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MARKOV STRATEGIES FOR OPTIMAL CONTROL PROBLEMS INDEXED BY A PARTIALLY ORDERED SET

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We define the notion of a Markov strategy for the general optimal control problem where the index set is partially ordered. We prove that the supremum over all strategies is always attained by a Markov strategy if and only if the structure of the probability space is that of a Markov random field.

1. Introduction. Several papers have appeared recently (Krengel and Sucheston, 1981, Mandelbaum and Vanderbei, 1981, Mazziotto and Szpirglas, 1981, Washburn and Willisky, 1981) which generalize the optimal stopping problem to the case where the index set S is partially ordered. In the paper by Mandelbaum and Vanderbei, it was shown that if the pay-off is a function of the state of a multi-time parameter Markov chain then the supremum over all policies coincides with the supremum over all "Markov policies." This means that the strategy and the stopping rule depend only on the current state of the process. In the case when $S = N = \{0, 1, 2, \cdots\}$, the converse to the above statement has been proved recently by Irle (1981). That is, the supremum over all stopping times coincides with the supremum over all "Markov" stopping times if and only if the structure of the probability space is that of a Markov chain.

The aim of this paper is to extend Irle's result to the case of a partially ordered index set. In this case it turns out that in addition to the reward obtained when we stop, we need to consider also a running reward. That is, the correct setting is optimal control as opposed to optimal stopping (in the case where $S = N$, these notions coincide).

In Section 2, we investigate the general optimal control problem. We define the notion of a Markov strategy in Section 3. We then prove that the supremum over all strategies is always attained by a Markov strategy if and only if the structure of the probability space is that of a Markov random field. Finally we apply this to the case of a family of independent stochastic processes and we see that the supremum is attained by a strategy which at each time is only a function of the current state of each process if and only if each process is Markov.

2. The optimal control problem. Throughout this paper we assume that S is a countable partially ordered set such that: (i) there is a unique minimal element 0 in S, (ii) the set of direct successors $U(s)$ of each point $s \in S$ is finite, and (iii) each point $s \in S$ has only finitely many predecessors.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $\mathscr{F} = \{\mathscr{F}_s\}_{s \in S}$ be an increasing family of sub-σ-algebras of $\mathscr{A}$. For any sub-σ-algebra $\mathscr{G}$ of $\mathscr{A}$ and any random variable Z, the notation $Z \in \mathscr{G}$ means that Z is $\mathscr{G}$-measurable.

A measurable mapping ν from a measurable subset of Ω into S (with the σ-algebra of all subsets) is called a *random point*. For a random point ν in S, let $\mathscr{F}_\nu$ be the σ-algebra generated by functions of the form $\sum_{s \in S} Z_s 1_{\{\nu = s\}}$, $Z_s \in \mathscr{F}_s$. A random point ν is called a *stopping point* if $\{\nu = s\} \in \mathscr{F}_s$ for all $s \in S$. If ν is a stopping point, the above definition of $\mathscr{F}_\nu$ is equivalent to the usual definition: a set A is in $\mathscr{F}_\nu$ if and only if $A \cap \{\nu = s\} \in \mathscr{F}_s$ for all $s \in S$.

Let $\boldsymbol{\sigma} = \{\sigma_t\}_{t \in N}$ be an increasing sequence of random points in S (i.e., $\sigma_t \le \sigma_{t+1}$ for all $t \in N$). Put $\tau = \inf\{t : \sigma_{t+1} = \sigma_t\}$ with the convention that $\inf \varnothing = \infty$. We say that $\boldsymbol{\sigma}$ is a

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strategy starting at s if:

(a) $\sigma_0 = s$;

(b) $\sigma_{t+1} \in U(\sigma_t)$ for $t < \tau$ and $\sigma_t = \sigma_\tau$ for $t \geq \tau$;

(c) $\sigma_{t+1} \in \mathscr{F}_{\sigma_t}$.

Usually we will be interested in strategies starting at 0. For such strategies we omit the phrase "starting at 0". The following properties of $\boldsymbol{\sigma}$ follow from (a), (b), and (c):

(d) For each $t \in N$, σ_t is a stopping point;

(e) σ_τ is a stopping point (defined on $\{\tau < \infty\}$);

(f) τ is a stopping time (with values in $N \cup \{\infty\}$) relative to the increasing sequence of σ-algebras $\mathscr{F}^\sigma_t = \mathscr{F}_{\sigma_t}$, $t \in N$.

In the case where $S = N$, there is a one to one correspondence between strategies and stopping times: to each strategy corresponds the stopping time $\nu = \sigma_\tau$ on $\{\tau < \infty\}$, $\nu = \infty$ on $\{\tau = \infty\}$ and to each stopping time ν corresponds the strategy $\sigma_t = \nu \wedge t$.

Let $Z^1_{s,u}$, $u \in U(s)$, $s \in S$, and Z^2_s, $s \in S$, be random variables such that $Z^1_{s,u} \in \mathscr{F}_u$, $Z^2_s \in \mathscr{F}_s$. The random variable $Z^1_{s,u}$ is the *running reward* obtained in going from s to u and Z^2_s is the *final reward* obtained if we stop at s. Since our goal is not to solve the most general optimal control problem, we will make the simplifying assumption that the running and final rewards are uniformly bounded

$$\text{(2.1)} \qquad \sup_{s\in S, u\in U(s)} \text{ess sup}_\Omega |Z^1_{s,u}| < \infty, \quad \sup_{s\in S} \text{ess sup}_\Omega |Z^2_s| < \infty.$$

For each strategy $\boldsymbol{\sigma}$ let

$$\mathscr{Z}(\boldsymbol{\sigma}) = \sum_{t=0}^{\tau-1} \alpha^t Z^1_{\sigma_t,\sigma_{t+1}} + \alpha^\tau Z^2_{\sigma_\tau},$$

denote the discounted *payoff* obtained using $\boldsymbol{\sigma}$. Here α is a real number strictly between zero and one—the *discount factor*. We use the convention that $\alpha^\tau Z^2_{\sigma_\tau} = 0$ on $\{\tau = \infty\}$. The problem is to find a strategy $\boldsymbol{\sigma}^*$ which is *optimal* in the sense that

$$\text{(2.2)} \qquad E\mathscr{Z}(\boldsymbol{\sigma}^*) = \sup_{\boldsymbol{\sigma}\in\Sigma} E\mathscr{Z}(\boldsymbol{\sigma}).$$

The supremum is taken over the class Σ of all strategies starting at 0.

Let X_s denote the highest reward possible using strategies starting from s:

$$\text{(2.3)} \qquad X_s = \text{ess sup}_{\boldsymbol{\sigma}\in\Sigma_s} E^{\mathscr{F}_s}\mathscr{Z}(\boldsymbol{\sigma}).$$

Here Σ_s denotes the collection of all strategies starting at s. The process X_s is called *Snell's envelope*. We have

THEOREM 1. *Snell's envelope satisfies the stochastic dynamic programming equation*

$$\text{(2.4)} \qquad X_s = \max\{Z^2_s, \max_{u\in U(s)} E^{\mathscr{F}_s}(Z^1_{s,u} + \alpha X_u)\}.$$

Suppose we are using some strategy and we have arrived at the point s. If Z^2_s attains the maximum in (2.4) we stop. If not, we proceed to a point of $u \in U(s)$ and, of course, we pick one which attains the maximum. In this way we construct a strategy $\boldsymbol{\sigma} = \{\sigma_t\}_{t\in N}$, starting at 0, such that

$$\text{(2.5)} \qquad X_{\sigma_t} = E^{\mathscr{F}_{\sigma_t}}(Z^1_{\sigma_t,\sigma_{t+1}} + \alpha X_{\sigma_{t+1}}) \text{ on } \{\tau > t\},$$

$$\text{(2.6)} \qquad X_{\sigma_\tau} = Z^2_{\sigma_\tau}.$$

Strategies starting from 0 which satisfy (2.5) and (2.6) will be called *admissible*.

Let $\boldsymbol{\sigma}^*$ be an admissible strategy and put $\tau^* = \inf\{t : \sigma^*_{t+1} = \sigma^*_t\}$. By summing the stochastic difference equation (2.5) we get

LEMMA 1. (*Dynkin's formula*).

$$\text{(2.7)} \qquad EX_0 = E \sum_{t=0}^{\tau^*-1} \alpha^t Z^1_{\sigma^*_t,\sigma^*_{t+1}} + E\alpha^{\tau^*} X_{\sigma^*_{\tau^*}}.$$

In the case where $S = \mathbb{N}$, $Z^1_{s,u} \equiv 0$, and $\alpha = 1$, formula (2.5) is the definition of a *martingale up to time* $\nu = \sigma_\tau$ and (2.7) is the martingale optional sampling theorem (as is well known, the optional sampling theorem does not hold for all stopping times but requires some additional restrictions—we have avoided these difficulties by requiring that α be strictly less than one and assuming (2.1)). Using (2.6), (2.7), and the intuitive meaning of X_0, we are tempted to write

$$E\mathscr{Z}(\sigma^*) = E\sum_{t=0}^{\tau^*-1}\alpha^t Z^1_{\sigma^*_t,\sigma^*_{t+1}} + E\alpha^{\tau^*}X_{\sigma^*_{\tau^*}} = EX_0 = \sup_{\sigma\in\Sigma}E\mathscr{Z}(\sigma).$$

This is the idea behind the proof of

THEOREM 2. *Every admissible strategy is optimal.*

The following theorem justifies our calling X_s Snell's envelope

THEOREM 3. *The process X_s is the minimal process which satisfies*

$$X_s \ge Z^2_s \qquad s \in S \tag{2.8}$$

$$X_s \ge E^{\mathscr{F}_s}(Z^1_{s,u} + \alpha X_u) \qquad u \in U(s),\ s \in S. \tag{2.9}$$

That is, if Y_s also satisfies (2.8) *and* (2.9), *then*

$$X_s \le Y_s \text{ a.s. } P \text{ for all } s \in S.$$

Theorems 1, 2, and 3, and Lemma 1 are straightforward generalizations of results found in Neveu (1975) and Mandelbaum and Vanderbei (1981), so the proofs have been omitted (in fact, introducing the discount factor α makes the proofs easier and eliminates the need of any technical assumptions other than (2.1)). The next theorem shows that Snell's envelope is the increasing limit of the best that can be done using k-step look-ahead strategies. The proof is quite similar to the case $S = \mathbb{N}$ (see e.g. Neveu (1975) Section VI-2).

THEOREM 4. *Put $X^{(0)}_s = Z^2_s$ and define $X^{(k)}_s$ recursively by the formula*

$$X^{(k+1)}_s = \max\{Z^2_s, \max_{u\in U(s)}E^{\mathscr{F}_s}(Z^1_{s,u} + \alpha X^{(k)}_u)\}.$$

Then $X^{(k)}_s$ increases with k and

$$X_s = \lim_{k\to\infty}X^{(k)}_s.$$

PROOF. We start by noting that $X^{(1)}_s \ge X^{(0)}_s$ and proceeding inductively we see that if $X^{(k)}_s \ge X^{(k-1)}_s$, for all $s \in S$, then

$$\begin{aligned} X^{(k+1)}_s &= \max\{Z^2_s, \max_{u\in U(s)}E^{\mathscr{F}_s}(Z^1_{s,u} + \alpha X^{(k)}_u)\} \\ &\ge \max\{Z^2_s, \max_{u\in U(s)}E^{\mathscr{F}_s}(Z^1_{s,u} + \alpha X^{(k-1)}_u)\} \\ &= X^{(k)}_s. \end{aligned}$$

The limit $X^{(\infty)}_s = \lim_{k\to\infty}X^{(k)}_s$ therefore satisfies the inequalities

$$X^{(\infty)}_s \ge Z^2_s \qquad s \in S, \tag{2.10}$$

$$X^{(\infty)}_s \ge E^{\mathscr{F}_s}(Z^1_{s,u} + \alpha X^{(\infty)}_u) \qquad u \in U(s),\ s \in S. \tag{2.11}$$

Suppose that Y_s is any other process which satisfies (2.10) and (2.11). By (2.10), $Y_s \ge X^{(0)}_s$ and again proceeding inductively we see that if $Y_s \ge X^{(k)}_s$, for all $s \in S$, then (2.11) implies that

$$Y_s \ge E^{\mathscr{F}_s}(Z^1_{s,u} + \alpha Y_u) \ge E^{\mathscr{F}_s}(Z^1_{s,u} + \alpha X^{(k)}_u),$$

and so

$$Y_s \geq \max\{Z_s^2, \max_{u\in U(s)} E^{\mathscr{F}_s}(Z_{s,u}^1 + \alpha X_u^{(k)})\} = X_s^{(k+1)}.$$

Hence, by Theorem 3, $X_s^{(\infty)}$ is Snell's envelope.

3. Markov strategies and the Markov property. In practice, when an optimal control problem is solved either by hand or on a computer the σ-algebras $\mathscr{F}_s$ are finite. Unfortunately, however, they tend to grow very fast (usually exponentially fast) and so it would be nice to discard unnecessary information. In this section we investigate under what conditions this is possible.

Let $\mathscr{G}_s$, $s \in S$, be a family of σ-algebras such that $\mathscr{F}_s = \bigvee_{u\leq s} \mathscr{G}_u$. The σ-algebra $\mathscr{G}_s$ represents the knowledge available at the "present" at point s. Let $\mathscr{H}_s = \bigvee_{u\geq s} \mathscr{G}_s$ represent the "future." For a random point ν in S, let $\mathscr{G}_\nu$ be the σ-algebra generated by functions of the form $\sum Z_s 1_{\{\nu=s\}}$, $Z_s \in \mathscr{G}_s$ (if, for each s, $\mathscr{G}_s$ is the σ-algebra generated by a mapping Y_s of $(\Omega, \mathscr{A})$ into a state space $(E, \mathscr{B})$, then $\mathscr{G}_\nu = \sigma\{Y_\nu, \nu\}$). A strategy $\boldsymbol{\sigma}$ is *Markov* if

$$\sigma_{t+1} \in G_{\sigma_t}. \tag{3.1}$$

THEOREM 5. *The following are equivalent*:

(a) *For every* $Z_{s,u}^1 \in \mathscr{G}_s \vee \mathscr{G}_u$ *and* $Z_s^2 \in \mathscr{G}_s$ *satisfying* (2.1), *the supremum over all* $\boldsymbol{\sigma} \in \Sigma$ *of*

$$E\{\textstyle\sum_{t=0}^{t-1} \alpha^t Z_{\sigma_t,\sigma_{t+1}}^1 + \alpha^\tau Z_{\sigma_\tau}^2\}$$

is attained by a Markov strategy.

(b) *For every* $s \in S$, $\mathscr{F}_s$ *and* $\mathscr{H}_s$ *are conditionally independent given* $\mathscr{G}_s$.

We remarked in Section 2 that in the case $S = N$ there is a one-to-one correspondence between strategies and stopping times. Hence it makes sense to call a stopping time τ *Markov* if $\tau \wedge (t+1) \in \mathscr{G}_{\tau\wedge t}$. It is easy to check that this definition is equivalent to the one given in Irle (1981): for every $t \in N$ there exists a $\mathscr{G}_t$-measurable set G_t such that

$$\{\tau = t\} = \{\tau \geq t\} \cap G_t.$$

PROOF OF THEOREM 5. (b) $\Rightarrow$ (a). Let $X_s^{(k)}$ be the process defined in Theorem 4. Note that, since $Z_s^2 \in \mathscr{G}_s$, $X_s^{(0)} \in G_s$. Suppose that $X_s^{(k)} \in \mathscr{G}_s$. Then, since $Z_{s,u}^1 \in \mathscr{G}_s \vee \mathscr{G}_u$, the Markov property (b) gives

$$X_s^{(k+1)} = \max\{Z_s^2, \max_{u\in U(s)} E^{\mathscr{G}_s}(Z_{s,u}^1 + \alpha X_u^{(k)})\}.$$

Hence $X_s^{(k+1)} \in \mathscr{G}_s$. Since $X_s = \lim_{k\to\infty} X_s^{(k)}$ we see that $X_s \in \mathscr{G}_s$ for all s. Now according to (2.4) we can find an admissible strategy which is Markov, and so, by Theorem 2, the supremum over all strategies is attained by a Markov strategy.

(a) $\Rightarrow$ (b). Suppose there is a point $v \in S$ for which $\mathscr{F}_v$ and $\mathscr{H}_v$ are not conditionally independent given $\mathscr{G}_v$. Then there exists a point $w \in S$, $w > v$, and a $Y \in \mathscr{G}_w$ such that $0 \leq Y \leq 1$ and $P\{E^{\mathscr{F}_v} Y \neq E^{\mathscr{G}_v} Y\} > 0$. In fact, since both conditional expectations have the same expected value, the set

$$B = \{E^{\mathscr{F}_v} Y < E^{\mathscr{G}_v} Y\} \tag{3.2}$$

must have positive probability.

Let $0 = r_0, r_1, \cdots, r_n = v, r_{n+1}, \cdots, r_{n+m} = w$ be a sequence of direct successors connecting 0 to v to w and chosen so that n and m are minimal. Put

$$Z_{s,u}^1 = \begin{cases} 1 & s = r_t,\ u = r_{t+1},\ 0 \leq t < n \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_s^2 = \begin{cases} \alpha^m E^{\mathscr{G}_v} Y & s = v \\ Y & s = w \\ 0 & \text{otherwise.} \end{cases}$$

The expected payoff obtained using the non-Markov strategy

$$\sigma_t = \begin{cases} r_{t\wedge n} & \text{on } B \\ r_{t\wedge(n+m)} & \text{on } B^c \end{cases}$$

can be estimated as follows:

$$\begin{aligned} E\mathscr{Z}(\boldsymbol{\sigma}) &= \textstyle\sum_{t=0}^{n-1} \alpha^t + E1_B\alpha^n\alpha^m E^{\mathscr{G}_v}Y + E1_{B^c}\alpha^{n+m}Y \\ &> \beta_n + \alpha^{n+m}EY, \end{aligned} \tag{3.3}$$

where $\beta_n = \sum_{t=0}^{n-1} \alpha^t$. The strict inequality follows from the definition of B.

Now consider any strategy $\boldsymbol{\sigma}$. Put

$$C = \{\sigma_t = r_t,\ 0 \le t \le n\}.$$

On the set C^c, the payoff $\mathscr{Z}(\boldsymbol{\sigma})$ is bounded above by $\beta_{n-1} + \alpha^n < \beta_n$ (we use here the fact that n is minimal to conclude $\alpha^\tau Z^2_{\sigma_\tau} \le \alpha^n Z^2_{\sigma_\tau} \le \alpha^n$). Hence,

$$\begin{aligned} E\mathscr{Z}(\boldsymbol{\sigma}) &= E\mathscr{Z}(\boldsymbol{\sigma})1_{C^c} + E\mathscr{Z}(\boldsymbol{\sigma})1_C \\ &\le \beta_n P(C^c) + E\{\beta_n + \alpha^\tau Z^2_{\sigma_\tau}\}1_C \\ &\le \beta_n + E\alpha^\tau Z^2_{\sigma_\tau}1_C. \end{aligned} \tag{3.4}$$

Since Z^2_s takes on non-zero values only at the points v and w,

$$E\alpha^\tau Z^2_{\sigma_\tau}1_C \le \alpha^{n+m}E1_C\{1_{\{\sigma_{n+1}=v\}}E^{\mathscr{G}_v}Y + 1_{\{\sigma_{n+1}\ne v\}}Y\}. \tag{3.5}$$

Now suppose that $\boldsymbol{\sigma}$ is Markov. Then, by the definition of $\mathscr{G}_{\sigma n}$, there is a $\mathscr{G}_v$-measurable set A such that

$$\{\sigma_n = v\} \cap \{\sigma_{n+1} = v\} = \{\sigma_n = v\} \cap A$$

$$\{\sigma_n = v\} \cap \{\sigma_{n+1} \ne v\} = \{\sigma_n = v\} \cap A^c.$$

Hence

$$E1_C\{1_{\{\sigma_{n+1}=v\}}E^{\mathscr{G}_v}Y + 1_{\{\sigma_{n+1}\ne v\}}Y\} \le E\{1_A E^{\mathscr{G}_v}Y + 1_{A^c}Y\} = EY. \tag{3.6}$$

Combining (3.4), (3.5), and (3.6) we get

$$E\mathscr{Z}(\boldsymbol{\sigma}) \le \beta_n + \alpha^{n+m}EY$$

for any Markov strategy $\boldsymbol{\sigma}$. By (3.3) we see that the supremum over all strategies is strictly greater than the supremum over Markov strategies.

The above result would not be true if we considered only a final reward, i.e., the case of optimal stopping. To see this we consider the following example. Let $S = \{0 = (0, 0), a = (1, 0), b = (0, 1), c = (1, 1)\}$ with the usual partial order. Let $(\Omega, \mathscr{A}, P)$ be any nontrivial probability space and put $\mathscr{G}_0 = \mathscr{G}_c = \mathscr{A}$ and $\mathscr{G}_a = \mathscr{G}_b = \{\varnothing, \Omega\}$. Clearly, if $s = a$ or b, $\mathscr{F}_s$ and $\mathscr{H}_s$ are not conditionally independent given $\mathscr{G}_s$. For any collection of final rewards $Z^2_s \in \mathscr{G}_s$, we wish to maximize $E\alpha^\tau Z^2_{\sigma_\tau}$ over all $\boldsymbol{\sigma} \in \Sigma$. It is easy to see in this example that by changing Z^2_s we can replace α by 1. For any strategy $\boldsymbol{\sigma}$,

$$EZ^2_{\sigma_\tau} \le E \max_{s\in S} Z^2_s.$$

We will give a Markov strategy which attains this upper bound. Put $M(\omega) = \max_s Z^2_s(\omega)$. Let A_0, A_a, A_b be an $\mathscr{A}$-measurable partition of Ω such that

$$A_0 \subset \{Z^2_0 = M\}$$

$$A_a \subset \{Z^2_a = M\} \cap \{Z^2_c = M, Z^2_a \le Z^2_b\}$$

$$A_b \subset \{Z^2_b = M\} \cap \{Z^2_c = M, Z^2_a > Z^2_b\}.$$

Let σ be the strategy defined by

$$\sigma_0 = 0$$

$$\sigma_1 = \begin{cases} 0 & \text{on } A_0 \\ a & \text{on } A_a \\ b & \text{on } A_b \end{cases}$$

$$\sigma_2 = \begin{matrix} 0 & \text{on } \{\sigma_1 = 0\} \\ a & \text{on } \{\sigma_1 = a\} \cap \{Z_a^2 > Z_b^2\} \\ b & \text{on } \{\sigma_1 = b\} \cap \{Z_a^2 \le Z_b^2\} \\ c & \text{otherwise} \end{matrix}$$

$$\sigma_t = \sigma_2, \quad t \geq 2.$$

It is easy to see that σ is Markov and $EZ_{\sigma_\tau}^2 = EM$.

It was pointed out in Mandelbaum and Vanderbei (1981) and Washburn and Willisky (1981), that the importance of the condition $\sigma_{t+1} \in \mathscr{F}_{\sigma_t}$ is that it preserves martingales. That is, if M_s, $s \in S$, is an $\mathscr{F}-$ martingale then $M_t^\sigma = M_{\sigma_t}$ is an $\mathscr{F}^\sigma$ – martingale.

Similarly, (3.1) preserves the Markov property. Put $\mathscr{G}_t^\sigma = \mathscr{G}_{\sigma_t} \mathscr{F}_t^\sigma = \bigvee_{r \le t} \mathscr{G}_r^\sigma$ and $\mathscr{H}_t^\sigma = \bigvee_{r \ge t} \mathscr{G}_t^\sigma$. If $\mathscr{F}_s$ and $\mathscr{H}_s$ are conditionally independent given $\mathscr{G}_s$, for all $s \in S$, and σ satisfies (3.1), then $\mathscr{F}_t^\sigma$ and $\mathscr{H}_t^\sigma$ are conditionally independent given $\mathscr{G}_t^\sigma$ for every $t \in N$. In the case $S = N$, condition (3.1) is the discrete analogue of the statement that σ_t is the inverse of a (time inhomogeneous) additive functional.

We now apply Theorem 5 to the case of several independent stochastic processes. For $i = 1, \cdots, k$, let $\{Y_{s^i}^i\}_{s^i \in N}$ be a stochastic process defined on a probability space $(\Omega^i, \mathscr{A}^i, P^i)$ and taking values in a state space (E^i, B^i). Let $\mathscr{G}_{s^i}^i = \sigma\{Y_{s^i}^i\}$, $\mathscr{F}_{s^i}^i = \bigvee_{u^i \le s^i} \mathscr{G}_{u^i}^i$, $\mathscr{H}_{s^i}^i = \bigvee_{u^i \ge s^i} \mathscr{G}_{u^i}^i$. Put

$$(\Omega, \mathscr{A}, P) = (\Omega^1, \mathscr{A}^1, P^1) \times \cdots \times (\Omega^k, \mathscr{A}^k, P^k)$$

$$(E, B) = (E^1, B^1) \times \cdots \times (E^k, B^k)$$

$$Y_s = (Y_{s^1}^1, \cdots, Y_{s^k}^k) \in E, \; s = (s^1, \cdots, s^k) \in N^k.$$

Applying Theorem 5 to $\mathscr{G}_s = \sigma\{Y_s\}$ and using the fact that the processes Y^i are independent we get

COROLLARY 1. *The following are equivalent*:

(a) *For every pair of bounded measurable functions* $h: N^k \times N^k \times E \times E \to \mathbb{R}$ *and* $f: N^k \times E \to \mathbb{R}$, *the supremum over all* $\sigma \in \Sigma$ *of*

$$E\{\textstyle\sum_{t=0}^{\tau-1} \alpha^t h(\sigma_t, \sigma_{t+1}, Y_{\sigma_t}, Y_{\sigma_{t+1}}) + \alpha^\tau f(\sigma_\tau, Y_{\sigma_\tau})\}$$

is attained by a Markov strategy.

(b) *For every* $i = 1, \cdots, k$ *and every* $s^i \in N$, $\mathscr{F}_{s^i}^i$ *and* $\mathscr{H}_{s^i}^i$ *are conditionally independent given* $\mathscr{G}_{s^i}^i$ (i.e., *each process* Y^i *is Markov*).

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