-1}} \notin U, X_{\tau_n} \in U\} = [Q_T 1_{U^c}]^{n-1}(Q_T 1_U)(x)$$

so that:

$$\| \mathbb{P}.\{X_{\tau_1} \notin U, \cdots, X_{\tau_{n-1}} \notin U, X_{\tau_n} \in U\} \|_2 \tag{2.5}$$
$$\leq [(1-\mu(U))^2 + c(T)^2]^{(n-1)/2}[\mu(U)^2 + c(T)^2]^{1/2}\mu(U)$$

by applying $n-1$ times (2.3) with 1_{U^c} and once with 1_U. Since $c(T)$ tends to 0 as T goes to ∞, we can fix T large enough so that the right hand side of (2.5) is bounded above by $(1-\sigma)^n$ for some $0 < \sigma < 1$ independent of n. We conclude by putting together (2.4) and (2.5) to get:

$$\| \mathbb{E}.\{e^{\alpha N}\} \|_2 \leq \sum_{n=1}^{\infty} e^{\alpha n}(1-\sigma)^n$$

and choosing $\alpha > 0$ small enough. □

We feel that it is worth completing the above result by the following facts.

REMARK 1. The above theorem may be regarded as unsatisfactory in the sense that it gives the finiteness of $\mathbb{E}_x\{e^{\alpha T_U}\}$ only for μ-almost every $x \in \mathcal{X}$. Nevertheless, in many cases it is possible to avoid this restriction. Let us assume for example that for some $t > 0$ and for all $x \in \mathcal{X}$ the measure $P_t(x, dy)$ is absolutely continuous with respect to μ with a square integrable density, say $p_t(x, y)$. Then, for every $x \in \mathcal{X}$ and for every U for which T_U is a stopping time:

$$\mathbb{E}_x\{e^{\alpha T_U}\} = \mathbb{E}_x\{e^{\alpha T_U}; T_U \leq t\} + \mathbb{E}_x\{e^{\alpha T_U}; T_U > t\}$$
$$\geq e^{\alpha t} + \int_{\mathcal{X}} p_t(x,y)\mathbb{E}_y\{e^{\alpha T_U}\}\, d\mu(y) < +\infty.$$

Note that the above applies to finite and some infinite dimensional Ornstein-Uhlenbeck (see 2.3 below) whereas the standard argument recalled in the introduction cannot be used in these cases.

REMARK 2. It might be interesting to know the exact range of $\alpha > 0$ for which the conclusion of the theorem holds. The proof above shows that α can be taken of the form $k\mu(U)$ for some $k > 0$ independent of U but we were not able to find the best possible constant k in general.

REMARK 3. We showed in Remark 1 that $\mathbb{E}_x\{e^{\alpha T_U}\}$ is very often finite for every $x \in \mathcal{X}$ rather than merely for μ-almost every $x \in \mathcal{X}$. In fact it is easy to see that this cannot be the case in general. Indeed we can always enlarge the state space and extend the transition kernel in such a way that the new states do not communicate with the original ones. For the x's which have been added this way we have now $\mathbb{E}_x\{e^{\alpha T_U}\} = \infty$ and the assumptions of the theorem are still satisfied since they involve only $L^2(\mathcal{X}, d\mu)$. Nevertheless the exceptional set of x's can be shown to be "very small." In fact it is possible to prove that it is μ-almost polar in the following sense: if U and α are as in the statement of Theorem 1, and if our process is a right-process (see [4. Sect. 9]), then we have:

$$\mathbb{P}_\mu\{T_{A_\infty} \leq T_U\} = 0$$

where $A_\infty = \{x \in \mathcal{X}; \mathbb{E}_x\{e^{\alpha T_U}\} = \infty\}$ and T_{A_∞} is its first hitting time.

2.2. *The discrete time case.* Let $R = \{R(x, A); x \in \mathcal{X}, A \in \mathscr{X}\}$ be a transition probability kernel on $(\mathcal{X}, \mathscr{X})$ and let $\mathbf{X} = \{X_n; n \geq 0\}$ be a stationary Markov chain with transition kernel R and invariant probability measure μ. We assume that this chain is *asymptotically uncorrelated* in the sense that:

$$\sup\left\{ \left\| \frac{1}{N} \sum_{n=1}^{N} R^n g - \int g\, d\mu \right\|_2 ; g \in L^2(\mathcal{X}, d\mu), \|g\|_2 \leq 1 \right\} < 1$$

for some $N \geq 1$. Note that this assumption is weaker than:

$$\sup\left\{ \|Rg\|_2;\, g \in L^2(\mathcal{X}, d\mu),\, \|g\|_2 = 1,\, \int g\, d\mu = 0 \right\} < 1$$

which has been extensively studied (see for example [16. Chap. VIII, Section 3]).

The proof of Theorem 1 above applies very simply to give the corresponding result in the discrete time case.

THEOREM 1′. *Let $X = \{X_n;\, n \geq 0\}$ be an asymptotically uncorrelated stationary Markov chain with transition kernel R and invariant probability distribution μ. Then for every $U \in \mathscr{X}$ such that $\mu(U) > 0$, there exist real numbers $\alpha > 1$ such that the function:*

$$\mathcal{X} \ni x \mapsto \mathbb{E}_x\{\alpha^N\} \in [0, \infty]$$

is in $L^2(\mathcal{X}, d\mu)$. Here N denotes the first hitting time of the set U, namely:

$$N = \inf\{n > 0;\, X_n \in U\}.$$

REMARK 4. The above result implies that for μ-almost every $x \in \mathcal{X}$, the exponential moment $\mathbb{E}_x\{\alpha^N\}$ is finite. Moreover in the same way as in the continuous case (recall Remark 1) we can get rid of the μ-almost everywhere restriction whenever for some $n > 0$ and all $x \in \mathcal{X}$ the measure $R^n(x, dx')$ has a square integrable density with respect to μ.

Finally, as we pointed out in Remark 2, α can be shown to depend only on the number $\mu(U)$ and not on the set U.

2.3. *Back to the neuron problem.* We already proved in the introduction that the assumptions and the conclusion of Theorem 1 were satisfied in Walsh's model for neural response. We note that the conditions of Remark 1 are also satisfied so that, if U and α are as in Theorem 1, $\mathbb{E}_x\{e^{\alpha T_U}\}$ is finite for all $x \in \mathcal{X}$. Indeed μ is a mean zero Gaussian measure on the function space $C[0, L]$ and $P_t(x, \cdot)$ is the mean zero Gaussian measure $P_t(0, \cdot)$ translated by $e^{tA}x$, so that knowing the covariances of these mean zero Gaussian measures we conclude using Feldman-Jacek's Theorem (see for example [14. Chapter VIII]) which gives explicit formulae for the densities. We omit the details because they have already been argued in [12].

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