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SOME RESULTS ON LIL BEHAVIOR

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We examine the relationship between the central limit theorem with Gaussian limit, and law of the iterated logarithm type behavior in the Banach space setting. Our results solve some problems posed by Kesten (1972) and generalize some results of Klass (1976, 1977).

1. Introduction. It is well known that the law of the iterated logarithm (LIL) is more or less just a refinement of the central limit theorem (CLT) with Gaussian limit, and here we present some recent results in this regard. Our main theorems are of three basic types. One type relates LIL behavior with CLT behavior for sums of independent identically distributed vector valued random variables, and solves some of the problems posed in Kesten (1972). These results assert the existence of normalizing sequences which produce detailed LIL behavior under a minimal CLT type assumption, but say nothing about the regularity of the normalizing sequence and are contained in Theorem 1. Theorem 2 involves a limited converse result in this setting. Another type of result generalizes some work of Klass (1976–1977), and here we investigate stability results with some attention focused on the behavior of the normalizing sequence (see Theorem 3 and Corollaries 1–5). Finally, we present some results which are more or less a combination of these types, and which are related to the methods in a recent paper of Pruitt (1981). These are contained in Theorems 4 and 5 and Corollaries 6 and 7.

In Kesten (1972) the regularity, as well as the existence, of a sequence which produces LIL behavior is investigated via methods which involve considerable analysis. The one thing which our approach emphasizes is that to do the existence problem, even in a general vector valued setting, simple probability inequalities are really all that is necessary. Furthermore, our methods allow us to answer some of the clustering phenomenon questions posed by Kesten (1972) in this setting, but to go beyond the "existence of a normalizing sequence" and to establish regularity properties on the sequence definitely requires more analysis. Much remains to be done in this regard, and the paper by Pruitt (1981) is an excellent source to these, as well as the related problems involving one-sided LIL behavior.

To make things more precise we now turn to some notation and a discussion of CLT and LIL behavior. We will see that CLT and LIL are related at various levels and a fundamental result is Theorem A below.

Throughout, B is a real separable Banach space with topological dual B^* and norm $\|\cdot\|$. We assume $X, X_1, X_2, \cdots$ are independent identically distributed B-valued random variables, and as usual $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. We use Lx to denote the function $\max(1, \log_e x)$ and we write L_2x to denote $L(Lx)$. The law of X is denoted by $\mathscr{L}(X)$, and X satisfies the classical central limit theorem in B if the sequence $\{\mathscr{L}(S_n/\sqrt{n})\}$ converges weakly to, say $\mathscr{L}(Z)$. Of course, Z must be a mean zero B-valued Gaussian random variable and X must have covariance structure identical to that of Z. We will use the notation

$$\mathscr{L}(S_n/\sqrt{n}) \to_{n\to\infty} \mathscr{L}(Z) \tag{1.1}$$

to denote the weak convergence of $S_n/\sqrt{n}$ to Z.

If X does not satisfy the classical central limit theorem, there are several other ways in

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which the partial sum process $\{S_n\}$, suitably normalized, might approximate a Gaussian law weakly. One possibility is to consider normalizations other than $\sqrt{n}$, and another is to pass to subsequences of the partial sums $\{S_n\}$. That is, we say X is in the domain of attraction (domain of partial attraction) of a Gaussian random variable Z if there exists a sequence $d_n \nearrow \infty$, $\{\delta_n\} \subseteq B$ (a subsequence of integers $n_k \nearrow$ and sequences $d_{n_k} \nearrow \infty$, $\{\delta_{n_k}\} \subseteq B$) such that

$$\mathscr{L}\left(\frac{S_n - \delta_n}{d_n}\right) \to_{n\to\infty} \mathscr{L}(Z)\left(\mathscr{L}\left(\frac{S_{n_k} - \delta_{n_k}}{d_{n_k}}\right) \to_{k\to\infty} \mathscr{L}(Z)\right).$$

If X is in the domain of attraction (of partial attraction) of a Gaussian random variable Z we write $X \in DA(Z)(X \in DPA(Z))$, and, of course, we always are assuming Z is a mean zero random variable which is not identically zero.

As a result, when saying X has CLT behavior, what we have in mind is that X satisfies the classical CLT, $X \in DA(Z)$, or $X \in DPA(Z)$ where Z is Gaussian.

Next we turn to the LIL. The classical normalizing constants in the LIL are

$$a_n = \sqrt{2nL_2n}, \tag{1.2}$$

and in the infinite dimensional setting there are two forms of the classical LIL which are of interest. They are the so-called bounded LIL (we write $X \in$ BLIL) and the compact LIL (we write $X \in$ CLIL). That is, $X \in$ BLIL if

$$\limsup_n \|S_n/a_n\| < \infty \quad \text{w.p. } 1, \tag{1.3}$$

and $X \in$ CLIL if there exists a non-random compact set $D \subseteq B$ such that

$$d\left(\frac{S_n(\omega)}{a_n}, D\right) \to 0 \quad \text{w.p. } 1, \tag{1.4}$$

and

$$D = C\left(\left\{\frac{S_n(\omega)}{a_n}\right\}\right) \quad \text{w.p. } 1 \tag{1.5}$$

where

$$d(x, A) = \inf_{y\in A} \|x - y\|,$$

and $C(\{S_n(\omega)/a_n\})$ denotes all limit points of the random sequence $\{S_n(\omega)/a_n\}$. The set D in the compact LIL is called the "limit-set".

If B is finite dimensional, then it is known that the conditions $X \in$ BLIL, $X \in$ CLIL, X satisfies the classical central limit theorem, and the moment conditions $EX = 0$, $E\|X\|^2 < \infty$ are all equivalent. Hence the classical CLT is equivalent to the classical LIL in finite dimensional spaces, but if B is infinite dimensional much more variety is available. For further details involving various implications between the classical CLT and the classical forms of the LIL, we suggest Kuelbs (1980) as a reference. Now, however, we turn to a generalization of the LIL and study its relationship to the generalized notions available in the central limit theorem.

Let X be a B-valued random variable. We say X has LIL behavior with respect to the centering sequence $\{\delta_n\} \subseteq B$ if there exists a normalizing sequence $\gamma_n \nearrow \infty$ such that

$$0 < \limsup_n \left\|\frac{S_n - \delta_n}{\gamma_n}\right\| < \infty \quad \text{w.p. } 1. \tag{1.6}$$

Of course, LIL behavior is much more general than that required in the classical LIL, and several remarks are in order.

First of all, if X is symmetric we are naturally interested in X having LIL behavior with respect to the centerings $\delta_n = 0$, $n \geq 1$. Secondly, we must control the magnitude of the sequence δ_n. That is, if $\limsup_n \|S_n\|/\|\delta_n\| = 0$ and γ_n is of order $\|\delta_n\|$, then the limiting behavior in (1.6) is not that of $\{S_n\}$ but is determined solely by $\{\delta_n\}$ and $\{\gamma_n\}$. Hence LIL behavior with respect to centering sequences $\{\delta_n\}$ which do not dominate $\{S_n\}$ is the item of interest. If X takes values in $\mathbb{R}^1$, then a fundamental result relating LIL behavior and CLT behavior appears in Kesten (1972) and implies the following.

THEOREM A. *Let X be real-valued. Then, X has* LIL *behavior with respect to the centering sequence $\delta_n = \text{med}(S_n)$, where $\text{med}(S_n)$ is any choice for median S_n*, iff *X is in the* DPA(Z) *where Z is a mean zero Gaussian random variable with variance one. That is, there exists a sequence $\gamma_n \nearrow \infty$ such that*

$$0 < \limsup_n \left| \frac{S_n - \text{med}(S_n)}{\gamma_n} \right| < \infty, \quad \text{w.p. } 1 \tag{1.7}$$

iff $X \in$ DPA(Z) *where Z is $N(0, 1)$. Further, for every fixed $\varepsilon > 0$ $\{\gamma_n\}$ can be chosen so that*

$$n^{-1/2+\varepsilon}\gamma_n \nearrow \infty. \tag{1.8}$$

The fact that LIL behavior with respect to the centerings $\delta_n = \text{med}(S_n)$, $n \ge 1$, implies X is in DPA(Z) where $Z = N(0, 1)$, appears in some work of Heyde (1969) and Rogozin (1968). The converse result and (1.8) are due to Kesten (1972) for centerings slightly more general than medians, and the recent results of Pruitt (1981) and Klass (1976–1977) mentioned previously furthers the investigation of such matters. Before we turn to statements of the results we prove here, we mention the following unsolved problem posed in Kesten (1972).

PROBLEM (Kesten, 1972). Find the accumulation points of

$$\left\{\frac{S_n - \text{med}(S_n)}{\gamma_n}\right\}, \tag{1.9}$$

and of the polygonal functions $\{\eta_n\}$ where

$$\eta_n(t) = \begin{cases} \dfrac{S_k - \text{med}(S_k)}{\gamma_n} & t = k/n,\ k = 0, \cdots, n \\ \text{linearly interpolated elsewhere for } 0 \le t \le 1. \end{cases} \tag{1.10}$$

If the polygonal functions $\{\eta_n\}$ defined in (1.10) have a nondegenerate limit set of functions we say X has functional LIL behavior.

We will solve Kesten's problem for a general separable Banach space B and for certain sequences $\{\gamma_n\}$ when the centerings $\{\text{med}(S_n)\}$ are replaced by truncated means. Actually, Kesten posed the problem for centerings slightly more general than medians, but in the Banach space setting it is more natural to work with truncated means. We consider this a positive aspect of our work as our centerings are much more transparent than medians anyway since they are directly computable from the law of X. Furthermore, once our result is proved, we will have LIL behavior for the symmetrization of X, and hence by Kesten's Lemma 1 (1972, page 721), whenever X is real valued, we will have LIL behavior with respect to his centerings as well.

2. Statements of the main results. Our first result contains Kesten's half of Theorem A (without (1.8)), and also indicates one solution for the problem of Kesten indicated at the end of section one. However, to state this result we must provide some indication of the set of accumulation points obtained in the solution of this problem.

The motivation for this "limit set" begins with the fundamental paper of Strassen (1964), and it is described in Kuelbs (1976). That is, if Z is a mean zero B-valued Gaussian random variable and K denotes the unit ball of the Hilbert space $H_{\mathscr{L}(Z)}$ described in Lemma 2.1 of Kuelbs (1976), then K is compact in B, and is the limit set we use in connection with the accumulation points of the sequence (1.9) when the centerings $\text{med}(S_n)$

are replaced by suitable truncated means. For the accumulation points of the polygonal functions in (1.10) (again with medians replaced by suitable truncated means) let $\mu = \mathscr{L}(Z)$ and consider the μ-Wiener measure W induced by μ on C_B. Here C_B denotes the B-valued continuous functions on [0, 1] with the sup-norm. Then W is a mean zero Gaussian measure on C_B and we use $\mathscr{K}$ to denote the unit ball of the Hilbert space $H_W \subseteq C_B$ as constructed in Lemma 2.1 of Kuelbs (1976). Further, $\mathscr{K}$ is compact in C_B and is the desired limit set for the polygonal functions of (1.10). For details regarding μ-Wiener measure and $\mathscr{K}$ the reader can also consult Kuelbs and Le Page (1973).

If (M, d) is a metric space and $A \subseteq M$ we define the distance from $x \in M$ to A by $d(x, A) = \inf_{y \in A} d(x, y)$. If $\{x_n\}$ is a sequence of points in M, then $C(\{x_n\})$ denotes the cluster set of $\{x_n\}$. That is, $C(\{x_n\}) = \{x : \liminf_n d(x, x_n) = 0\}$. We will use the notation $\{x_n\} \twoheadrightarrow A$ if both $\lim_n d(x_n, A) = 0$ and $C(\{x_n\}) = A$.

THEOREM 1. *Let X be B-valued and in* DPA(Z) *where Z is a mean-zero Gaussian variable. Let K and $\mathscr{K}$ be as described above. Then, there exists a subsequence of integers $\{n_k\}$ and normalizing constants $d_k \nearrow \infty$ such that if*

$$\delta_n = nE(XI(\|X\| \le d_k)) \quad n \in (n_{k-1}, n_k] \tag{2.1}$$

and

$$\gamma_n = \sqrt{2Lk}\, d_k \quad n \in (n_{k-1}, n_k], \tag{2.2}$$

then

$$\mathscr{L}\left(\frac{S_{n_k} - \delta_{n_k}}{d_k}\right) \to_{k\to\infty} \mathscr{L}(Z), \tag{2.3}$$

and

$$P\left(\left\{\frac{S_n - \delta_n}{\gamma_n}\right\} \twoheadrightarrow K\right) = 1. \tag{2.4}$$

Furthermore, the polygonal functions $\{\eta_n\}$ defined in (1.10), *with the medians replaced by suitable modifications of the truncated means $\{\delta_n\}$ (see* (3.5) *for details), are such that*

$$P(\{\eta_n\} \twoheadrightarrow \mathscr{K}) = 1. \tag{2.5}$$

The converse situation in Theorem 1 is to show that LIL behavior implies some sort of central limit behavior. However, in this regard very little is known and there are examples of random variables X taking values in infinite dimensional Banach spaces such that X satisfies the CLIL (with classical normalizing constants) yet X does not satisfy the classical CLT. What we can show, however, is that the analytic condition

$$\liminf_{u\to\infty} \frac{u^2 P(\|X\| > u)}{\int_{\|x\| \le u} \|x\|^2 \, dP_X(x)} = 0 \tag{2.6}$$

does hold whenever X is symmetric and has LIL behavior in a type 2 Banach space. Recall that a Banach space B is said to be of type 2 (cotype 2) if for all n and all $X_1, \cdots, X_n$ independent mean zero B-valued random variables there is some constant A, $0 < A < \infty$, such that

$$E\|X_1 + \cdots + X_n\|^2 \le (\ge)\, A \textstyle\sum_{j=1}^n E\|X_j\|^2.$$

Furthermore, it is well known that the condition (2.6) is necessary and sufficient for X to be in the DPA of a Gaussian law when X is real-valued, but it is clearly only part of the story in more than one dimension. Some Hilbert space valued random variables producing rather unexpected behavior are presented in the final section of the paper indicating some of the subtleties in a converse type result. For now we state:

THEOREM 2. *Let X by symmetric and B-valued where B is a type 2 Banach space, and assume X has* LIL *behavior with respect to the centerings $\delta_n = 0$. Then X satisfies the analytic condition* (2.6). *Furthermore, if* (2.6) *fails, then for $\gamma_n \nearrow$*

$$\limsup_n \left\| \frac{S_n}{\gamma_n} \right\| = 0 \quad \text{or} \quad \infty \tag{2.7}$$

according as $\sum P(\|X_n\| > \gamma_n) < \infty$ or $= \infty$.

REMARK. If B is a Hilbert space, X is B-valued and symmetric, and X has LIL behavior with respect to the centerings $\delta_n = 0$, then we can prove more than the analytic condition (2.6). However, it is not necessarily true, as it is when B is the real line, that such an X is in the DPA of a Gaussian law (see the examples of Section 7). What can be proved (see the end of Section 4) is that under these circumstances there exist $\{d_k\} \nearrow \infty$ and $\{n_k\} \nearrow \infty$ such that

$$\{S_{n_k}/d_k\} \tag{2.8}$$

is stochastically bound in H, and

$$\lim_k E \left\|\sum_{j=1}^{n_k} X_j I(\|X_j\| \le d_k)\right\|^2 / d_k^2 = 1. \tag{2.9}$$

The conditions (2.8) and (2.9) can be interpreted as a weak form of central limit behavior and, as the examples of Section 7 demonstrate, perhaps one can hope for little more in the way of a converse in the infinite dimensional setting.

The results of the second type mentioned in the introduction of the paper involve stability of the normalized partial sums with some attention to the behavior of the normalizing sequence, but without regard to clustering.

To motivate the normalizations in these results we recall that if X is real-valued then the conditions $EX = 0$ and $EX^2 < \infty$, X satisfies the classical BLIL, and $E\,|S_n| \approx \sqrt{n}$ are all equivalent. Here we write $a_n \approx b_n$ to mean there exists constants A, B such that $0 < A < a_n/b_n < B < \infty$ for all n. Hence for real valued X we have X satisfying the classical BLIL iff $EX = 0$, $EX^2 < \infty$, and then with probability one

$$\limsup_n \frac{|S_n|}{L_2 n E\,|S_{n/L_2 n}|} < \infty \tag{2.10}$$

where $S_t = S_{[t]}$ and $[t]$ denotes the greatest integer in t.

This formulation of the LIL suggests normalizations which exist whenever $E\|X\| < \infty$, but of course it requires estimating $E\|S_n\|$. This is a difficult problem in an arbitrary Banach space, but for many purposes the $K(\cdot)$ function introduced by M. Klass (1976–1977) is of great use even though it does not always measure $E\|S_n\|$ in arbitrary spaces.

Given a B-valued random variable X such that $0 < E\|X\| < \infty$, we define the strictly increasing absolutely continuous function $G(y)$ for $y > 0$ by

$$G(y) = y^2 \Big/ \int_0^y E(\|X\| I(\|X\| > u))\, du. \tag{2.11}$$

Letting $K(\cdot)$ denote the inverse of G we see that $K(y)$ satisfies

$$K^2(y) = y \int_0^{K(y)} E(\|X\| I(\|X\| > u))\, du. \tag{2.12}$$

Thus from (2.11) and (2.12), $K(y)$ is a strictly increasing absolutely continuous function such that

$$K(y)/y \searrow 0 \tag{2.13}$$

and

$$K^2(y)/y \nearrow E\|X\|^2 \text{ (possibly infinite).} \tag{2.14}$$

Further, since

$$E(\|X\|^2 I(\|X\| \le r)) = -rE(\|X\| I(\|X\| > r)) + \int_0^r E(\|X\| I(\|X\| > u))\,du. \tag{2.15}$$

we have by setting $r = K(y)$ and combining (2.12) and (2.15) that

$$K^2(y) = yE(\|X\|^2 I(\|X\| \le K(y))) + yK(y)E(\|X\| I(\|X\| > K(y))). \tag{2.16}$$

The equation (2.16) uniquely determines $K(y)$ among the positive continuous strictly increasing functions, and is used in an important way in the proof of Theorem 3. K, of course, is a function of X, and in arguments involving more than one K-function we will use K_X to denote the K-function determined by the random variable X.

For a mean zero real-valued X with $0 < E|X| < \infty$, Klass (1976–1980) has shown that

$$\tfrac{1}{2}E|S_n| \le K(n) \le 3E|S_n|, \quad (n \ge 1). \tag{2.17}$$

Hence the normalization constants in (2.10) are equivalent to $L_2nK(n/L_2n)$ and Klass's results (1976–1977) extend the LIL to mean zero random variables without variance. The connection between $K(n)$ and $E\|S_n\|$ in Banach spaces is given by the following lemma.

LEMMA 1. *If X is a mean zero B-valued random variable with $0 < E\|X\| < \infty$, then*

$$B \text{ is type 2 iff } E\|S_n\| \le A_1K(n), \quad (n \ge 1) \tag{2.18}$$

for some constant $A_1 < \infty$,

$$B \text{ is cotype 2 iff } E\|S_n\| \ge A_2K(n), \quad (n \ge 1) \tag{2.19}$$

for some constant $A_2 > 0$, and B is isomorphic to Hilbert space iff

$$C_1K(n) \le E\|S_n\| \le C_2K(n), \quad (n \ge 1) \tag{2.20}$$

for constants $0 < C_1 < C_2 < \infty$.

In connection with Theorem 3 and its corollaries we set

$$\alpha_n = L_2nK(n/L_2n), \quad (n \ge 1), \tag{2.21}$$

and for $\beta > 1$, let

$$n_k = n_k(\beta) = [\beta^k] \tag{2.22}$$

where $[t]$ denotes the greatest integer in t. For all $t \ge 0$ we define $S_t = S_{[t]}$ with $S_0 = 0$.

THEOREM 3. *Let X denote a B-valued random variable with mean zero and $0 < E\|X\| < \infty$. Let $\{\gamma_n\}$ denote a nondecreasing sequence with*

$$\gamma_n = \sqrt{n}\,h(n) \ge \alpha_n \tag{2.23}$$

where $h(n)$ satisfies

$$\inf_r \inf_{k \ge r} \frac{h(n_k)}{h(n_r)} \ge c > 0. \tag{2.24}$$

Then, for all $\beta > 1$ such that (2.24) holds, we have

$$\limsup_n \frac{\|S_n\| - E\|S_{\beta n}\|}{\gamma_n} < \infty \quad \text{w.p. } 1 \tag{2.25}$$

provided

$$P(\|X_n\| > M\gamma_n \text{ i.o.}) = 0 \tag{2.26}$$

for some $M < \infty$. *Further, if* $E\|X\|^2 < \infty$ *the regularity condition in* (2.24) *is unnecessary for* (2.26) *to imply* (2.25).

REMARKS. (1) The condition (2.24) holds with $c = 1$ if h is non-decreasing.

(2) The subtraction of $E\|S_{\beta n}\|$ from $\|S_n\|$ rather than only $E\|S_n\|$ to obtain (2.25) is perhaps unnecessary, but we cannot prove such a result. Furthermore, there are examples (one is given in Kuelbs-Zinn, 1981) of vector valued random variables and normalization sequences $\{\gamma_n\}$ satisfying the conditions of Theorem 3 yet

$$\lim_n \frac{\|S_n\|}{\gamma_n} = \infty, \quad \text{w.p. 1.} \tag{2.27}$$

Hence it is clear that in the vector valued case we must center somewhere, and what we show is that $E\|S_{\beta_n}\|$ always works. Corollary 2 will say more in this regard, and since B is separable, the strong law of large numbers easily applies to yield (2.25) whenever $EX = x \neq 0$ as well. That is, if $E(X) = x \neq 0$, then $E\|S_n\| \geq n\|x\|$, so for $\beta > 1$ we have

$$\begin{aligned} P(\|S_n\| - E\|S_{\beta n}\| > b_n \text{ i.o}) &\leq P(\|S_n\| > b_n + [n\beta]\|x\| \text{ i.o.}) \\ &\leq P(\|S_n\| > (n\beta - 1)\|x\| \text{ i.o.}) \\ &= 0 \text{ for all positive sequences } \{b_n\}. \end{aligned}$$

(3) Since the normalizations $\{\gamma_n\}$ in Theorem 3 satisfy (2.23) where $\{\alpha_n\}$ is as in (2.21), it follows from (2.14) that $\gamma_n = O(\sqrt{nL_2 n})$ is possible only if $E\|X\|^2 < \infty$. Hence Theorem 3 applies to the classical BLIL iff $E\|X\|^2 < \infty$, and Corollary 3 gives the complete picture in this situation. However, it is well known that $E\|X\|^2 < \infty$ is not necessary for the classical BLIL even in Hilbert space, and Corollary 4 and the remark following it contains some additional information on this.

(4) From the definition of α_n in (2.21) and that $K(n)/\sqrt{n} \nearrow$ it easily follows that $\alpha_n/\sqrt{n}$ is non-decreasing and hence α_n satisfies the regularity condition (2.24).

Some corollaries of Theorem 3 are the following.

COROLLARY 1. *Let* X *and* $\{\gamma_n\}$ *be as in Theorem* 3. *Then*

$$\limsup_n \frac{\|S_n\|}{\gamma_n} < \infty \quad \text{w.p. 1} \tag{2.28}$$

iff

a) $P(\|X_n\| > M\gamma_n \text{ i.o}) = 0$ *for some* $M < \infty$, *and* (2.29)

b) $\{S_n/\gamma_n\}$ *is bounded in probability.*

COROLLARY 2. *Let* B *be a type* 2 *Banach space and assume* X *and* $\{\gamma_n\}$ *are as in Theorem* 3. *Then*

$$\limsup_n \frac{\|S_n\|}{\gamma_n} < \infty \quad \text{w.p. 1} \tag{2.30}$$

iff

$$P(\|X_n\| > M\gamma_n \text{ i.o.}) = 0 \tag{2.31}$$

for some $M < \infty$.

COROLLARY 3. *Let* X *be mean zero and assume* $E\|X\|^2 < \infty$. *If* $\{\gamma_n\}$ *is any non-decreasing sequence such that for some positive constant* ρ

$$\sqrt{nL_2 n} \leq \rho\gamma_n \quad (n \geq 1), \tag{2.32}$$

then

$$\limsup_n \left\| \frac{S_n}{\gamma_n} \right\| < \infty \text{ w.p. 1 iff } \left\{ \frac{S_n}{\gamma_n} \right\} \textit{ is bounded in probability.} \tag{2.33}$$

Further, if $\{\gamma_n\}$ *is any non-decreasing sequence such that*

$$\lim_n \frac{\gamma_n}{\sqrt{nL_2 n}} = \infty, \tag{2.34}$$

then

$$\lim_n \left\| \frac{S_n}{\gamma_n} \right\| = 0 \text{ w.p. 1 iff } \frac{S_n}{\gamma_n} \to_{\text{prob}} 0. \tag{2.35}$$

REMARKS. (1) If $EX = 0$ and $E\|X\|^2 < \infty$, then Corollary 3 immediately implies X satisfies the BLIL iff $\{S_n/\sqrt{nL_2 n}\}$ is bounded in probability. This was first proved in Kuelbs (1977).

(2) Assuming $EX = 0$ and $0 < E\|X\|^2 < \infty$, Corollary 3 also implies that if $\{\gamma_n\} = \{\sqrt{L_2 n}\, E\|S_n\|\}$ or $\{\gamma_n\} = \{L_2 nE\|S_{n/L_2 n}\|\}$ then

$$\limsup_n \|S_n/\gamma_n\| < \infty \quad \text{w.p. 1.} \tag{2.36}$$

To verify (2.36) we first note that in both cases $\{\gamma_n\}$ is nondecreasing and a trivial application of Chebyshev's inequality implies $\{S_n/\gamma_n\}$ is bounded in probability. For example,

$$P(\|S_n\| \geq \Lambda\, L_2 nE\|S_{n/L_2 n}\|) \leq \frac{E\|S_n\|}{\Lambda L_2 nE\|S_n/L_2 n\|} = O\left(\frac{1}{\Lambda}\right)$$

where the first inequality is Chebyshev's, and the second follows since for all sufficiently large n the triangle inequality implies $E\|S_n\| \leq 2L_2 nE\|S_{n/L_2 n}\|$ (recall $S_t = S_{[t]}$). Hence to verify (2.36) in these cases we need only show (2.32) applies and this is easy. That is, since $EX = 0$, $0 < E\|X\|^2 < \infty$, there exists $f \in B^*$ such that $\|f\|_{B^*} = 1$ and $Ef(X) = 0$, $0 < Ef^2(X) < \infty$. Therefore

$$E\|S_n\| \geq E|f(S_n)| \approx \sqrt{n}$$

by applying (2.20) with $K = K_{f(X)}$, and then recalling (2.14) which implies $K_{f(X)}(n) \approx \sqrt{n}$. Hence (2.32) holds for each sequence $\{\gamma_n\}$ as claimed.

(3) Again, assuming $EX = 0$ and $0 < E\|X\|^2 < \infty$, the condition $\liminf_n(E\|S_n\|/\sqrt{nL_2 n}) > 0$ implies that for all sequences $\{d_n\} \nearrow \infty$ we have $\limsup_n (\|S_n\|/d_n E\|S_n\|) = 0$ w.p. 1. To see this, set $\gamma_n = d_n E\|S_n\|$. Then $\{S_n/\gamma_n\} \to_{\text{prob}} 0$ and (2.34) holds, so (2.35) yields the claim. Of course, if X takes values in a type 2 space, then by applying (2.18) and (2.14) we see the three conditions $EX = 0$, $0 < E\|X\|^2 < \infty$, and $\liminf(E\|S_n\|/\sqrt{nL_2 n}) > 0$ are incompatible. However, in some non-type 2 spaces there are examples satisfying these three conditions, and hence the above conclusions are non-trivial.

Thus Corollary 3 gives us a fairly complete picture when $EX = 0$ and $E\|X\|^2 < \infty$, and we now turn to some situations when $E\|X\|^2$ is possibly infinite.

COROLLARY 4. *Let* $EX = 0$ *and assume* $E(\|X\|^2 g(\|X\|)/L_2\|X\|) < \infty$ *where* $g:[0, \infty) \to (0, \infty)$ *is such that*

$$g(t)/L_2 t \textit{ is non-increasing on } [0, \infty), \tag{2.37}$$

$$tg(t)/L_2 t \textit{ is eventually non-decreasing as } t \to \infty, \tag{2.38}$$

and

$$x \geq \frac{L_2 x}{g(x)} \textit{ for all sufficiently large } x. \tag{2.39}$$

If

$$\{\gamma_n\} = \left\{ \sqrt{nL_2n} \sqrt{\frac{L_2n}{g(n)}} \right\},$$

then

(2.40) $$\limsup_n \left\| \frac{S_n}{\gamma_n} \right\| < \infty \text{ w.p. 1 iff } \left\{ \frac{S_n}{\gamma_n} \right\} \textit{ is bounded in probability.}$$

Furthermore, if

(2.41) $$\lim_n \frac{L_2n}{g(n)} = +\infty,$$

then

(2.42) $$\limsup_n \left\| \frac{S_n}{\gamma_n} \right\| = 0 \text{ w.p. 1 iff } \left\{ \frac{S_n}{\gamma_n} \right\} \to_{\text{prob}} 0.$$

REMARKS. (1) If $g : [0, \infty) \to (0, \infty)$ satisfies (2.37) and (2.38), then for some constant c, $0 < c < \infty$, $cg(t)$ satisfies (2.39) as well, and applying the corollary to $cg(t)$ yields the result for g.

(2) If $g(t) = L_2t$ we have $E\|X\|^2 < \infty$ and (2.40) reduces to the BLIL being equivalent to $\{S_n/\sqrt{nL_2n}\}$ bounded in probability. If $g(t) = 1$, then we have $E(\|X\|^2/L_2\|X\|) < \infty$ and (2.40) becomes

(2.43) $$\limsup_n \frac{\|S_n\|}{\sqrt{nL_2n}} < \infty \text{ w.p. 1 iff } \{S_n/\sqrt{nL_2n}\} \text{ is bounded in probability,}$$

and (2.42) becomes

(2.44) $$\lim_n \frac{\|S_n\|}{\sqrt{nL_2n}} = 0 \text{ w.p. 1 iff } \frac{S_n}{\sqrt{nL_2n}} \to_{\text{prob}} 0.$$

There are, of course, numerous other interesting cases, but we mention the above because of their relationship to the classical LIL. For example, if B is a type 2 space and X is mean zero with $E(\|X\|^2/L_2\|X\|) < \infty$, then we automatically have

$$\frac{S_n}{\sqrt{nL_2n}} \to_{\text{prob}} 0$$

(see Goodman, Kuelbs, Zinn (1981), Proposition 7.2). Further, the condition $E(\|X\|^2/L_2\|X\|) < \infty$ is known to be necessary for the classical BLIL, so Corollary 4 can be viewed as a result demonstrating that it is near being sufficient in type 2 spaces as well.

COROLLARY 5. *Let B denote a co-type 2 Banach space and assume $EX = 0$, $0 < E\|X\| < \infty$. Then*

(2.45) $$\limsup_n \frac{\|S_n\|}{L_2nE\|S_{n/L_2n}\|} < \infty \quad \text{w.p.1}$$

iff

(2.46) $$P(\|X_n\| > ML_2nE\|S_{n/L_2n}\| \text{ i.o.}) = 0$$

for some $M < \infty$.

REMARK. Corollary 5 is a generalization of part of Klass (1976), but we do not obtain a bound for (2.45) as Klass does.

Now we turn to the results which are somewhat of a combination of the types found in Theorem 1 and Theorem 3. Loosely speaking, to obtain these results we assume a rate of growth on S_n along a subsequence and we assume a "regularity" of the tails of X associated with this growth rate; then we combine these to obtain a suitable normalizing sequence. For example, we have the following theorem and a corollary which deals with the situation when X is in the DPA(Z).

THEOREM 4. *Assume that q is a Borel measurable semi-norm which induces a separable topology on B, and that $r_k \nearrow \infty$ and $d_k \nearrow \infty$ are such that*

$$\sup_k P\left(q\left(\frac{S_{r_k} - r_k E(XI(q(X) \leq d_k))}{d_k}\right) \geq 1\right) \leq \frac{1}{16e^2}, \tag{2.47}$$

and

$$\textstyle\sum_k (Lk) r_k P(q(X) > d_k) < \infty. \tag{2.48}$$

Then, putting $n_k = [Lk] r_k = p_k r_k$,

$$\begin{aligned} \gamma_n = \gamma(n) &= p_k d_k \quad \text{if} \quad n_{k-1} < n \leq n_k, \quad \textit{and} \\ \lambda(n) &= d_k \qquad\quad \text{if} \quad n_{k-1} < n \leq n_k, \end{aligned} \tag{2.49}$$

we have

$$\limsup_n q\left(\frac{S_n - nE(XI(q(X) \leq \lambda(n)))}{\gamma(n)}\right) \leq 416e^2 \quad \text{w.p.1.} \tag{2.50}$$

Further, if there exists a q-continuous linear functional h on B such that

$$\mathscr{L}\left(h\left(\frac{S_{r_k} - r_k E(XI(q(X) \leq d_k))}{d_k}\right)\right) \to \mathscr{L}(g) \tag{2.51}$$

where g is mean zero Gaussian random variable with non-zero variance and $n_{k+1}/n_k \geq 40$, then

$$\limsup_n q\left(\frac{S_n - nE(XI(q(X) \leq \lambda(n)))}{\gamma(n)}\right) > 0 \quad \text{w.p.1.} \tag{2.52}$$

As a corollary to Theorem 4 we have the following result which is somewhat like the conclusion (2.4) in Theorem 1. It differs from (2.4) in that we cannot identify the limit set precisely, but it does allow us to relate the regularity of the tails of X to the normalizing sequence which eventually provides LIL behavior.

COROLLARY 6. *Let $X \in$ DPA(Z) where Z is a mean zero Gaussian variable. Then, there exists $d_k \nearrow \infty$, integers $r_k \nearrow \infty$, and a compact convex symmetric set $D \subseteq B$ such that*

$$\textstyle\sum_k (Lk) r_k P(X \notin d_k D^\delta) < \infty \tag{2.53}$$

for all $\delta > 0$,

$$r_{k+1}/r_k \geq 40, \tag{2.54}$$

and

$$\mathscr{L}\left(\frac{S_{r_k} - r_k E(XI(\|X\| \leq d_k))}{d_k}\right) \to_{k\to\infty} \mathscr{L}(Z). \tag{2.55}$$

Further, for any $r_k \nearrow \infty$ and $d_k \nearrow \infty$ satisfying the above three conditions, and $\gamma(n)$ and $\lambda(n)$ as in (2.49), we have with probability one that

$$\left\{\frac{S_n - nE(XI(\|X\| \leq \lambda(n)))}{\gamma(n)}\right\}$$

is relatively compact with cluster set not equal to $\{0\}$.

Now the regularity of the normalizing sequences $\{\gamma_n\}$ as constructed in Theorem 4 and Corollary 6 depends on several quantities as given in (2.47) and (2.48). The following question then arises. If (2.47) holds for $r_k = k$, what is an appropriate modification for (2.48)? Using the arguments of the proof of Theorem 4.1 of Goodman, Kuelbs, Zinn (1981) we have that if

$$\left\{\frac{S_n}{\sqrt{n}}\right\} \text{ is stochastically bounded,} \tag{2.56}$$

then

$$\textstyle\sum_n P(\|X\| > \sqrt{nL_2 n}) < \infty \quad \left(\text{i.e. } E\left\{\frac{\|X\|^2}{L_2\|X\|}\right\} < \infty\right) \tag{2.57}$$

is necessary and sufficient for

$$\limsup_n \|S_n\|/\sqrt{nL_2 n} < \infty \quad \text{w.p.1.} \tag{2.58}$$

Now (2.57) is easily seen to be equivalent to

$$\textstyle\sum_k (Lk)2^k P(\|X\| > \sqrt{2^k}\,(Lk)) < \infty, \tag{2.59}$$

and when X is symmetric (2.56) implies we can take $r_k = 2^k$, $d_k = \sqrt{2^k}$ in (2.47), so (2.57) is then equivalent to

$$\textstyle\sum_k (Lk) r_k P(\|X\| > d_k(Lk)) < \infty \tag{2.60}$$

which is a weaker condition than (2.48) when $q(x) = \|x\|$.

We now turn to Theorem 5 which shows that an analogous weakening of (2.48) is sufficient in wide generality. Corollary 7 deals with the case that $X \in \mathrm{DA}(Z)$ and perhaps is more transparent to the first time reader.

We use the notation $\alpha(t) = t/L_2 t$, and hence $\alpha^{-1}(t) \sim tL_2 t$ as $t \to \infty$. As possible choices for $d(t)$ in Theorem 5 the reader can try $d(t) = t^p$, $\frac{1}{2} \le p < 1$. To better understand the conditions of Theorem 5 we suggest first considering the proof of Corollary 7.

Theorem 5. *Assume there exists an increasing continuous function $d: \mathbb{R}^+ \to \mathbb{R}^+$ with inverse g, a countable set A and constants, $0 < c$, $\ell < \infty$ such that for all $t \ge \ell$, g' exists on A^c and*

(2.61)

(i) $0 < g(t)/t \le g'(t) \le cg(t)/t$,

(ii) $\dfrac{g(\alpha(t))}{g(t)} \le c/((L_3 t)^2 L_2 t)$,

(iii) $\dfrac{d^2(t)}{t}$ *increases, and*

(iv) *for a Borel measurable semi-norm q which induces a separable topology on B, there exists $t_0 < \infty$ such that for all n sufficiently large*

$$P\left(q\left(\frac{S_n - nE(XI(q(X) \le d(n)))}{d(n)}\right) \ge t_0\right) \le \frac{1}{16e^2}.$$

Then, the following are equivalent:

(2.62)

(i) $$\limsup_n q\left(\frac{S_n - nE(XI(q(X) \le \alpha^{-1}d\alpha(n)))}{\alpha^{-1}d\alpha(n)}\right)$$
$$\le 26\, t_0[32e^2 + 96(2+\sqrt{2})] \quad \text{w.p.1,}$$

(ii) $$\limsup_n q\left(\frac{S_n - nE(XI(q(X) \le \alpha^{-1}d\alpha(n)))}{\alpha^{-1}d\alpha(n)}\right) < \infty \quad \text{w.p.1,}$$

and

$$E(\alpha^{-1}g\alpha(q(X))) < \infty. \tag{2.63}$$

As a corollary to Theorem 5 we have:

COROLLARY 7. *Let $X \in \mathrm{DA}(Z)$ where Z is Gaussian and B-valued. Then there exists a strictly increasing function $d: \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$d(n) \approx \bar{d}(n)$$

where $\{\bar{d}(n): n \geq 1\}$ is such that

$$\mathscr{L}\left(\frac{S_n - nE(XI(\|X\| \leq \bar{d}(n)))}{\bar{d}(n)}\right) \to \mathscr{L}(Z),$$

and with probability one

$$\left\{\frac{S_n - nE(X)}{\alpha^{-1}d\alpha(n)}\right\} \tag{2.64}$$

is relatively compact with cluster set which is non-empty and not $\{0\}$ iff

$$E(\alpha^{-1}d^{-1}\alpha(\|X\|)) < \infty. \tag{2.65}$$

REMARK. Recall that if $X \in \mathrm{DA}(Z)$ where Z is Gaussian, then it is well known that $E\|X\|^p < \infty$ for $0 \leq p < 2$. Hence $E(X)$ is well defined.

A lemma which is of some independent interest is the following: we use it to reduce some of our proofs to the symmetric case. Throughout the lemma and its proof we assume $S_n' = X_1' + \cdots + X_n' (n \geq 1)$ where $\{X_j'\}$ is an independent copy of the sequence $\{X_j\}$, and $\{\gamma_n\}$ is a sequence of positive numbers.

LEMMA 2. *Let $q(\cdot)$ denote a Borel measurable semi-norm on B which induces a separable topology on B. Assume X is a random variable such that $P(q(X - x_0) = 0) < 1$ for all $x_0 \in B$, and consider the following hypotheses:*

(a) $\limsup_n q\left(\frac{S_n - S_n'}{\gamma_n}\right) = M < \infty$ w.p.1

(b) $\left\{\frac{S_n - S_n'}{\gamma_n}\right\}$ *is q-relatively compact and for some convex, symmetric q-compact set D we have*

$$P\left(C_q\left(\left\{\frac{S_n - S_n'}{\gamma_n}\right\}\right) \subseteq D\right) = 1. \tag{2.66}$$

Here, of course q-relatively compact and $C_q(\{x_n\})$ are defined with regard to the topology induced by the semi-norm q.

(c) $\limsup_n q\left(\frac{S_n - S_n'}{\gamma_n}\right) = M > 0$ w.p.1.

We then have the following conclusions:

(i) (a) *implies $\gamma_n \to \infty$ as $n \to \infty$.*

(ii) (c) *implies that for all $\{c_n\} \subseteq B$*

$$\limsup_{n\to\infty} q\left(\frac{S_n - c_n}{\gamma_n}\right) \geq \frac{M}{2} > 0 \quad \text{w.p.1.}$$

(iii) (a) *implies there exists $\{c_n\} \subseteq B$ such that*

$$\lim_{n\to\infty} q\left(\frac{S_n - c_n}{\gamma_n}\right) = M \quad \text{w.p.1.}$$

(iv) (b) *implies there exists* $\{c_n\} \subseteq B$ *such that* $\left\{\frac{S_n - c_n}{\gamma_n}\right\}$ *is q-relatively compact* w.p.1 *and*

$$P\left(C_q\left(\left\{\frac{S_n - c_n}{\gamma_n}\right\}\right) \subseteq D\right) = 1.$$

(v) (a) *implies*

$$\limsup_{n\to\infty} q(X_n)/\gamma_n \leq 2M \quad \text{w.p.1.}$$

We now assume $\gamma_n \nearrow$.

(vi) *If* (a) *holds and* $P\left(q\left(\frac{S_n - S_n'}{\gamma_n}\right) \geq t_0\right) < \frac{1}{24}$, *then for all n sufficiently large and all* $\lambda > 0$

$$P\left(q\left(\frac{{}^\rho S_n}{\gamma_n}\right) \geq \lambda\right) \leq \frac{1}{\lambda}[24\, t_0 + 12\rho] + nP(q(X) > \rho\gamma_n)$$

(2.67) *where*

$${}^\rho S_n = \textstyle\sum_{j=1}^n [X_j - E(X_j I(q(X_j) \leq \rho\gamma_n))]$$

and

$$\rho > \limsup_n q(X_n)/\gamma_n.$$

(vii) (b) *implies*

$$q(\textstyle\sum_{j=1}^n [X_j - EX_j I(q(X_j) \leq \rho\gamma_n)]/\gamma_n) \to 0$$

in probability for all $\rho > \limsup_n q(X_n)/\gamma_n$.

(viii) *If* (a) *holds,* $\delta > 0$ *and*

$$P(q((S_n - d_n)/\gamma_n) \geq A_\delta) \leq 1 - \delta \quad \textit{for all} \quad n \geq n_0,$$

then

$$\limsup_n q(S_n - d_n)/\gamma_n \leq 2M + A_\delta \quad \text{w.p.1.}$$

(ix) *If* (b) *holds,* $\delta > 0$, *and for all* $\eta > 0$

$$P((S_n - d_n)/\gamma_n \notin A_\delta D^\eta) \leq 1 - \delta \quad \textit{for all} \quad n \geq n_0(\eta),$$

where $D^\eta = \{y : q(x - y) < \eta$ *for some* $x \in D\}$, *then* $(S_n - d_n)/\gamma_n$ *is q-relatively compact* w.p.1 *with q-cluster set contained in*

$$(2 + A_\delta)D.$$

(x) (a) *implies*

$$\limsup_n q(S_n - nE(XI(q(X) \leq \rho\gamma_n)))/\gamma_n \leq 26M + 12\rho$$

for all $\rho > \limsup_n q(X_n)/\gamma_n$.

(xi) (b) *implies that*

$$\left\{\frac{S_n - nE(XI(q(X) \leq \rho\gamma_n))}{\gamma_n}\right\}$$

is q-relatively compact w.p.1 *with q-cluster set contained in* $2D$.

3. Proof of Theorem 1. Our first task is to choose the sequence $\{n_k\}$. Since $X \in \mathrm{DPA}(Z)$ there is a subsequence $\{n'\}$, shifts $\{c_{n'}\}$, and normalizing constants $d_{n'} \nearrow \infty$ such that

(3.1) $$\mathscr{L}\left(\frac{S_{n'} - c_{n'}}{d_{n'}}\right) \to_{n'\to\infty} \mathscr{L}(Z).$$

Further, we know from Araujo, de Acosta, and Giné (1978), Corollary 2.12, that

$$[c_{n'} - n'E(XI(\|X\| \le d_{n'}))]/d_{n'} \to_{n'\to\infty} 0. \tag{3.2}$$

Hence for *any* subsequence $\{n_k\}$ of $\{n'\}$ if we let $d_k = d_{n_k}$ (for ease of notation) and define for $n \in (n_{k-1}, n_k]$

$$\delta_n = nE(XI(\|X\| \le d_k)), \tag{3.3}$$

$$\gamma_n = d_k\sqrt{2Lk}, \tag{3.4}$$

$$\eta_n(t) = \begin{cases} \dfrac{S_j - j\delta_n/n}{\gamma_n} & t = j/n, j = 0, \dots, n \\ \text{linearly interpolated elsewhere for } 0 \le t \le 1, \end{cases} \tag{3.5}$$

then by (3.1) we have (2.3) holding and by the invariance principle in Kuelbs (1973) we also have

$$\mathscr{L}(\sqrt{2Lk}\,\eta_{n_k}(\cdot)) \to_k \mathscr{L}(W) \tag{3.6}$$

where W is the Wiener measure induced by $\mathscr{L}(Z)$ on the Banach space C_B.

Actually, to apply the invariance principle given in Theorem 1 of Kuelbs (1973) two points must be clarified. First, the three conditions on the triangular array required in the statement of that theorem are redundant, and one only needs the first two conditions as the third then follows (this was kindly pointed out to the author by Harold Dehling and is quite easy to verify). Secondly, the theorem as stated is for polygonal processes defined by triangular arrays such that the nth row has n independent identically distributed random variables and hence it does not apply directly to the polygonal processes $\{\sqrt{2Lk}\,\eta_{n_k}\}$. However, this is an easy problem to handle since the polygonal process $\sqrt{2Lk}\,\eta_{n_k}(t)$ can be thought of as that process built from the n_kth row with the n_k i.i.d. elements in the row being

$$\left\{\frac{X_j - E(XI(\|X\| \le d_k))}{d_k} : j = 1, \cdots, n_k\right\},$$

and for $n \ne n_k$ one fills the nth row of the triangular array with n independent copies of $Z/\sqrt{n}$. Then the result of Kuelbs (1973) applies to this "filled in setup" and the subsequence $\{\mathscr{L}(\sqrt{2Lk}\,\eta_{n_k}(\cdot))\}$ converges to $\mathscr{L}(W)$ as claimed.

If U and V are random vectors we let $\rho(U, V)$ denote the Prokhorov distance between $\mathscr{L}(U)$ and $\mathscr{L}(V)$. Further, if $\{n_k\}$ is *any* strictly increasing subsequence of integers we let $\tilde{\eta}_{n_k}$ denote the polygonal process defined by

$$\tilde{\eta}_{n_k}(t) = \begin{cases} 0 & t = j/n_k, 0 \le j \le n_{k-1} \\ \dfrac{S_j - S_{n_{k-1}} - j\delta_{n_k}/n_k}{\gamma_{n_k}} & t = j/n_k, n_{k-1} < j \le n_k \\ \text{linearly interpolated elsewhere for } 0 \le t \le 1. \end{cases} \tag{3.7}$$

Then the processes $\{\tilde{\eta}_{n_k}\}$ are independent, and we now choose our subsequence $\{n_k\}$ of $\{n'\}$ such that

$$\max_{1\le j\le n_{k-1}} \frac{\|S_j\|}{d_k} \to_{k\to\infty} 0 \quad \text{w.p.1}, \tag{3.8}$$

and if

$$\beta_k = \max\{\rho(\sqrt{2Lk}\,\eta_{n_k}^{(\cdot)}, W), \rho(\sqrt{2Lk}\,\tilde{\eta}_{n_k}^{(\cdot)}, W)\}, \tag{3.9}$$

then

$$\textstyle\sum_k \beta_k < \infty. \tag{3.10}$$

Then, by Theorem 4.3 of Kuelbs (1976) we have

$$P(\{\tilde{\eta}_{n_k}(\cdot)\} \twoheadrightarrow \mathscr{K}) = 1 \tag{3.11}$$

and since (3.8) holds (3.11) implies

$$P(\{\eta_{n_k}(\cdot)\} \twoheadrightarrow \mathscr{K}) = 1. \tag{3.12}$$

Thus (2.5) holds if we can prove

$$\lim_n \| \eta_n - \mathscr{K} \|_{C_B} = 0 \quad \text{w.p.1.} \tag{3.13}$$

Here, of course, $\| \eta_n - \mathscr{K} \|_{C_B} = \inf_{g \in \mathscr{K}} \| \eta_n - g \|_{C_B}$. Further, by Lemma 4-c and the argument used in the proof of Corollary 2 of the paper of Kuelbs and LePage (1973) we have

$$K = \{f(1) : f \in \mathscr{K}\}, \tag{3.14}$$

and hence (2.5) implies (2.4). Thus the proof is complete once (3.13) is established.

To prove (3.13) we recall the maps $\Pi_N : B \to H_{\mathscr{L}(Z)}$ given in Lemma 2.1 of Kuelbs (1976), and then we proceed in two steps. First, we fix $\varepsilon > 0$ and prove that there exists an N such that if I is the identity map on B, then

$$\lim_n \| (I - \Pi_N)\eta_n(\cdot) \|_{C_B} \le \varepsilon \quad \text{w.p.1.} \tag{3.15}$$

Of course, if $f \in C_B$ by $\Pi_N f$ we mean the function such that $(\Pi_N f)(t) = \Pi_N(f(t))$. After establishing (3.15) we show

$$\limsup_n \| \Pi_N \eta_n - \mathscr{K} \|_{C_B} \le \varepsilon \quad \text{w.p.1,} \tag{3.16}$$

and since (3.15) and (3.16) yield (3.13) the proof will be complete.

To verify (3.15) set

$$B_k = \{\| (I - \Pi_N)\eta_n^{(\cdot)} \|_{C_B} > \varepsilon \quad \text{for some } n \in (n_{k-1}, n_k]\} \tag{3.17}$$

where N is such that

$$\textstyle\sum_k P(\| (I - \Pi_N)Z \| > (\varepsilon/4)\sqrt{2Lk}) < \infty. \tag{3.18}$$

The existence of an N such that (3.18) holds is proved on the bottom of page 758 and the top of page 759 of Kuelbs (1976). Fixing N such that (3.18) holds we next prove that for all sufficiently large k

$$\begin{aligned} P(B_k) &= P(\sup_{n_{k-1} < n \le n_k} \sup_{1 \le j \le n} \| (I - \Pi_N)(S_j - j\delta_{n_k}/n_k) \| > \varepsilon\, d_k \sqrt{2Lk}) \\ &= P\left(\sup_{j \le n_k} \| (I - \Pi_N)\left(S_j - j\frac{\delta_{n_k}}{n_k}\right) \| > \varepsilon\, d_k \sqrt{2Lk}\right) \\ &\le 2P\left(\| (I - \Pi_N)(S_{n_k} - \delta_{n_k}) \| > \frac{\varepsilon}{2}\, d_k \sqrt{2Lk}\right). \end{aligned} \tag{3.19}$$

To verify that (3.19) holds note that the first two equalities are a matter of the definitions of the quantities involved, so we must only verify the inequality in (3.19). To do this we observe that if $T_1, \cdots, T_N$ are successive sums of independent B-valued random variables such that

$$\sup_{1 \le j \le N} P(\| T_N - T_j \| > \varepsilon/2) = c < 1,$$

then by the same proof as used when B is the real line we have

$$P(\sup_{1 \le j \le N} \| T_j \| \ge \lambda + \varepsilon) \le \frac{1}{1 - c} P(\| T_N \| \ge \lambda + \varepsilon/2).$$

Hence the inequality in (3.19) holds if we show for all sufficiently large k that

$$\sup_{1 \le n \le n_k} P\left(\| S_{n_k} - \delta_{n_k} - S_n + \frac{n\delta_{n_k}}{n_k} \| > (\varepsilon/2)\gamma_{n_k}\right) < 1/2. \tag{3.20}$$

To verify (3.20) we define

$$U_n = \sum_{j=1}^{n} X_j I(\|X_j\| \le d_k) \quad (1 \le n \le n_k)$$

and set

$$V_n = S_n - U_n \quad (1 \le n \le n_k).$$

Then $E(U_{n_k} - U_n) = \delta_{n_k} - n\delta_{n_k}/n_k$ and

$$\begin{aligned} P\left(\left\|S_{n_k} - \delta_{n_k} - S_n + n\frac{\delta_{n_k}}{n_k}\right\| > (\varepsilon/2)\gamma_{n_k}\right) &\le P(\| U_{n_k} - U_n - E(U_{n_k} - U_n)\| > (\varepsilon/4)\gamma_{n_k}) \\ &\quad + P(\| V_{n_k} - V_n\| > (\varepsilon/4)\gamma_{n_k}) \\ &\le 4E\| U_{n_k} - U_n - E(U_{n_k} - U_n)\|/\varepsilon\gamma_{n_k} \\ &\quad + (n_k - n)P(\|X\| > d_k). \\ &\le 4E\| U_{n_k} - E(U_{n_k})\|/\varepsilon\gamma_{n_k} + n_k P(\|X\| > d_k). \end{aligned} \tag{3.21}$$

Since (2.3) holds we have by Araujo, de Acosta, Giné (1978), Corollary 2.12, that

$$\lim_k n_k P(\|X\| > d_k) = 0, \tag{3.22}$$

and

$$\mathscr{L}\left(\frac{U_{n_k} - EU_{n_k}}{d_k}\right) \to \mathscr{L}(Z). \tag{3.23}$$

Since U_{n_k} is a sum of independent random variables truncated at d_k and (3.23) holds we have by standard arguments (see, for example, Kuelbs (1977) pages 787–788 for details or Lemma 6.1a below) that

$$\sup_k E\frac{\| U_{n_k} - EU_{n_k}\|}{d_k} < \infty. \tag{3.24}$$

Hence for sufficiently large k (3.21), (3.22), and (3.24) imply (3.20) and thus (3.19) holds.

Now let $\theta: C_B[0, 1] \to B$ be defined by $\theta(f) = f(1)$ and note that

$$(I - \Pi_N)(\theta\eta_{n_k}) = (I - \Pi_N)\frac{(S_{n_k} - \delta_{n_k})}{d_k\sqrt{2Lk}}.$$

Further, since $(I - \Pi_N)(\theta)$ is linear and continuous it has a finite norm, call it M, such that

$$\|(I - \Pi_N)\theta(f - g)\| \le M\|f - g\|_{C_B}. \tag{3.25}$$

Hence by (3.19), (3.25), and the definition of the Prokhorov metric, (3.9) implies for all sufficiently large k that

$$\begin{aligned} P(B_k) &\le 2P(\|(I - \Pi_N)(S_{n_k} - \delta_{n_k})\| > (\varepsilon/2)\cdot d_k\sqrt{2Lk}) \\ &\le 2P((I - \Pi_N)(\theta(\sqrt{2Lk}\ \eta_{n_k}^{(\cdot)})) \in F_k) \quad \text{where } F_k = \{x:\| x\| \ge (\varepsilon/2)\cdot\sqrt{2Lk}\} \\ &= 2P(\sqrt{2Lk}\ \eta_{n_k} \in \psi^{-1}(F_k)) \quad \text{where } \psi = (I - \Pi_N)\circ\theta \\ &\le 2P(W \in (\psi^{-1}(F_k))^{2\beta_k}) + 2\beta_k \quad \text{where } E^{\delta} = \{y:\inf_{x\in E} d(x, y) < \delta\} \\ &\le 2P(\psi(W) \in F_k^{2M\beta_k}) + 2\beta_k \\ &= 2P((I - \Pi_N)(Z) \in F_k^{2M\beta_k}) + 2\beta_k \\ &\le 2P((I - \Pi_N)(Z) \in F_k^{\varepsilon/4}) + 2\beta_k \\ &\le 2P(\|(I - \Pi_N)(Z)\| \ge \varepsilon/4\ \sqrt{2Lk}) + 2\beta_k. \end{aligned} \tag{3.26}$$

Hence by (3.10) and (3.18) we have $\sum_k P(B_k) < \infty$, and hence (3.15) holds.

To prove (3.16) recall that N is fixed, and that from Lemma 4 of Kuelbs and LePage (1973) we have $\Pi_N \mathscr{K} \subseteq \mathscr{K}$. Then as a result of (3.12) and the continuity of Π_N we have

$$\|\Pi_N \eta_{n_k} - \Pi_N \mathscr{K}\|_{C_B} \to 0 \quad \text{w.p.1}, \tag{3.27}$$

and hence (3.16) will follow (since $\Pi_N \mathscr{K} \subseteq \mathscr{K}$) once we establish that

$$\sup_{n_{k-1} < n \leq n_k} \|\Pi_N \eta_n - \Pi_N \mathscr{K}\|_{C_B} \leq \|\Pi_N \eta_{n_k} - \Pi_N \mathscr{K}\|_{C_B}. \tag{3.28}$$

To verify (3.28) set $m_k = E(XI(\|X\| \leq d_k))$ and define the maps $\Gamma_n : C_B[0, 1] \to C_B[0, 1]$ as

$$(\Gamma_n f)(t) = \begin{cases} f(j/n) & t = j/n, j = 0, 1, \cdots, n \\ \text{linear interpolated elsewhere on } [0, 1]. \end{cases}$$

Then for any $n \in (n_{k-1}, n_k]$ we have

$$\|\Pi_N \eta_n - \Pi_N \mathscr{K}\|_{C_B} = \inf_{h \in \mathscr{K}} \|\Pi_N \eta_n - \Pi_N h\|_{C_B} = \inf_{h \in \mathscr{K}} \|\Pi_N \eta_n - \Pi_N \Gamma_n h\|_{C_B}, \tag{3.29}$$

where the first equality (3.29) is a matter of definition and the second results from the fact that $\Pi_N \eta_n$ is a polygonal process. That is, by applying Lemma 4 of Kuelbs and LePage (1973) we see that $h \in \mathscr{K}$ implies $\Pi_N \Gamma_n h \in \mathscr{K}$, and hence since $\Pi_N^2 = \Pi_N$ we have

$$\inf_{h \in \mathscr{K}} \|\Pi_N \eta_n - \Pi_N h\|_{C_B} \leq \inf_{h \in \mathscr{K}} \|\Pi_N \eta_n - \Pi_N \Gamma_n h\|_{C_B}. \tag{3.30}$$

On the other hand,

$$\begin{aligned} \|\Pi_N \eta_n - \Pi_N h\|_{C_B} &= \sup_{0 \leq t \leq 1} \|\Pi_N \eta_n(t) - \Pi_N h(t)\| \\ &\geq \sup_{1 \leq j \leq n} \|\Pi_N \eta_n(j/n) - \Pi_N h(j/h)\| \\ &= \|\Pi_N \eta_n - \Pi_N \Gamma_n h\|_{C_B}, \end{aligned} \tag{3.31}$$

so combining (3.30) and (3.31) we have (3.29). Letting $n = n_k$ in (3.29) and recalling $\mathscr{K}$ is compact in $C_B[0, 1]$ there exists $g_0 \in \mathscr{K}$ such that for all $n \in (n_{k-1}, n_k]$

$$\begin{aligned} \|\Pi_N \eta_{n_k} - \Pi_N \mathscr{K}\|_{C_B} &= \|\Pi_N \eta_{n_k} - \Pi_N \Gamma_{n_k} g_0\|_{C_B} \\ &= \sup_{1 \leq j \leq n_k} \left\|\Pi_N\left(\frac{S_j - jm_k}{d_k \sqrt{2Lk}}\right) - \Pi_N g_0(j/n_k)\right\| \\ &\geq \sup_{1 \leq j \leq n} \left\|\Pi_N\left(\frac{S_j - jm_k}{d_k \sqrt{2Lk}}\right) - \Pi_N h_0(j/n)\right\| \\ &= \|\Pi_N \eta_n - \Pi_N \Gamma_n h_0\|_{C_B} \end{aligned} \tag{3.32}$$

where $h_0(t) = g_0(tn/n_k)$, $0 \leq t \leq 1$. Applying Lemma 4 of Kuelbs and Le Page (1973) again, it is easy to see that $g_0 \in \mathscr{K}$ implies $\Pi_N h_0 \in \mathscr{K}$, and hence (3.32) implies

$$\begin{aligned} \|\Pi_N \eta_{n_k} - \Pi_N \mathscr{K}\|_{C_B} &\geq \sup_{n_{k-1} < n \leq n_k} \|\Pi_N \eta_n - \Pi_N \Gamma_n h_0\|_{C_B} \\ &\geq \sup_{n_{k-1} < n \leq n_k} \inf_{h \in \mathscr{K}} \|\Pi_N \eta_n - \Pi_N h\|_{C_B} \end{aligned} \tag{3.33}$$

since $\Pi_N h_0 \in \mathscr{K}$ implies $\Pi_N \Gamma_n h_0 \in \mathscr{K}$. Now the last expression in (3.33) equals $\sup_{n_{k-1} < n \leq n_k} \|\Pi_N \eta_n - \Pi_N \mathscr{K}\|_{C_B}$, so (3.33) implies (3.28) and Theorem 1 is proved.

REMARK. It was pointed out in the proof of Theorem 1 that (2.5) implies (2.4), but it might be useful to indicate a direct proof of (2.4) as well.

What needs to be done to prove (2.4) is to choose a subsequence $\{n_k\}$ with the properties

$$S_{n_{k-1}}/d_k \to_{k \to \infty} 0 \quad \text{w.p.1}, \tag{3.34}$$

and, if

$$\alpha_k = \max\left\{\rho\left(\frac{S_{n_k} - S_{n_{k-1}} - \delta_{n_k}}{d_k}, Z\right), \rho\left(\frac{S_{n_k} - \delta_{n_k}}{d_k}, Z\right)\right\}, \tag{3.35}$$

then

$$\textstyle\sum_k \alpha_k < \infty. \tag{3.36}$$

Then, setting

$$Y_k = (S_{n_k} - S_{n_{k-1}} - \delta_{n_k})/d_k \quad (k \geq 1), \tag{3.37}$$

we have by Theorem 4.3 of Kuelbs (1976) that

$$P(Y_k/\sqrt{2Lk} \twoheadrightarrow K) = 1. \tag{3.38}$$

Hence by applying (3.34) we have

$$P((S_{n_k} - \delta_{n_k})/\sqrt{2Lk}\, d_k \twoheadrightarrow K) = 1. \tag{3.39}$$

Hence (2.4) holds if we can prove

$$\lim_n \| (S_n - \delta_n)/\gamma_n - K \| = 0 \quad \text{w.p.1.} \tag{3.40}$$

To establish (3.40) fix $\varepsilon > 0$. Then for $k \geq 1$ we define

$$C_k = \left\{\frac{S_n - \delta_n}{\gamma_n} \notin K^\varepsilon \text{ for some } n \in (n_{k-1}, n_k]\right\},$$

and hence (3.40) holds if

$$\textstyle\sum_k P(C_k) < \infty. \tag{3.41}$$

To verify (3.41) we argue as in (3.19) to show that for k sufficiently large we have

$$P(C_k) \leq 2P\left(\frac{S_{n_k} - \delta_{n_k}}{\gamma_{n_k}} \notin K^{\varepsilon/2}\right).$$

Now by our choice of $\{\alpha_k\}$ in (3.35)

$$P\left(\frac{S_{n_k} - \delta_{n_k}}{\gamma_{n_k}} \notin K^{\varepsilon/2}\right) \leq P(Z \notin K^{\varepsilon/4}\sqrt{2Lk}) + \alpha_k$$

for all sufficiently large k, and since $\sum_k \alpha_k < \infty$ we have (3.41) if

$$\textstyle\sum_k P(Z \notin K^{\varepsilon/4}\sqrt{2Lk}) < \infty. \tag{3.42}$$

That (3.42) holds follows from the proof of Corollary 3.2 of Kuelbs (1976), and hence the proof of (2.4) is complete.

4. Proof of Theorem 2. If (2.6) fails there exists $c > 0$ such that for all $u > 0$

$$u^2 P(\|X\| > u) \geq c \int_{\|x\| \leq u} \|x\|^2 \, dP_X(x), \tag{4.1}$$

and hence for $\rho > 1$, $u > 0$

$$u^2[P(\|X\| > u) - P(\|X\| > \rho u)] \leq \int_{u < \|x\| \leq \rho u} \|x\|^2 \, dP_X(x) \leq \rho^2 u^2 P(\|X\| > \rho u)/c. \tag{4.2}$$

Thus for $\rho > 1$, $u > 0$

$$P(\|X\| > \rho u) \leq P(\|X\| \leq u) \leq (1 + \rho^2/c)P(\|X\| > \rho u), \tag{4.3}$$

and hence for $0 < \varepsilon < M$ we have

$$P(\|X\| > M\gamma_n) \le P(\|X\| \ge \varepsilon\gamma_n) \le \left(1 + \frac{M^2}{\varepsilon^2 c}\right) P(\|X\| > M\gamma_n). \tag{4.4}$$

Thus if (2.6) fails (4.4) implies that

$$\textstyle\sum_n P(\|X_n\| > \varepsilon\gamma_n) < \infty \tag{4.5}$$

for some $\varepsilon > 0$ iff it converges for all $\varepsilon > 0$.

Hence $\sum_n P(\|X\| > \gamma_n) = \infty$ implies $\sum_n P(\|X_n\| > M\gamma_n) = \infty$ for all $M > 0$, and thus with probability one

$$\limsup_n \frac{\|S_n - S_{n-1}\|}{\gamma_n} = \limsup_n \frac{\|X_n\|}{\gamma_n} \ge M. \tag{4.6}$$

Thus

$$\limsup_n \frac{\|S_n\|}{\gamma_n} \ge \frac{M}{2} \quad \text{w.p.1},$$

and since M was arbitrary we have

$$\limsup_n \frac{\|S_n\|}{\gamma_n} = \infty \quad \text{w.p.1}.$$

On the other hand, if $\sum_n P(\|X_n\| > \varepsilon\gamma_n) < \infty$ for all $\varepsilon > 0$, we set

$$Y_k = \begin{cases} X_k & \text{if} \quad \|X_k\| \le \gamma_k \\ 0 & \text{if} \quad \|X_k\| > \gamma_k \end{cases}$$

and let

$$T_n = \textstyle\sum_{k=1}^n Y_k.$$

Then

$$\textstyle\sum_k P(\|Y_k - X_k\| > 0) = \sum_k P(\|X_k\| > \gamma_k) < \infty$$

so

$$\limsup_n \frac{\|S_n - T_n\|}{\gamma_n} = 0 \quad \text{w.p. 1}. \tag{4.7}$$

Now (4.1) implies

$$E\|Y_k\|^2 = E(\|X_k\|^2 I(\|X_k\| \le \gamma_k)) \le \frac{\gamma_k^2}{c} P(\|X_k\| \ge \gamma_k)$$

and hence $\sum_n P(\|X_k\| \ge \gamma_k) < \infty$ implies we have

$$\textstyle\sum_k \dfrac{E\|Y_k\|^2}{\gamma_k^2} < \infty. \tag{4.8}$$

Now B type 2 and (4.8) implies $\sum_k (Y_k/\gamma_k)$ converges with probability one. Since $\gamma_n \nearrow \infty$ Kronecker's lemma implies

$$\lim_n \frac{\|T_n\|}{\gamma_n} = 0 \quad \text{w.p. 1}. \tag{4.9}$$

Combining (4.7) and (4.9) we have

$$\lim_n \frac{\|S_n\|}{\gamma_n} = 0 \quad \text{w.p. 1}. \tag{4.10}$$

Hence (2.7) holds.

Furthermore, if (2.6) fails the above proves that LIL behavior with respect to the centerings $\delta_n = 0$ is impossible. That is, if

$$0 < \limsup_n \frac{\| S_n \|}{\gamma_n} < \infty \quad \text{w.p. } 1, \tag{4.11}$$

then

$$\limsup_n \frac{\| X_n \|}{\gamma_n} < \infty \quad \text{w.p. } 1 \tag{4.12}$$

and hence there is an $M < \infty$ such that

$$\textstyle\sum_n P(\| X_n \| > M\gamma_n) < \infty. \tag{4.13}$$

Thus by the above truncation argument we have (4.10) which contradicts (4.11). Hence LIL behavior with respect to the centerings $\delta_n = 0$, implies (2.6) holds as claimed. Thus Theorem 2 is proved.

Proof of the remark following Theorem 2. If B is a Hilbert space and X has LIL behavior with respect to the centerings $\delta_n = 0$, then the above implies (2.6). Hence choose $d_k \nearrow \infty$ such that

$$\lim_k \frac{d_k^2 P(\| X \| > d_k)}{U(d_k)} = 0 \tag{4.14}$$

where $U(t) = \int_{\|x\| \le t} \| x \|^2 \, dP_X(x)$. Now $\lim_{t \to \infty} U(t)/t^2 = 0$ so choose $n_k \nearrow \infty$ such that

$$\lim_k n_k \frac{U(d_k)}{d_k^2} = 1. \tag{4.15}$$

Thus

$$\begin{aligned} P\left(\frac{\| S_{n_k} \|}{d_k} > \lambda\right) &\le n_k P(\| X \| > d_k) + P\left(\frac{\| T_{n_k} \|}{d_k} > \lambda\right) \\ &\le n_k P(\| X \| > d_k) + E \| T_{n_k} \|^2 / \lambda^2 d_k^2 \end{aligned} \tag{4.16}$$

where $T_{n_k} = \sum_{j=1}^{n_k} X_j I(\| X_j \| \le d_k)$. Now B a Hilbert space and X symmetric implies

$$P\left(\frac{\| S_{n_k} \|}{d_k} > \lambda\right) \le n_k P(\| X \| > d_k) + \frac{n_k}{\lambda^2 d_k^2} E(\| X \|^2 I(\| X \| \le d_k)), \tag{4.17}$$

and hence (4.14) and (4.15) imply that

$$\limsup_k P\left(\frac{\| S_{n_k} \|}{d_k} > \lambda\right) \le \limsup_k \frac{n_k U(d_k)}{d_k^2} \left[\varepsilon(d_k) + \frac{1}{\lambda^2}\right] \le \frac{1}{\lambda^2} \tag{4.18}$$

where $\varepsilon(d_k) \to 0$ as $k \to \infty$. Hence (2.8) holds, and since B is a Hilbert space and X symmetric we have

$$\frac{E \| T_{n_k} \|^2}{d_k^2} = n_k E(\| X \|^2 I(\| X \| \le d_k))/d_k^2 = \frac{n_k U(d_k)}{d_k^2} \to_{k \to \infty} 1$$

by (4.15). Thus (2.9) holds and the remark is proved.

5. The proof of Lemma 1, Theorem 3, and its corollaries.

First we will prove Lemma 1. The notation is that established in (2.11)–(2.16) and (2.21)–(2.24).

5.1. *Proof of Lemma* 1. If B is type 2, there is a constant A, $1 \le A < \infty$, such that

$$E \| X_1 + \cdots + X_n \|^2 \le A \textstyle\sum_{j=1}^n E \| X_j \|^2$$

whenever $X_1, \cdots, X_n$ are independent with $EX_j = 0$ $(1 \le j \le n)$. Thus if $X, X_1, X_2, \cdots$ are i.i.d. with $EX = 0$ we will first show that

$$E\|S_n\| \le \tfrac{5}{2}A^{1/2}K(n). \tag{5.1}$$

To establish (5.1) fix $b > 0$ and set $S_n(b) = \sum_{j=1}^n X_j I(\|X_j\| \le b)$, $U_n(b) = \sum_{j=1}^n X_j I(\|X_j\| > b)$. Then $S_n = S_n(b) + U_n(b)$, and $ES_n = 0$ implies $ES_n(b) = -EU_n(b) = -nE(XI(\|X\| > b))$. Hence

$$\begin{aligned} E\|S_n\| &= E\|S_n(b) + U_n(b)\| \\ &\le E\|S_n(b) - ES_n(b)\| + \|ES_n(b)\| + E\|U_n(b)\| \\ &\le (E\|S_n(b) - ES_n(b)\|^2)^{1/2} + 2nE(\|X\| I(\|X\| > b)) \\ &\le \{AnE\|XI(\|X\| \le b) - E(XI(\|X\| \le b))\|^2\}^{1/2} \\ &\quad + 2nE(\|X\| I(\|X\| > b)) \\ &\le 2A^{1/2}\{nE(\|X\|^2 I(\|X\| \le b))\}^{1/2} + 2nE(\|X\| I(\|X\| > b)). \end{aligned} \tag{5.2}$$

Setting $b = K(n)$ in (5.2) we have from (2.16) that

$$E\|S_n\| \le A^{1/2}[\lambda_n K^2(n)]^{1/2} + 2(1 - \lambda_n)K(n) \tag{5.3}$$

where $\lambda_n K^2(n) = nE(\|X\|^2 I(\|X\| \le K(n)))$ and $0 \le \lambda_n \le 1$. Therefore, since $1 \le A$ we have

$$E\|S_n\| \le A^{1/2}K(n)[\{4\lambda_n\}^{1/2} + 2(1 - \lambda_n)] = A^{1/2}K(n)[2(1 - \lambda_n + \lambda_n^{1/2})] \le \tfrac{5}{2}A^{1/2}K(n)$$

by maximizing the function $g(\lambda) = 1 - \lambda + \lambda^{1/2}$, $0 \le \lambda \le 1$. Hence if B is type 2 we have

$$E\|S_n\| \le A_1 K(n) \tag{5.4}$$

where $A_1 = \tfrac{5}{2}A^{1/2} < \infty$.

Now assume (5.4) holds whenever $X, X_1, X_2, \cdots$ are i.i.d. with $EX = 0$. To show B is type 2 we also assume $E\|X\|^2 < \infty$. Then, from (2.14) we have $K^2(n) \le nE\|X\|^2$, and since we are assuming $E\|S_n\| \le AK(n)$ we see that

$$E\|S_n\| \le A\sqrt{n}\,(E\|X\|^2)^{1/2}. \tag{5.5}$$

Since the inequality (5.5) applies to all mean zero B valued X, standard approximation and tightness arguments easily imply that B has the property that, for each B-valued random variable X with $EX = 0$, $E\|X\|^2 < \infty$, we have X satisfying the classical CLT. Hence by Hoffman-Jørgensen and Pisier (1976) we have B of type 2.

To establish (2.19) first assume B is of cotype 2. Then, with $\{\varepsilon_i\}$ independent Rademacher random variables, we have

$$\begin{aligned} E\|S_n\| &\ge A'E[(\textstyle\sum_{j=1}^n \|X_j\|^2)^{1/2}] && \text{since } B \text{ is cotype 2} \\ &\ge A''E_X E_\varepsilon |\textstyle\sum_{j=1}^n \|X_j\| \varepsilon_j| && \text{by Khintchine's inequality} \\ &\ge A'''K_{\|X\|\varepsilon}(n) \end{aligned} \tag{5.6}$$

where the constants A', A'', A''' are positive, the first inequality follows since B is cotype 2, the second inequality results from Khintchine's inequality, and the final inequality follows from Klass's Theorem 1.1 (Klass, 1976). Now $K_X = K_{\|X\|\varepsilon}$, so (5.6) implies $E\|S_n\| \ge A_2K(n)$ for some constant $A_2 > 0$ whenever B is cotype 2.

Hence (2.19) will hold if we show

$$E\|S_n\| \ge A_2K(n) \quad (n \ge 1) \tag{5.7}$$

for some constant $A_2 > 0$, implies B is cotype 2. To show B is cotype 2 assume X satisfies the classical CLT. Then we have a mean zero Gaussian random variable Z such that

$$\lim_n E\left\|\frac{S_n}{\sqrt{n}}\right\| = E\|Z\| < \infty. \tag{5.8}$$

Combining (5.7) and (5.8) we see that

$$\limsup_n \frac{K(n)}{\sqrt{n}} \le \frac{E\|Z\|}{A_2} < \infty.$$

and hence $E\|X\|^2 < \infty$ by (2.14). Hence X satisfying the classical CLT implies $E\|X\|^2 < \infty$, and as a result of Jain (1977) we have B of cotype 2. Thus (2.19) holds, and to verify that (2.20) is equivalent to B being isomorphic with Hilbert space we simply apply (2.18), (2.19), and Kwapien's result (1972).

5.2. *Proof of Theorem* 3. The proof of Theorem 3 will proceed via several lemmas. The first is:

LEMMA 5.1. *If Theorem* 3 *holds whenever* (2.26) *holds with* $M = 1$, *then it holds in general.*

PROOF. If $M \le 1$, then (2.26) holding for M implies it also holds for $M = 1$. As a result of our assumption, (2.25) then holds whenever $M \le 1$. Hence we assume (2.26) holds for some $M > 1$. Thus we have

$$P\left(\frac{\|X_n\|}{M} > \gamma_n \text{ i.o.}\right) = 0.$$

Hence by our assumption we have (2.25) holding when $\{X_j\}$ is replaced by $\{X_j/M\}$ provided

$$\gamma_n \ge L_2 n K(n/L_2 n) \tag{5.9}$$

implies

$$\gamma_n \ge L_2 n K_{X/M}(n/L_2 n). \tag{5.10}$$

To see that (5.9) implies (5.10) it suffices to observe that the function G as defined in (2.11) satisfies

$$G_{X/M}(y) = G_X(My), \tag{5.11}$$

and hence the inverse functions K are easily seen to satisfy

$$K_{X/M}(y) = \frac{1}{M} K_X(y). \tag{5.12}$$

Hence (5.9) and (5.12) with $M > 1$ imply (5.10) and the lemma is proved.

Our goal now is to establish (2.25) provided (2.26) holds with $M = 1$, and (2.23) and (2.24) are also satisfied.

Now for $\Lambda' > 0$

$$\begin{aligned} &P(\|S_n\| - E\|S_{\beta n}\| > \Lambda'\gamma_n \text{ i.o.}) \\ &\qquad = \lim_{r\to\infty} P(\cup_{n>n_r}\{\|S_n\| > \Lambda'\gamma_n + E\|S_{\beta n}\|\}) \\ &\qquad \le \lim_{r\to\infty} \textstyle\sum_{k\ge r} P(\{\max_{n_k<n\le n_{k+1}} \|S_n\| \ge \Lambda'\gamma_{n_k} + E\|S_{\beta n_k}\|\}). \end{aligned} \tag{5.13}$$

To estimate

$$P(\{\max_{n_k<n\le n_{k+1}} \|S_n\| > \Lambda'\gamma_{n_k} + E\|S_{\beta n_k}\|\})$$

we define for $1 \le j \le n_{k+1}$ the truncated random variables (depending on k)

$$u_j = X_j I(\|X_j\| \le K(n_k/L_2 n_k)), \quad v_j = X_j I(K(n_k/L_2 n_k) < \|X_j\| \le \gamma_{n_k})$$

$$w_j = X_j I(\gamma_{n_k} < \|X_j\|),$$

and set

$$U_n = \sum_{j=1}^n u_j, \quad V_n = \sum_{j=1}^n v_j, \quad W_n = \sum_{j=1}^n w_j.$$

Then $S_n = U_n + V_n + W_n$, and hence

$$P(\{\max_{n_k<n\le n_{k+1}} \| S_n \| > \Lambda'\gamma_{n_k} + E\| S_{\beta n_k} \|\}) \le I_{1,k} + I_{2,k}, + I_{3,k} \tag{5.14}$$

where

$$\begin{aligned} I_{1,k} &= P(\{\max_{n_k<n\le n_{k+1}} \| U_n \| \ge \Lambda\gamma_{n_k} + E\| S_{\beta n_k} \|\}) \\ I_{2,k} &= P(\{\max_{n_k<n\le n_{k+1}} \| V_n \| \ge \Lambda\gamma_{n_k}\}) \\ I_{3,k} &= P(\{\max_{n_k<n\le n_{k+1}} \| W_n \| \ge \Lambda\gamma_{n_k}\}) \end{aligned} \tag{5.15}$$

and $\Lambda = \Lambda'/3$.

In view of (5.13), (5.14), and (5.15) we can establish (2.25), completing the proof of the theorem, provided there exists $\Lambda < \infty$ such that

$$\lim_{r\to\infty} \sum_{k\ge r} I_{j,k} = 0 \tag{5.16}$$

for $j = 1, 2, 3$.

Since $\beta > 1$ and $n_k = [\beta^k]$ we have for all k sufficiently large, say $k \ge k_0$, that $\beta^k > 3$ and

(5.17) (i) $n_{k+1} < 2\beta n_k \quad (k \ge k_0)$
(ii) there exist finite constants b and c such that

$$1 < b < \frac{n_{k+1}}{n_k} < c \quad (k \ge k_0).$$

As a result of (5.17-ii) elementary computations imply

$$n_{k+1} \le \frac{bc}{b-1}(n_k - n_k) \quad (k \ge k_0). \tag{5.18}$$

LEMMA 5.2. *If $\Lambda > 0$, then* $\lim_{r\to\infty} \sum_{k\ge r} I_{3,k} = 0$.

PROOF. Since

$$P(\max_{n_k<n\le n_{k+1}} \| W_n \| > \Lambda\gamma_{n_k}) \le n_{k+1}P(\| X \| > \gamma_{n_k}),$$

$\gamma_n \nearrow \infty$, and (5.18) holds for $k \ge k_0$, we have for $r \ge k_0$ that

$$\begin{aligned} \sum_{k\ge r} I_{3,k} &\le \sum_{k\ge r} n_{k+1}P(\| X \| > \gamma_{n_k}) \le \frac{bc}{b-1}\sum_{k\ge r}(n_k - n_{k-1})P(\| X \| > \gamma_{n_k}) \\ &\le \frac{bc}{b-1}\sum_{n\ge n_{r-1}} P(\| X > \gamma_n). \end{aligned}$$

Since (2.26) holds with $M = 1$ we have $\sum_n P(\| X \| > \gamma_n) < \infty$, and hence the lemma is proved.

LEMMA 5.3. *If $\Lambda > 4\beta$ and the regularity condition* (2.24) *holds, then* $\lim_{r\to\infty} \sum_{k\ge r} I_{2,k} = 0$.

PROOF. Using (2.16) and (5.17) we have for all $k \ge k_0$ and n such that $n_k < n \le n_{k+1}$ that

$$\begin{aligned} E\| V_n \| &\le n_{k+1}E(\| X \| I(K(n_k/L_2n_k) < \| X \|)) \\ &\le 2\beta n_k E(\| X \| I(K(n_k/L_2n_k) < \| X \|)) \\ &\le 2\beta L_2 n_k K(n_k/L_2n_k) \le 2\beta\gamma_{n_k}. \end{aligned} \tag{5.19}$$

Thus for $k \geq k_0$

$$
\begin{aligned}
I_{2,k} &\leq P(\max_{n_k<n\leq n_{k+1}} \| V_n - EV_n \| + \max_{n_k<n\leq n_{k+1}} \| EV_n \| \geq \Lambda \gamma_{n_k}) \\
&\leq P(\max_{n_k<n\leq n_{k+1}} \| V_n - EV_n \| \geq (\Lambda - 2\beta)\gamma_{n_k}) \\
&= P(\max_{n_k<n\leq n_{k+1}} \| V_n - EV_n \| - E\| V_{n_{k+1}} - EV_{n_{k+1}} \| \\
&\qquad \geq (\Lambda - 2\beta)\gamma_{n_k} - E\| V_{n_{k+1}} - EV_{n_{k+1}} \|) \\
&\leq P(\max_{n_k<n\leq n_{k+1}} \| V_n - EV_n \| - E\| V_{n_{k+1}} - EV_{n_{k+1}} \| \geq (\Lambda - 4\beta)\gamma_{n_k}).
\end{aligned} \tag{5.20}
$$

Now $\{\| V_n - EV_n \| : n_k < n \leq n_{k+1}\}$ is a submartingale and hence so is $\{\phi(\| V_n - EV_n \| - \theta) : n_k < n \leq n_{k+1}\}$ for any constant θ where $\phi(t) = t^4$ for $t \geq 0$ and zero otherwise. Hence by the maximal inequality for submartingales we have for $k \geq k_0$ and $\Lambda > 4\beta$ that

$$
\begin{aligned}
I_{2,k} &\leq E[\phi(\| V_{n_{k+1}} - EV_{n_{k+1}} \| - E\| V_{n_{k+1}} - EV_{n_{k+1}} \|)]/(\Lambda - 4\beta)^4\gamma_{n_k}^4 \\
&\leq E|\| V_{n_{k+1}} - EV_{n_{k+1}} \| - E\| V_{n_{k+1}} - EV_{n_{k+1}} \||^4/(\Lambda - 4\beta)^4\gamma_{n_k}^4.
\end{aligned} \tag{5.21}
$$

Now by applying the generalization of Rosenthal's inequality indicated in A. de Acosta (1981) to $\{v_j - Ev_j : 1 \leq j \leq n_{k+1}\}$ we have for all $\Lambda > 4\beta$ a constant C_Λ independent of k for $k \geq k_0$ such that

$$
\begin{aligned}
I_{2,k} &\leq \frac{C_\Lambda}{\gamma_{n_k}^4} \{(\textstyle\sum_{j=1}^{n_{k+1}} E\| v_j \|^2)^2 + \sum_{j=1}^{n_{k+1}} E\| v_j \|^4\} \\
&= \frac{C_\Lambda}{\gamma_{n_k}^4} \{(n_{k+1}E(\| X \|^2 I(X \in A_k)))^2 + n_{k+1}E(\| X \|^4 I(X \in A_k))\}
\end{aligned} \tag{5.22}
$$

where $A_k = \{x : K(n_k/L_2n_k) < \| x \| \leq \gamma_{n_k}\}$. Then for $k \geq k_0$ the Cauchy-Schwartz inequality implies

$$
\begin{aligned}
z_k &\equiv [n_{k+1}E(\| X \|^2 I(X \in A_k))]^2 \\
&\leq n_{k+1}^2 E(\| X \| I(X \in A_k))E(\| X \|^3 I(X \in A_k)) \\
&\leq 2\beta^2 n_k^2 E(\| X \| I(X \in A_k))E(\| X \|^3 I(X \in A_k))
\end{aligned} \tag{5.23}
$$

since $n_k^2 \geq (\beta^k - 1)^2 \geq \beta^{2k}/2 \geq (n_{k+1})^2/2\beta^2$ whenever $\beta^k \geq 3$. Now by using the definition of the function $K(\cdot)$ we see that

$$n_k E(\| X \| I(X \in A_k)) \leq K(n_k/L_2n_k)L_2n_k = \alpha_{n_k} \leq \gamma_{n_k},$$

so we have for $k \geq k_0$ that

$$z_k \leq 2\beta^2 n_k E(\| X \|^3 I(X \in A_k))\gamma_{n_k}. \tag{5.24}$$

Since $E(\| X \|^4 I(X \in A_k)) \leq \gamma_{n_k} E(\| X \|^3 I(X \in A_k))$ it follows from (5.23) and (5.24) that

$$I_{2,k} \leq 4C_\Lambda \beta^2 n_k E(\| X \|^3 I(X \in A_k))/\gamma_{n_k}^3.$$

Now let $\theta(k) = \inf\{j : \gamma_{n_j} \geq K(n_k/L_2n_k)\}$. Since $\gamma_n \geq \alpha_n$, we see $\gamma_{n_j} \geq \alpha_{n_j} \geq L_2n_jK(n_j/L_2n_j) \geq K(n_k/L_2n_k)$ if $j \geq k$, so $\theta(k) \leq k$. Hence for $\theta(r) \geq k_0$ we have

$$
\begin{aligned}
\textstyle\sum_{k\geq r} I_{2,k} &\leq 4C_\Lambda \beta^2 \sum_{k\geq r} \frac{n_k}{(\gamma_{n_k})^3} E(\| X \|^3 I(X \in A_k)) \\
&\leq 4C_\Lambda \beta^2 \sum_{k\geq r} \frac{n_k}{(\gamma_{n_k})^3} \sum_{j=\theta(k)}^{k} E(\| X \|^3 I(\gamma_{n_{j-1}} < \| X \| \leq \gamma_{n_j})) \\
&\leq 4C_\Lambda \beta^2 \sum_{j=\theta(r)}^{\infty} E(\| X \|^3 I(\gamma_{n_{j-1}} < \| X \| \leq \gamma_{n_j})) \sum_{k\geq j} \frac{n_k}{\gamma_{n_k}^3}.
\end{aligned} \tag{5.25}
$$

Now since $\gamma_n = \sqrt{n}h(n)$ where $h(n)$ satisfies (2.24) we see that

$$\sum_{k\ge j} \frac{n_k}{(\gamma_{n_k})^3} = \frac{n_j}{(\gamma_{n_j})^3} \sum_{k\ge j} \frac{n_k}{(\gamma_{n_k})^3} \frac{(\gamma_{n_j})^3}{n_j} = \frac{n_j}{(\gamma_{n_j})^3} \sum_{k\ge j} \frac{n_k}{n_j} \frac{(n_j)^{3/2}}{(n_k)^{3/2}} \left(\frac{h(n_j)}{h(n_k)}\right)^3$$

$$\le \frac{1}{c^3} \frac{n_j}{(\gamma_{n_j})^3} \sum_{k\ge j} \left(\frac{n_j}{n_k}\right)^{1/2} \le \frac{1}{c^3} \frac{n_j}{(\gamma_{n_j})^3} \sum_{k\ge j} \left(\frac{\beta^j}{\beta^k - 1}\right)^{1/2} \tag{5.26}$$

$$\le \frac{2n_j}{c^3(\gamma_{n_j})^3} \sum_{k\ge j} (\beta^{j-k})^{1/2} = \frac{2\sqrt{\beta}}{c^3(\sqrt{\beta}-1)} \frac{n_j}{(\gamma_{n_j})^3}$$

since $\beta^j/(\beta^k - 1) = \beta^{j-k}/(1-\beta^{-k}) \le 4\beta^{j-k}$ if $4(1-\beta^{-k}) \ge 1$. Now for $k \ge \theta(r) \ge k_0$ we have $\beta^k \ge 3$ and hence for such k, $4(1-\beta^{-k}) \ge 1$. Thus for $\theta(r) \ge k_0$ we have

$$\sum_{k\ge r} I_{2,k} \le \frac{8C_\Lambda \beta^2\sqrt{\beta}}{c^3(\sqrt{\beta}-1)} \sum_{j=\theta(r)}^\infty n_j E\left(\left(\frac{\|X\|}{\gamma_{n_j}}\right)^3 I(\gamma_{n_{j-1}} < \|X\| \le \gamma_{n_j})\right)$$

$$\le \frac{8C_\Lambda \beta^2\sqrt{\beta}}{c^3(\sqrt{\beta}-1)} \sum_{j=\theta(r)}^\infty n_j P(\gamma_{n_{j-1}} < \|X\| \le \gamma_{n_j}) \tag{5.27}$$

$$\le \frac{8C_\Lambda \beta^{9/2}}{c^3(\sqrt{\beta}-1)(\beta-1)} \sum_{j=\theta(r)}^\infty (n_{j-1} + 1 - n_{j-2}) P(\gamma_{n_{j-1}} < \|X\| < \gamma_{n_j})$$

since $n_j \le \beta^j = (\beta^2/(\beta-1))\, \beta^{j-2}(\beta-1) \le (\beta^2/(\beta-1))(n_{j-1} + 1 - n_{j-2})$. Hence (5.27) implies

$$\sum_{k\ge r} I_{2,k} \le \frac{8C_\Lambda \beta^{9/2}}{c^3(\sqrt{\beta}-1)(\beta-1)}$$

$$\cdot\left[\sum_{j=\theta(r)}^\infty (n_{j-1} - n_{j-2}) P(\gamma_{n_{j-1}} < \|X\| < \gamma_{n_j}) + P(\|X\| > \gamma_{n_{\theta(r)-1}})\right] \tag{5.28}$$

$$\le \frac{8C_\Lambda \beta^{9/2}}{c^3(\sqrt{\beta}-1)(\beta-1)}$$

$$\cdot\left[\sum_{n\ge n_{\theta(r)-2}} P(\|X\| > \gamma_n) + P(\|X\| > \gamma_{n_{\theta(r)-1}})\right] \to 0 \quad \text{as} \quad r\to\infty$$

since $\theta(r) \to \infty$ as $r \to \infty$ and (2.26) holds with $M = 1$. Hence Lemma 5.3 is proved.

A suitable replacement for Lemma 5.3 when $E\|X\|^2 < \infty$ and the regularity assumption (2.24) is not necessarily satisfied is the following.

LEMMA 5.4. *Using the notation of Lemma* 5.3 *and assuming* $EX = 0$, $0 < E\|X\|^2 < \infty$, *we have for all* $\Lambda > 2\beta$ *that*

$$\lim_{r\to\infty} \sum_{k\ge r} I_{2,k} = 0. \tag{5.29}$$

PROOF. From (5.20) and the fact that $\{\|V_n - EV_n\| : n_k < n \le n_{k+1}\}$ is a submartingale we have

$$I_{2,k} \le E\|V_{n_{k+1}} - EV_{n_{k+1}}\|/(\Lambda - 2\beta)\gamma_{n_k}$$

$$\le 2n_{k+1}E(\|X\| I(\|X\| > K(n_k/L_2n_k)))/(\Lambda - 2\beta)\gamma_{n_k}. \tag{5.30}$$

Since $n_{k+1} \le \beta^{k+1} = (\beta^2/(\beta-1)).\, \beta^{k-1}(\beta-1) \le (\beta^2/(\beta-1))(n_k + 1 - n_{k-1})$ and $E\|X\|^2 < \infty$ implies $K(n) \approx \sqrt{n}$, there are constants Γ and $\varepsilon > 0$ such that

$$I_{2,k} \le \Gamma(n_k - n_{k-1})E\left(\|X\| I\left(\|X\| > \varepsilon\sqrt{\frac{n_k}{L_2n_k}}\right)\right) \Big/ \sqrt{n_kL_2n_k}. \tag{5.31}$$

Hence for all sufficiently large r

$$
\begin{aligned}
\sum_{k\ge r} I_{2,k} &\le \Gamma \sum_{k\ge 1} (n_k - n_{k-1}) E\left(\|X\| I\left(\|X\| > \varepsilon\sqrt{\frac{n_k}{L_2 n_k}}\right)\right) \Big/ \sqrt{n_k L_2 n_k} \\
&\le \Gamma \sum_{n\ge n_{r-1}} E\left(\|X\| I\left(\|X\| > \varepsilon\sqrt{\frac{n}{L_2 n}}\right)\right) \Big/ \sqrt{n L_2 n}.
\end{aligned} \tag{5.32}
$$

The conclusion (5.29) will follow from (5.23) since

$$\sum_{n\ge 1} E\left(\|X\| I\left(\|X\| > \varepsilon\sqrt{\frac{n}{L_2 n}}\right)\right) \Big/ \sqrt{n L_2 n} < \infty. \tag{5.33}$$

To verify (5.33) note that

$$
\begin{aligned}
&\sum_{n\ge 1} E\left(\|X\| I\left(\|X\| > \varepsilon\sqrt{\frac{n}{L_2 n}}\right)\right) \Big/ \sqrt{n L_2 n} \\
&\qquad = E\left(\|X\| \sum_{n\ge 1} I\left(\|X\| > \varepsilon\sqrt{\frac{n}{L_2 n}}\right) \Big/ \sqrt{n L_2 n}\right)
\end{aligned} \tag{5.35}
$$

and for a sufficiently large constant Γ' we have for all n that

$$I\left(\|X\| \ge \varepsilon\sqrt{\frac{n}{L_2 n}}\right) \le I(n \le \Gamma' \|X\|^2 L_2 \|X\|).$$

Hence there is a constant Γ'' such that

$$\sum_{n\ge 1} I\left(\|X\| \ge \varepsilon\sqrt{\frac{n}{L_2 n}}\right) \Big/ \sqrt{n L_2 n} \le \sum_{n=1}^{\Gamma' \|X\|^2 L_2 \|X\|} \frac{1}{\sqrt{n L_2 n}} \le \Gamma''(\|X\| + 1), \tag{5.35}$$

and since $E\|X\|^2 < \infty$, combining (5.35) and (5.34) we obtain (5.33), so the lemma is proved.

To complete the proof of Theorem 3 it suffices to prove:

LEMMA 5.5. *For all Λ sufficiently large we have*

$$\lim_{r\to\infty} \sum_{k\ge r} I_{1,k} = 0.$$

PROOF. First we observe that for $\varepsilon > 0$ and all k sufficiently large

$$
\begin{aligned}
I_{1,k} &= P(\max_{n_k < n \le n_{k+1}} \|U_n\| \ge \Lambda \gamma_{n_k} + E\|S_{\beta n_k}\|) \\
&\le P(\max_{n_k < n \le n_{k+1}} \|U_n - EU_n\| + \max_{n_k < n \le n_{k+1}} \|EU_n\| \ge \Lambda \gamma_{n_k} + E\|S_{\beta n_k}\|) \\
&\le P(\max_{n_k < n \le n_{k+1}} \|U_n - EU_n\| - E\|U_{n_{k+1}} - EU_{n_{k+1}}\| \ge \Lambda \gamma_{n_k} + E\|S_{\beta n_k}\| \\
&\qquad - E\|U_{n_{k+1}} - EU_{n_{k+1}}\| - \max_{n_k < n \le n_{k+1}} \|EU_n\|). \\
&\le P(\max_{n_k < n \le n_{k+1}} \|U_n - EU_n\| - E\|U_{n_{k+1}} - EU_{n_{k+1}}\| \ge (\Lambda - 8\beta - \varepsilon)\gamma_{n_k}) \\
&= P(\exp\{\gamma(\max_{n_k < n \le n_{k+1}} \|U_n - EU_n\| - E\|U_{n_{k+1}} - EU_{n_{k+1}}\|)\} \\
&\qquad \ge \exp\{\gamma(\Lambda - 8\beta - \varepsilon)\gamma_{n_k}\}).
\end{aligned} \tag{5.36}
$$

The last inequality in (5.36) results from the fact that $EU_n = -E(V_n + W_n)$, $\beta n_k \ge \beta^{k+1} - \beta \ge n_{k+1} - \beta$, and the estimate in (5.19) implies for all k sufficiently large that

$$
\begin{aligned}
&E\|S_{\beta n_k}\| - E\|U_{n_{k+1}} - EU_{n_{k+1}}\| - \max_{n_k < n \le n_{k+1}} \|EU_n\| \\
&\qquad \ge E\|U_{n_{k+1}} + V_{n_{k+1}} + W_{n_{k+1}}\| - \beta E\|X\| - E\|U_{n_{k+1}}\| - \|EU_{n_{k+1}}\| - 2\beta\gamma_{n_k} \\
&\qquad \ge -8\beta\gamma_{n_k} - \beta E\|X\| \ge -(8\beta + \varepsilon)\gamma_{n_k}.
\end{aligned}
$$

Since $\{\| U_n - EU_n \| : n_k < n \leq n_{k+1}\}$ is a submartingale and $e^{\gamma x}$ is increasing and convex for $\gamma > 0$ we have by the maximal inequality for submartingales that for k sufficiently large

$$I_{1,k} \leq \exp\{-\gamma(\Lambda - 8\beta - \varepsilon)\gamma_{n_k}\} E(\exp(\gamma(\| U_{n_{k+1}} - EU_{n_{k+1}} \| - E\| U_{n_{k+1}} - EU_{n_{k+1}} \|))). \tag{5.37}$$

Now recall from the definition of u_j that

$$\begin{aligned} \| u_j - Eu_j \| &\leq 2K(n_k/L_2 n_k) \quad (1 \leq j \leq n_{k+1}) \\ &= c_{n_k} \eta_{n_k} \end{aligned}$$

where

$$c_{n_k} = 2/(\sqrt{L_2 n_k}\,\sqrt{\Lambda - 8\beta - \varepsilon}), \quad \eta_{n_k} = \sqrt{L_2 n_k}\, K\left(\frac{n_k}{L_2 n_k}\right) \sqrt{\Lambda - 8\beta - \varepsilon}.$$

Let $\varepsilon_{n_k} = \sqrt{L_2 n_k}\,\sqrt{\Lambda - 8\beta - \varepsilon}/2$ and set $\gamma = \varepsilon_{n_k}/2\eta_{n_k}$. Then $\varepsilon_{n_k} c_{n_k} \leq 1$, and hence by (2.4) in Kuelbs (1977) we have

$$I_{1,k} \leq \exp\{-\varepsilon_{n_k}(\Lambda - 8\beta - \varepsilon)\gamma_{n_k}/2\eta_{n_k}\} \exp\left\{\frac{3\varepsilon_{n_k}^2}{4} \sum_{j=1}^{n_{k+1}} \frac{E\| u_j - Eu_j \|^2}{\eta_{n_k}^2}\right\}. \tag{5.38}$$

Since $\gamma_n \geq \alpha_n = L_2 n K(n/L_2 n)$ we have

$$I_{1,k} \leq \exp\{-\,(\Lambda - 8\beta - \varepsilon) L_2 n_k/4\}$$

$$\exp\left\{\frac{3\varepsilon_{n_k}^2 n_{k+1} E\left\| XI\left(\|X\| \leq K\left(\frac{n_k}{L_2 n_k}\right)\right) - E\left(XI\left(\|X\| \leq K\left(\frac{n_k}{L_2 n_k}\right)\right)\right)\right\|^2}{4 L_2 n_k K^2\left(\frac{n_k}{L_2 n_k}\right)(\Lambda - 8\beta - \varepsilon)}\right\}. \tag{5.39}$$

Now the function $K(\cdot)$ satisfies (2.16) and hence

$$K^2(x) \geq E(\|X\|^2 I(\|X\| \leq K(x))),$$

so

$$\frac{n_k}{L_2 n_k} E\left(\left\| XI\left(\|X\| \leq K\left(\frac{n_k}{L_2 n_k}\right)\right) - E\left(XI\left(\|X\| \leq K\left(\frac{n_k}{L_2 n_k}\right)\right)\right)\right\|^2\right) \leq 4K^2\left(\frac{n_k}{L_2 n_k}\right).$$

Now for $k \geq k_0$ (5.17) implies

$$n_{k+1} \leq 2\beta n_k$$

and hence for all sufficiently large k (5.39) implies

$$I_{1,k} \leq \exp\{[-\,(\Lambda - 8\beta - \varepsilon)/4 + 3\beta/2] L_2 n_k\}.$$

Choosing Λ such that

$$(\Lambda - 8\beta - \varepsilon)/4 - 3\beta/2 > 1$$

we have $\sum_k I_{1,k} < \infty$, and hence Lemma 5.5 is proved.

As was pointed out in (5.16), the proof of Theorem 3 is now complete.

5.3 *Proof of the Corollaries to Theorem* 3. To prove Corollary 1 we first observe that (2.28) and Kolmogorov's zero-one law easily imply (2.29a) and (2.29b). Hence it suffices to show the converse, and this easily follows from Theorem 3 and (2.25), if we can show (2.29) implies

$$\sup_n \frac{E\| S_{\beta n} \|}{\gamma_n} < \infty. \tag{5.40}$$

To verify (5.40) choose an integer ℓ such that $\beta < 2^\ell$. Then $EX = 0$ implies

$$E\|S_{\beta n}\| \le E\|S_{2^\ell n}\| \le 2^\ell E\|S_n\|$$

and hence (5.40) holds if (2.29) implies

$$\sup_n \frac{E\|S_n\|}{\gamma_n} < \infty. \tag{5.41}$$

To verify (5.41) we first assume $M = 1$ in (2.29a) and that X, and hence the sequence $\{X_j\}$, is symmetric.

If $n_k < n \le n_{k+1}$ we define $u_j, v_j, w_j (1 \le j \le n_{h+1})$ and U_n, V_n, W_n as prior to (5.14). Then, for $n_k < n \le n_{k+1}$

$$E\|S_n\| \le E\|U_n\| + E\|V_n + W_n\|,$$

and for all $k \ge k_0$ the argument yielding (5.19) implies

$$\sup_{n_k<n\le n_{k+1}} E\|V_n + W_n\| \le 2\beta\gamma_{n_k}.$$

Since $EX = 0$ we have $E\|S_n\|$ increasing in n and hence $\{\gamma_n\} \nearrow$ implies

$$\begin{aligned} \sup_n \frac{E\|S_n\|}{\gamma_n} &= \sup_k \sup_{n_k<n\le n_{k+1}} \frac{E\|S_n\|}{\gamma_n} \le \sup_k \frac{E\|S_{n_{k+1}}\|}{\gamma_{n_k}} \\ &\le 2\beta + \sup_k \frac{E\|U_{n_{k+1}}\|}{\gamma_{n_k}}. \end{aligned} \tag{5.42}$$

Since $E(u_j) = 0$ and there exists an integer ℓ such that $n_{k+1} \le 2^\ell n_k$, we have $E\|U_{n_{k+1}}\| \le 2^\ell E\|\sum_{j=1}^{n_k} u_j\|$, and hence (5.41) follows from (5.42) if

$$\sup_k \frac{E\|\sum_{j=1}^{n_k} u_j\|}{\gamma_{n_k}} < \infty. \tag{5.43}$$

Now the u_j, $1 \le j \le n_k$, are independent and symmetric with

$$\max_{1\le j\le n_k} \|u_j\|/\gamma_{n_k} \le 1/L_2 n_k,$$

and we also have the well known inequality

$$P(\textstyle\sum_{j=1}^{n_k} u_j\| > t\gamma_{n_k}) \le 2P(\|S_{n_k}\| > t\gamma_{n_k}).$$

Hence (2.29-b), along with a standard argument involving (3.3) of Hoffman-Jørgensen (1974) (see also Lemma 6.1-a below), easily implies (5.43).

Hence (2.29) implies (5.41), and also (5.40), provided $M = 1$ in (2.29a) and X is assumed symmetric. For the general case, let $\{X'_n\}$ be an independent copy of $\{X_n\}$ and set $Y_n = (X_n - X'_n)/2M$ $(n \ge 1)$ where $0 < M < \infty$ is such that (2.29a) holds for $\{X_n\}$. Then

$$P(\|Y_n\| > \gamma_n \text{ i.o.}) = 0$$

and

$$\{\textstyle\sum_{j=1}^{n} Y_j/\gamma_n\} \text{ is bounded in probability.}$$

Hence the previous case implies

$$\sup_n E\|\textstyle\sum_{j=1}^{n} Y_j/\gamma_n\| < \infty,$$

and since $E\|S_n\| \le E\|\sum_{j=1}^{n} (X_j - X'_j)\| = 2ME\|\sum_{j=1}^{n} Y_j\|$, this implies (5.41). Hence Corollary 1 is proved.

PROOF OF COROLLARY 2. To prove Corollary 2, we simply apply Lemma 1 and Corollary 1. That is, in type 2 spaces, Lemma 1 provides a constant $A < \infty$ such that $E\|S_n\| \le AK(n)$, and hence, recalling that $S_t = S_{[t]}$, we have for all $n \ge n_0$ and all $\Lambda > 0$

that

$$P\left(\frac{\|S_n\|}{\gamma_n} > \Lambda\right) \le \frac{E\|S_n\|}{\Lambda\gamma_n} \le 2L_2nE\,\frac{\|S_n\|}{L_2n}\Big/\Lambda\gamma_n \tag{5.44}$$
$$\le 2AL_2nK(n/L_2n)/\Lambda\gamma_n \le \frac{2A}{\Lambda}.$$

Letting $\Lambda \to \infty$ we see (5.44) implies $\{S_n/\gamma_n\}$ is bounded in probability. Since (2.31) is assumed, Corollary 1 applies and the proof is complete.

PROOF OF COROLLARY 3. To prove Corollary 3 we first point out that if $E\|X\|^2 = 0$ the corollary is obvious. If $0 < E\|X\|^2 < \infty$, then the Borel-Cantelli lemma easily implies $P(\|X_n\| > \sqrt{n} \text{ i.o.}) = 0$, (2.14) implies $K(n) \approx \sqrt{n}$, and hence

$$\alpha_n = L_2nK\left(\frac{n}{L_2n}\right) \approx \sqrt{nL_2n}.$$

Thus Theorem 3 immediately implies

$$\limsup_n \frac{\|S_n\| - E\|S_{\beta n}\|}{\alpha_n} < \infty \quad \text{w.p. } 1$$

for any $\beta > 1$. Since $\{\gamma_n\}$ satisfies (2.32) and $\alpha_n \approx \sqrt{nL_2n}$ we thus have with probability one that

$$\limsup_n \frac{\|S_n\| - E\|S_{\beta n}\|}{\gamma_n} < \infty \text{ (resp.} \le 0 \text{ if (2.34) holds).}$$

Using the argument in the proof of Corollary 1 it easily follows that

$$\limsup_n E\|S_{\beta n}\|/\gamma_n < \infty \; (=0)$$

if $\{S_n/\gamma_n\}$ is bounded in probability (resp. $S_n/\gamma_n \to_{\text{prob}} 0$ and (2.34) holds). Thus Corollary 3 is proved.

PROOF OF COROLLARY 4. To prove Corollary 4 we first observe that if t is sufficiently large the inequality

$$t^2g(t)/L_2t \le x \tag{5.45}$$

implies

$$t \le x^{1/2}\sqrt{\frac{L_2x}{g(x)}}. \tag{5.46}$$

To see (5.46) follows from (5.45) for large t we use the fact that (2.38) implies $\psi(t) = t^2g(t)/L_2t$ is eventually non-decreasing, and observe that

$$\psi\left(x^{1/2}\sqrt{\frac{L_2x}{g(x)}}\right) = x\frac{L_2x}{g(x)}g\left(x^{1/2}\sqrt{\frac{L_2x}{g(x)}}\right)\Big/L_2\left(x^{1/2}\sqrt{\frac{L_2x}{g(x)}}\right) \ge x \tag{5.47}$$

since (2.39) implies $x \ge x^{1/2}\sqrt{L_2x/g(x)}$ for large x and $L_2t/g(t)$ is assumed to be non-decreasing on $[0, \infty)$.

Since (5.45) implies (5.46) for large t, we have

$$\textstyle\sum_n P(\|X\|^2g(\|X\|)/L_2\|X\| \ge n) < \infty \tag{5.48}$$

implying that

$$\sum_n P\left(\|X\| \ge n^{1/2}\sqrt{\frac{L_2n}{g(n)}}\right) < \infty. \tag{5.49}$$

Now $E(\|X\|^2 g(\|X\|)/L_2\|X\|) < \infty$ implies (5.48), so (5.49) holds.

Now we apply Corollary 1. That is, (5.49) implies $\{\gamma_n\}$ satisfies (2.26) for all M, $0 < M < \infty$, and since $\{\gamma_n\}$ and $\{\gamma_n/\sqrt{n}\}$ are both non-decreasing we have $\{\gamma_n\}$ satisfying (2.24) as well. Thus we can apply Corollary 1 if we show $\{\rho\gamma_n\}$ satisfies (2.23) for some positive constant ρ.

In this regard we see that (2.16), (2.37), and (2.38) imply

$$K^2(n) \le n\frac{L_2K(n)}{g(K(n))}I + \frac{nK(n)L_2K(n)}{K(n)g(K(n))}I \tag{5.50}$$

where $I = E(\|X\|^2 g(\|X\|)/L_2\|X\|) < \infty$.

Now (5.50) implies

$$K^2(n)g(K(n))/L_2K(n) \le 2In, \tag{5.51}$$

and hence for large n (recall $K(n) \nearrow \infty$) (5.45) implying (5.46) gives us that

$$K(n) \le (2I)^{1/2}n^{1/2}\sqrt{\frac{L_2(2In)}{g(2In)}}. \tag{5.52}$$

Hence

$$\alpha_n = L_2nK\left(\frac{n}{L_2n}\right) = O\left(L_2n\left(\frac{n}{L_2n}\right)^{1/2}\sqrt{\frac{L_2\left(2I\frac{n}{L_2n}\right)}{g\left((2I\frac{n}{L_2n}\right)}}\right) = O\left(\sqrt{nL_2n}\sqrt{\frac{L_2n}{g(n)}}\right),$$

since $2In/L_2n \le n$ for large n and $L_2t/g(t)$ is non-decreasing.

Hence there exists a constant $\rho < \infty$ such that $\{\rho\gamma_n\}$ satisfies (2.23). By Corollary 1 we now have

$$\limsup_n \left\|\frac{S_n}{\rho\gamma_n}\right\| < \infty \text{ w.p. 1 iff } \left\{\frac{S_n}{\rho\gamma_n}\right\} \text{ is bounded in probability,} \tag{5.53}$$

and thus (2.40) holds.

If (2.41) holds and $\{S_n/\gamma_n\} \to_{\text{prob}} 0$ we choose $\delta(t) \searrow 0$ such that if $\delta_n = \delta(n)$, then

$$\begin{aligned} &\text{(a) } \sqrt{\delta_n}\gamma_n \nearrow \infty \\ &\text{(b) } \frac{S_n}{\sqrt{\delta_n}\gamma_n} \to_{\text{prob}} 0 \\ &\text{(c) } \sqrt{\delta_n}\,\gamma_n \ge \alpha_n \text{ for all } n \text{ sufficiently large.} \\ &\text{(d) } E(\|X\|^2 g(\|X\|)/(L_2\|X\|\cdot\delta(\|X\|))) < \infty. \end{aligned} \tag{5.54}$$

Obtaining the function $\delta(t)$ is easy once we show

$$\alpha_n = o(\gamma_n). \tag{5.55}$$

That is, one can choose strictly positive functions $\varepsilon_1(t)$, $\varepsilon_2(t)$, $\varepsilon_3(t)$, $\varepsilon_4(t)$ satisfying (a), (b), (c), (d), of (5.54), respectively, and then $\delta(t) = \max_{1\le j\le 4}\varepsilon_j(t)$ will suffice.

Now (5.55) will follow from (2.37), (2.38), and (2.41) since (2.16) then implies that

$$\begin{aligned} K^2(n) &= \frac{nL_2K(n)}{g(K(n))}\left[E\left(\frac{\|X\|^2 g(\|X\|)}{L_2\|X\|}\left(\frac{g(K(n))}{L_2K(n)}\Big/\frac{g(\|X\|)}{L_2\|X\|}\right)I(\|X\| \le K(n))\right)\right. \\ &\quad \left. + E\left(\|X\|^2\frac{g(\|X\|)}{L_2\|X\|}\left(\frac{K(n)g(K(n))}{L_2K(n)}\Big/\|X\|\frac{g(\|X\|)}{L_2\|X\|}\right)I(\|X\| > K(n))\right)\right] \\ &= o(nL_2K(n)/g(K(n))) \end{aligned} \tag{5.56}$$

by the dominated convergence theorem. Seeing that the dominated convergence theorem implies the little-oh condition in (5.56) is easy, since (2.37), (2.38) and (2.41) imply that the random variables

$$\left\{\frac{g(K(n))}{L_2K(n)}\Big/\frac{g(\|X\|)}{L_2\|X\|}\right\}I(\|X\| \le K(n))$$

and

$$\{(K(n)g(K(n))/L_2K(n)/K(X)g(\|X\|)/L_2\|X\|)I(\|X\| \ge K(n))\}$$

are eventually bounded by one and converge pointwise to zero as $n \to \infty$.

Now that (5.54) holds we choose a function $\varepsilon(t) \downarrow 0$ such that $\varepsilon(t) \ge \delta(t)$ and $g(t)/(L_2t\cdot \varepsilon(t)) \downarrow 0$ at $t \to \infty$. The existence of $\varepsilon(t)$ follows from the following easily proved lemma applied to $h(t) = g(t)/L_2t$.

LEMMA. *Let $h(t)$, $\delta(t)$ be strictly positive functions on $[0, \infty)$ such that $h(t) \downarrow 0$ and $\delta(t) \downarrow 0$ as $t \to \infty$. Let $\rho(t) = \inf_{0\le u\le t} h(u)/\delta(u)$. Then*

$$\varepsilon(t) = h(t)/\rho(t)$$

is such that $\varepsilon(t) \downarrow 0$, $\varepsilon(t) \ge \delta(t)$, and $h(t)/\varepsilon(t) \downarrow 0$ as $t \to \infty$.

Given $\varepsilon(t)$ as above by applying (2.40) with $g(t)$ replaced by $g(t)/\varepsilon(t)$ we have that

$$\limsup_n \|S_n/\sqrt{\varepsilon_n}\cdot\gamma_n\| = 0 \quad \text{w.p. } 1, \tag{5.57}$$

and hence

$$\lim_n \left\|\frac{S_n}{\gamma_n}\right\| = 0 \quad \text{w.p. } 1.$$

Hence (2.42) holds, and the proof of Corollary 4 is complete.

PROOF OF COROLLARY 5. To prove Corollary 5 we apply Corollary 1 to the sequence $\{\gamma_n\} = \{\rho L_2nE\|S_{n/L_2n}\|\}$ for some positive constant $\rho < \infty$.

First of all the sequence $\{L_2nE\|S_{n/L_2n}\|\}$ is non-decreasing, and for some constant $c > 0$

$$L_2nE\|S_{n/L_2n}\| \ge cE\|S_n\|. \tag{5.58}$$

Now (5.58) implies $\{S_n/L_2nE\|S_{n/L_2n}\|\}$ is bounded in probability, and hence from (2.46) we see $\{\gamma_n\}$ satisfies (2.26). Further, since B is of cotype 2, and $EX = 0$, $0 < E\|X\| < \infty$, we have from (2.19) that there is a $\rho < \infty$ such that

$$\rho L_2nE\|S_{n/L_2n}\| \ge L_2nK\left(\frac{n}{L_2n}\right).$$

Thus there is a $\rho < \infty$ such that $\{\gamma_n\}$ satisfies both (2.23) and (2.26), and hence Corollary 1 applies if we show (γ_n) satisfies (2.24).

To verify (2.24) we note that since B is of cotype 2 we have a constant A, independent of the law of X and n, such that $EX = 0$ implies

$$E\|S_n\| \ge AE[(\textstyle\sum_{j=1}^n \|X_j\|^2)^{1/2}] \ge AE\left(\sum_{j=1}^n \frac{\|X_j\|}{\sqrt{n}}\right) = A\sqrt{n}E\|X\|.$$

Hence we have for all mean zero random variables X

$$\inf_n E\frac{\|S_n\|}{\sqrt{n}} \ge AE\|X\|, \tag{5.59}$$

and the function $h(n)$ of (2.23) and (2.4) is

$$h(n) = \frac{\gamma_n}{\sqrt{n}} = \rho L_2 n E\|S_n/L_2 n\|/\sqrt{n}.$$

Thus we have (2.24) since (5.59) applied to the mean zero random variable $S_{n_r/L_2 n_r}$ implies there is a $\theta > 0$ such that

$$\begin{aligned}
\inf_r \inf_{k\geq r} \frac{h(n_k)}{h(n_r)} &= \inf_r \inf_{k\geq r} \frac{L_2 n_k E\|S_{n_k/L_2 n_k}\|/\sqrt{n_k}}{L_2 n_r E\|S_{n_r/L_2 n_r}\|/\sqrt{n_r}} \\
&\geq \frac{\inf_r \inf_{k\geq r} \sqrt{\frac{n_r}{n_k}}\, AE\|S_{n_r/L_2 n_r}\| \sqrt{\left[\left[\frac{n_k}{L_2 n_k}\right]\Big/\left[\frac{n_r}{L_2 n_r}\right]\right]}\,(L_2 n_k)}{L_2 n_r E\|S_{n_r/L_2 n_r}\|} \\
&\geq \theta > 0.
\end{aligned}$$

Thus Corollary 1 implies (2.45) and Corollary 5 holds.

6. The Proof of Theorems 4 and 5, their corollaries, and Lemma 2. We first provide some notation and necessary lemmas.

If q is a Borel measurable semi-norm on B and Z is a B-valued random variable we let

$$q_p(Z) = (E(q(Z))^p)^{1/p}, \quad 1 \leq p < \infty$$

$$q_\infty(Z) = \text{ess} - \sup q(Z)$$

$$q_{e^*}(Z) = \inf\{a : E\{\exp\{q(Z)/a\}\} \leq e\},$$

and

$$J_q(Z, a) = \inf\{t > 0 : P(q(Z) \geq t) \leq a\}.$$

Further, if $\{Y_j\}$ are independent B-valued random variables we let

$$S_n = \textstyle\sum_{j=1}^n Y_j \quad \text{and} \quad M_n = \max_{1\leq j\leq n} q(Y_j).$$

The first two lemmas below are essentially given in Pisier (1975). We give a proof of the first and refer the reader to Pisier (1975) for the proof of the second.

LEMMA 6.1. *Let $\{Y_j\}$ be independent, symmetric B-valued random variables and q a Borel measurable semi-norm on B. Then for each $r > 0$*

$$\text{(a)}\ E(q^r(S_n)) \leq 2\cdot 3^r\left[4J_q^r\left(S_n, \frac{1}{8\cdot 3^r}\right) + E(M_n^r)\right],$$

and

$$\text{(b)}\ q_{e^*}(S_n) \leq 2[J_q(S_n, (2e)^{-2}) + \sup_{1\leq j\leq n} q_\infty(Y_j)].$$

PROOF. We prove (b) since (a) follows easily from the methods and the result (3.3) in Hoffman-Jørgenson (1974) which states that

$$P(q(S_n) \geq 2t + s) \leq [2P(q(S_n) \geq t)]^2 + P(M_n \geq s). \tag{6.1}$$

To prove (b) we assume without loss of generality that $\sup_{j\leq n} q_\infty(Y_j) < \infty$. Then, setting $t > J_q(S_n, (2e)^2)$, $s > \sup_{j\leq n} q_\infty(Y_j)$, and iterating (6.1) we obtain

$$\begin{aligned}
P(q(S_n) \geq 2^r(t+s)) &\leq P(q(S_n) \geq 2^r t + \textstyle\sum_{j=0}^{r-1} 2^j s) \\
&\leq 2^{\sum_{j=1}^{r} 2^j}[P(q(S_n) \geq t)]^{2^r} \\
&\leq 2^{2^{r+1}-1}((2e)^{-2})^{2^r}.
\end{aligned} \tag{6.2}$$

Hence,

$$E(\exp(q(S_n)/2(t+s))) = \int_0^\infty \frac{1}{2(t+s)} \exp\left\{\frac{u}{2(t+s)}\right\} P(q(S_n) \ge u)\, du$$

$$\le \left[\exp\left(\frac{1}{2}\right) - 1\right] + \textstyle\sum_{r=1}^{\infty} \displaystyle\int_{2^{r-1}(t+s)}^{2^r(t+s)} \exp\left\{\frac{u}{2(t+s)}\right\} \Big/ 2(t+s) \cdot P(q(S_n) \ge 2^{r-1}(t+s))\, du$$

$$\le \left[\exp\left(\frac{1}{2}\right) - 1\right] + \textstyle\sum_{r=1}^{\infty} 2^{2^{r-1}}[(2e)^{-2}]^{2^{r-1}} \exp\{2^{r-1}\} \le e^{1/2} \le e,$$

which concludes the proof.

Lemma 6.2. *For $\{Y_j\}$ independent, mean zero B-valued random variables we have*

$$q_N(S_N) \le N \sup_{j \le N} q_{e^*}(Y_j).$$

Proof of Theorem 4. We first assume X is symmetric and $\sup_k P(q(S_{r_k}/d_k) \ge 1) \le 1/8e^2$. Then for $n_{k-1} < i \le n_k$ we let

$$Y_i = X_i I(q(X_i) \le d_k)$$

and let

$$V_n = \textstyle\sum_{i=1}^n Y_i, \quad T_m^n = V_n - V_m (n \ge m).$$

Now put $p_k = [Lk]$. Then for $p_k \ge 2$,

$$P(\max_{n_{k-1} < n \le n_k} q(V_n) \ge 2Ap_k\, d_k)$$

$$\le E[q^{p_k}(V_{n_k})]/(Ap_k\, d_k)^{p_k} \text{ (by the submartingale inequality)}$$

$$= \left[\frac{1}{Ap_k\, d_k} q_{p_k}\left(\textstyle\sum_{j=1}^{p_k} T_{(j-1)r_k}^{jr_k}\right)\right]^{p_k} \tag{6.3}$$

$$\le [p_k \sup_{j \le p_k} q_{e^*}(T_{(j-1)r_k}^{jr_k})/Ap_k\, d_k]^{p_k} \quad \text{(by Lemma 6.2)}$$

$$\le [\sup_{j \le p_k} 2\{J_q(T_{(j-1)r_k}^{jr_k}, (2e)^{-2}) + \sup_{(j-1)r_k < i \le jr_k} q(Y_i)\}/A\, d_k]^{p_k}$$

by Lemma 6.1 (b).

Now $q(Y_i) \le d_k$ if $i \le n_k$ and since X is symmetric the truncated variables satisfy

$$P(q(T_{(j-1)r_k}^{jr_k}) \ge d_k) \le 2P(q(S_{r_k}) \ge d_k) \le (2e)^{-2}.$$

Hence $J_q(T_{(j-1)r_k}^{jr_k}, (2e)^{-2}) \le d_k$ and hence the last term in (6.3) is dominated by

$$[4\, d_k/A\, d_k]^{p_k} = \exp\{-p_k \log(A/4)\} \le \exp\{-\tfrac{1}{2}(\log A/4)Lk\},$$

which is summable if $A > 4e^2$. Hence, if $A > 4e^2$, by the very definition of γ_n

$$\textstyle\sum_k P(\max_{n_{k-1} < n \le n_k} q(V_n)/\gamma_n \ge 2A) < \infty.$$

Therefore,

$$\limsup_n q(V_n)/\gamma_n \le 8e^2 \quad \text{w.p. } 1,$$

and by (2.48)

$$\limsup_n q\left(\frac{S_n}{\gamma_n}\right) \le 8e^2 \quad \text{w.p. } 1.$$

Now in the general case, (2.47) and the previous argument implies that the symmetrizations, $S_n - S_n'$, are such that

$$\limsup_n q\left(\frac{S_n - S_n'}{\gamma_n}\right) \le 16e^2 \quad \text{w.p. 1.}$$

Further, (2.48) implies

$$\limsup_n q(X_n)/\gamma_n = 0 \quad \text{w.p. 1.}$$

Applying Lemma 2–x, for all $\rho > 0$, we have

$$\limsup_n q\left(\frac{S_n - nEXI(q(X) \le \rho\gamma_n)}{\gamma_n}\right) \le 26(16e^2) + 12\rho \quad \text{w.p. 1.}$$

Further, for $n_{k-1} < n \le n_k$

$$q\left(\frac{nE(XI(q(X) \le \rho\gamma_n))}{\gamma_n} - \frac{nE(XI(q(X) \le d_k))}{\gamma_n}\right) \le n_k\rho\gamma_n P(q(X) > d_k)/\gamma_n \to 0$$

by (2.48), so we have

$$\limsup_n q\left(\frac{S_n - nE(XI(q(X) \le \lambda(n)))}{\gamma_n}\right) \le 416e^2 \quad \text{w.p. 1.}$$

For the second part of the proof of Theorem 4 we first observe that if X is not symmetric then the symmetrization of X, namely $X - X'$, satisfies (2.51), and hence by Lemma 2-(ii) it suffices to assume X is symmetric.

Let $\eta = h(X)$. Then η is also symmetric and (2.52) will hold for X if

$$\limsup_n \frac{h(S_n)}{\gamma_n} > 0 \quad \text{w.p. 1.}$$

To verify this we show η satisfies the condition (4.11) of Lemma 4.2 of Pruitt (1981).

Now by scalar multiplication we can also assume $E(g^2) = 1$. Further, we have that Pruitt's β_n is our γ_n and thus his $u_k = (p_k/Lk)\, d_k$, and, of course, $n_k = p_k r_k$. Hence, in view of (2.48), it suffices to show that

$$f(u_k) \sim n_k^{-1}\log k \quad \left(\text{which is} \sim \frac{1}{r_k}\right)$$

where $f(t) = E(\eta^2 \wedge t^2)/t^2$, $(t > 0)$.

Now Corollary 2.12 of Araujo, de Acosta, and Giné (1978) implies

$$\lim_k r_k E\left(\left|\frac{\eta}{d_k}\right|^2 I(|\eta| \le \varepsilon\, d_k)\right) = E(g^2) = 1$$

independently of $\varepsilon > 0$. Since $u_k \le d_k$ and $u_k \sim d_k$ as $k \to \infty$ we have

$$E\left(\eta^2 I\left(|\eta| \le \frac{1}{2}\, d_k\right)\right) \le E(\eta^2 I(|\eta| \le u_k)) \le E(\eta^2 I(|\eta| \le d_k)).$$

Thus the above implies

$$\lim_k r_k \frac{E(\eta^2 I(|\eta| \le u_k))}{u_k^2} = 1$$

and Pruitt's result applies. Hence the theorem is proved.

PROOF OF COROLLARY 6. First we note that $X \in \mathrm{DPA}(Z)$ implies (see Corollary 2.12 of Araujo, de Acosta, and Giné (1978)) that there exists $r_k \nearrow \infty$ and $d_k \nearrow \infty$ such that

$$W_k = \frac{S_{r_k} - r_k E(XI(\|X\| \le d_k))}{d_k}$$

converges weakly to Z. Therefore, there exists a compact, convex, symmetric set $D \subseteq B$

such that

$$\sup_k P(W_k \notin D) < (4e)^{-2},$$

and the Corollary 2.12 mentioned above also implies

$$\lim_k r_k P(X \notin d_k D^\delta) = 0$$

for all $\delta > 0$. Hence, by using a diagonal argument we can also assume

$$\frac{r_{k+1}}{r_k} \geq 40$$

and for all $\delta > 0$

$$\sum_k (Lk) r_k P(X \notin d_k D^\delta) < \infty.$$

Now for each $\delta > 0$ define the norm $q = q_\delta$ on B by

$$q(x) = \inf\{t > 0 : x/t \in D^\delta\}.$$

Then q is equivalent to $\|\cdot\|$ (so $q(x) < \infty$ for all $x \in B$) and putting $p_k = [Lk]$, $n_k = p_k r_k$, $\lambda(n) = d(k)$ and $\gamma(n) = p_k d_k$ for $n_{k-1} < n \leq n_k$ we have by Theorem 4 that

$$0 < \limsup_n q\left(\frac{S_n - nE(XI(\|X\| \leq \lambda(n)))}{\gamma(n)}\right) \leq 416e^2 \quad \text{w.p.1.} \tag{6.4}$$

Hence,

$$\left\{\frac{S_n - E(XI(\|X\| \leq \lambda(n)))}{\gamma(n)}\right\}$$

is relatively compact with probability one with limit set contained in $416e^2 D$. That the limit set is not $\{0\}$ follows immediately from the fact that the limit in (6.4) is positive with probability one. Thus Corollary 6 is proved.

Before we prove Theorem 5 we need some additional lemmas.

LEMMA 6.3. *Let X be symmetric and assume d is an increasing function taking $(0, \infty)$ into $(0, \infty)$ with an inverse such that $d^{-1}(2u) \leq 2d^{-1}(u)$. Further, if*

$$\sup_{n \geq n_0} P\left(q\left(\frac{S_n}{d_n}\right) \geq 1\right) \leq a < \frac{1}{2},$$

then

$$\sup_{n \geq n_0} nP(q(X) \geq 2d_n) \leq \frac{2a}{1 - 2a},$$

and

$$d^{-1}(u)P(q(X) \geq u) \leq \frac{8a}{1 - 2a}$$

for all $u \geq 2d(n_0)$.

PROOF. If $n \geq n_0$ we have

$$1 - [1 - P(q(X) \geq 2d_n)]^n = P(\max_{k \leq n} q(X_k) \geq 2d_n) \leq P(\max_{k \leq n} q(S_k) \geq d_n)$$
$$\leq 2P(q(S_n) \geq d_n)$$

by Lévy's inequality. The inequalities $1 - t \leq e^{-t}$ and $1 - e^{-t} \geq t/(1 + t)$ and interpolation between $2d(n)$ and $2d(n + 1)$ complete the proof.

LEMMA 6.4. *Suppose that* $\gamma(t) \nearrow \infty$ *and for all sufficiently large* t

$$c_2\gamma(t) \le \gamma(2t) \le c_1\gamma(t)$$

for some $1 < c_2 < c_1 < \infty$. *Let* $n_k = 2^k$ *and* $I_k = (n_{k-1}, n_k]$. *For a sequence of independent symmetric random variables put*

$$\Lambda(k) = \sum_{j\in I_k} E(q^2(X_j)I(q(X_j) \le \delta\gamma(n_k)))/\gamma^2(n_k)$$

where $0 < \delta < \infty$ *and* q *is a Borel measurable semi-norm on* B. *If*

(a) $\sum_{k=1}^{\infty}\sum_{j\in I_k} P(q(X_j) \ge \delta\gamma(n_k)) < \infty$,

(b) $\sum_{k=1}^{\infty}\Lambda^r(k) < \infty$ *for some* $0 < r < \infty$, *and*

(c) *for all* k *sufficiently large*

$$P(q(\textstyle\sum_{j\in I_k} X_jI(q(X_j) \le \Lambda^{1/4}(k)\gamma(n_k))) \ge \tau_0\gamma(n_k)) < 1/24\,,$$

then

$$\limsup_n q(S_n)/\gamma(n) \le \frac{c_1c_2}{c_2-1}\{24\tau_0 + A\} \quad \text{w.p.1}$$

where $0 < A < \infty$ *satisfies* $[A/\delta] \ge 2r$.

PROOF. We first note that by the proof of Theorem 3.4.1 of Stout (1974)

$$\limsup_{k\to\infty} q(\textstyle\sum_{j\in I_k} X_j)/\gamma(n_k) \le M \quad \text{w.p.1}$$

implies

$$\limsup_n q(S_n)/\gamma(n) \le \frac{c_1c_2M}{c_2-1} \quad \text{w.p.1.}$$

We now indicate how the proof of Theorem 1 of Kuelbs and Zinn (1979) can be modified to accommodate this more general result. Fix $0 < \delta < \infty$ and define

$$X_j' = X_jI(q(X_j) \le \Lambda(k)^{1/4}\gamma(n_k)), \qquad j \in I_k$$
$$U_k^1 = q(\textstyle\sum_{j\in I_k} X_j')$$
$$U_k^2 = q(\textstyle\sum_{j\in I_k} X_jI(q(X_j) > \delta\gamma(n_k)))$$
$$U_k^3 = q(\textstyle\sum_{j\in I_k} X_jI(\Lambda^{1/4}(k)\gamma(n_k) < q(X_j) \le \delta\gamma(n_k))).$$

Put

$$b_k = \gamma(n_k)\Lambda(k)^{1/8}, \qquad c_k = \Lambda(k)^{1/8}$$
$$\varepsilon_k = (1+\eta)\Gamma\gamma(n_k)/b_k = (1+\eta)\Gamma\Lambda^{-1/8}(k)$$

where $\Gamma = \limsup_k E(U_k^1)/\gamma(n_k)$ and $\eta > 0$.

Since $\Lambda(k) \to 0$ as $k \to \infty$ we now use Lemma 6.1a to show $\Gamma \le 24\tau_0$, and using the proof of Theorem 1 of Kuelbs and Zinn (1979) we easily establish

$$\limsup_k \frac{U_k^1}{\gamma(n_k)} \le \Gamma \quad \text{w.p.1},$$

$$\lim_k \frac{U_k^2}{\gamma(n_k)} = 0 \quad \text{w.p.1},$$

and

$$\limsup_k \frac{U_k^3}{\gamma(n_k)} \le A \quad \text{w.p.1.}$$

Combining the above with $M = A + \Gamma$, the lemma is now proved.

LEMMA 6.5. *Let d, g, α, and α^{-1} be as in Theorem 5, and let $\gamma = \alpha^{-1} d\alpha$. Then we have the following holding:*

(i) *there is a constant $c_1 > 1$ such that*

$$tL_2 t \le \alpha^{-1}(t) \le c_1 t L_2 t$$

and

$$\alpha^{-1}(t) \sim t L_2 t.$$

(6.5) (ii) *$g(t)$ is continuous and $\frac{g(t)}{t} \nearrow$ for $t \ge \ell$.*

(iii) *$t/d(t) \nearrow$ and $d(2t) \le 2d(t)$ for $t \ge g(\ell)$.*

(iv) $$\limsup_{t\to\infty} \frac{\gamma(2t)}{2\gamma(t)} \le 1.$$

(v) *for $t \ge \max(\ell, g(\ell))$ we have positive constants c_2 and c_3 such that*

$$c_2 t^{1/2} \le d(t) \le c_3 t,$$

and hence $L_2(d\alpha(t)) \sim L_2 t$, so $\gamma(t) \sim L_2 t\, d(t/L_2 t)$.

(vi) *there exists a constant $\lambda < \infty$ such that for $s \le t$ sufficiently large*

$$\frac{t}{[\gamma(t)]^2} \le \lambda \frac{s}{[\gamma(s)]^2}$$

(vii) $\liminf_{t\to\infty} \dfrac{\gamma(t)}{d(t)} \ge 1.$

PROOF. (i) is obvious since $\alpha(t) = t/L_2 t$. Further, since d is continuous, its inverse g is also continuous. Thus by (2.61-i) and Theorem 264 of Kestelman (1937) we have for all $b > a > \ell$ that

$$\frac{g(b)}{b} - \frac{g(a)}{a} = \int_a^b \frac{d}{dt}\left(\frac{g(t)}{t}\right) dt = \int_a^b \left[\frac{tg'(t) - g(t)}{t^2}\right] dt \ge 0.$$

Hence $g(t)/t$ increases for $t \ge \ell$, and $g(d(t))/d(t) = t/d(t)$ also increases for $t \ge g(\ell)$. As a result, $t/d(t) \le 2t/d(2t)$, or $d(2t) \le 2d(t)$ for $t \ge g(\ell)$, so (ii) and (iii) of Lemma 6.5 hold.

To establish (iv) we observe that

$$\gamma(2t) = \alpha^{-1} d\alpha(2t) = \alpha^{-1} d\left(\frac{2t}{L_2 2t}\right) \le \alpha^{-1} d\left(\frac{2t}{L_2 t}\right) \le \alpha^{-1}(2d(t/L_2 t))$$

$$\sim 2d(t/L_2 t) L_2(2d(t/L_2 t)) \sim 2d(t/L_2 t) L_2(d(t/L_2 t)) \sim 2\alpha^{-1} d\alpha(t) = 2\gamma(t).$$

Now (2.61-iii) and (iii) easily yield (v) so we now turn to (vi).

Let $s \le t$ be taken sufficiently large. Then (i) implies

$$\frac{t}{[\alpha^{-1} d\alpha(t)]^2} \le \frac{t}{[d\alpha(t) L_2(d\alpha(t))]^2} = \frac{\alpha(t) L_2 t}{[d\alpha(t)]^2 [L_2 d\alpha(t)]^2}$$

$$\le \frac{\alpha(s)}{[d(\alpha(s))]^2} \frac{L_2 t}{[L_2 d\alpha(t)]^2} \text{ by (2.61-iii)}$$

$$\le \frac{c_4 \alpha(s)}{[d(\alpha(s))]^2} \frac{1}{L_2 s} \text{ for some } c_4 < \infty \text{ by using (v)}$$

$$\le \frac{c_5 \alpha(s)}{[d(\alpha(s))]^2} \frac{L_2 s}{[L_2(d(\alpha(s)))]^2} \text{ for some } c_5 < \infty \text{ by using (v)}$$

$$\le \frac{c_6 s}{[\alpha^{-1} d\alpha(s)]^2} \text{ by (i) for some } c_6 < \infty.$$

Hence (vi) holds and by using (i), (v), and (iii) we have that

$$\alpha^{-1}d\alpha(t) \geq d(\alpha(t))L_2(d\alpha(t)) \geq \frac{d(t)}{L_2 t} L_2(d\alpha(t)) \sim d(t) \text{ as } t \to \infty.$$

Thus (vii) is verified and Lemma 6.5 is proved.

LEMMA 6.6. *Let d, g, α, and α^{-1} be as in Theorem* 5, *and let $\gamma = \alpha^{-1}d\alpha$. Then*

$$E(\alpha^{-1}g\alpha(q(X))) < \infty \text{ iff } E(\alpha^{-1}g\alpha(q(\rho X))) < \infty \tag{6.6}$$

for some (*and hence all*) $\rho > 0$. *Further* (6.6) *holds* iff *for some* (*and hence all*) $\rho > 0$

$$\textstyle\sum_{k=1}^{\infty} \sum_{j\in I_k} P(q(X_j) > \rho\gamma(n_k)) < \infty. \tag{6.7}$$

where $n_k = 2^k$, $I_k = (n_{k-1}, n_k]$ for $k \geq 1$.

PROOF. Since $\gamma^{-1} = \alpha^{-1}g\alpha \nearrow$, (6.6) holds because Lemma 6.5–vi implies there exist $u_0 > 0$ and $\lambda > 0$ such that

$$\forall\rho \geq 1, \quad \forall u \geq u_0, \quad \gamma^{-1}(\rho u) \leq \lambda\rho^2\gamma^{-1}(u).$$

Now (6.6) is true if and only if there exists $\rho > 0$ such that

$$\textstyle\sum_{k=1}^{\infty} \sum_{j\in I_k} P(q(X_j) > \rho\gamma(j)) < \infty.$$

Hence (6.6) is equivalent to (6.7) since

$$\gamma(n_k) \geq \gamma(j) \geq \tfrac{1}{2}\,\gamma(n_k)$$

whenever $j \in I_k$ and $k \geq k_0$ by Lemma 6.5–iv. Thus the lemma is proved.

PROOF OF THEOREM 5. Let $n_k = 2^k$, $I_k = (n_{k-1}, n_k]$, $p_k = [Lk]$, $r_k = [n_k/p_k]$, and $\gamma(t) = \alpha^{-1}d\alpha(t)$.

If (2.62–ii) holds and if $\{X_j':j \geq 1\}$ is an independent copy of $\{X_j:j \geq 1\}$, then

$$\limsup_n q((S_n - S_n')/\gamma(n)) = M < \infty \quad \text{w.p.1.} \tag{6.8}$$

Hence (2.63) holds by part one of Lemma 6.6.

Now assume (2.63) holds. Then (6.7) holds and hence

$$\begin{aligned}&\textstyle\sum_k \sum_{j\in I_k} P(q(X_j - X_j') > 2\gamma(n_k))\\ &\qquad \leq \textstyle\sum_k \sum_{j\in I_k} [P(q(X_j) > \gamma(n_k)) + P(q(X_j') > \gamma(n_k))] < \infty.\end{aligned} \tag{6.9}$$

Applying Lemma 6.6 to $X - X'$ we thus have

$$E(\alpha^{-1}g\alpha(q(X - X'))) < \infty. \tag{6.10}$$

If, in addition, we assume (2.61–iv) for X, then we also have $n_0 < \infty$ such that

$$\sup_{n\geq n_0} P\left(q\left(\frac{S_n - S_n'}{d(n)}\right) > 2t_0\right) \leq \frac{1}{8e^2}. \tag{6.11}$$

Thus if X satisfies (2.63) and (2.61–iv) we have the symmetrization $X - X'$ satisfying (6.10) and (6.11). Our next goal is to prove that for X symmetric, (2.63) and the condition

$$\sup_{n\geq n_0} P\left(q\left(\frac{S_n}{d(n)}\right) > 2t_0\right) \leq \frac{1}{8e^2} \tag{6.12}$$

together imply

$$\limsup_n q\left(\frac{S_n}{\gamma(n)}\right) \leq [32e^2 + 96(2 + \sqrt{2})]t_0 \quad \text{w.p.1.} \tag{6.13}$$

From this we then will have (6.10) and (6.11) implying

$$\limsup_n q\left(\frac{S_n - S'_n}{\gamma(n)}\right) \le [32e^2 + 96(2+\sqrt{2})]t_0 \quad \text{w.p.1.} \tag{6.14}$$

Using Lemma 2, (2.67–x), we thus have (2.62) since

$$\limsup_n \frac{q(X_n)}{\gamma(n)} = 0 \quad \text{w.p.1}$$

(by using Lemma 6.6), and for all $\rho > 0$, $0 < \rho < 1$,

$$q\left(\frac{nE(XI(q(X) \le \gamma(n))) - nE(XI(q(x) \le \rho\gamma(n)))}{\gamma(n)}\right) = \frac{n}{\gamma(n)} q(E(XI(\rho\gamma(n) < q(X) \le \gamma(n))))$$

$$\le nP(q(X) > \rho\gamma(n)) \to 0 \quad \text{as} \quad n \to \infty$$

(by using Lemma 6.6 since (2.63) holds).

Hence we assume X is symmetric, X satisfies (2.63), and (6.12) is satisfied. If $2t_0 > 1$ we replace X by $X/2t_0$ and using Lemma 6.6 it suffices to show (2.63) and

$$\sup_{n \ge n_0} P(q(S_n/d(n)) \ge 1) \le \tfrac{1}{8}e^2 \tag{6.15}$$

imply

$$\limsup_n q\left(\frac{S_n}{\gamma(n)}\right) \le [16e^2 + 48(2+\sqrt{2})] \quad \text{w.p.1.} \tag{6.16}$$

Now $d(\alpha(n_k)) = d\left(\frac{2^k}{L_2 2^k}\right) \ge d\left(\frac{2^k}{Lk}\right) \ge d\left(\frac{2^k}{p_k}\right) \ge d(r_k)$ so (6.15) implies there exists $k_0 < \infty$ such that

$$\sup_{k \ge k_0} P\left(q\left(\frac{S_{r_k}}{d\alpha(n_k)}\right) > 1\right) \le \frac{1}{8}e^2. \tag{6.17}$$

Now we set

$$\begin{aligned} Y_j &= X_j I(q(X_j) \le d\alpha(n_k)) \quad (j \in I_k) \\ Z_j &= X_j - Y_j \quad (j \in I_k). \end{aligned} \tag{6.18}$$

Next we define

$$V_n = \textstyle\sum_{j=1}^n Y_j \quad \text{and} \quad T_m^n = V_n - V_m$$

for $n \le 1$. The proof will be completed by showing

$$\begin{aligned} &\text{(i)} \ \limsup_n q(V_n/\gamma(n)) \le 16e^2 \quad \text{w.p.1} \\ &\text{(ii)} \ \limsup_n q(\textstyle\sum_{j=1}^n Z_j/\gamma(n)) \le 48(2+\sqrt{2}) \quad \text{w.p.1.} \end{aligned} \tag{6.19}$$

To prove (6.19–i) we will apply the argument used in the proof of Theorem 4 and to prove (6.19–ii) we will apply Lemma 6.4 to $\{Z_j\}$.

Applying (6.3) with $d_k = d\alpha(n_k)$ we obtain for $A > 4e^2$ that

$$\textstyle\sum_k P(\max_{n \in I_k} q(V_n) \ge 2Ap_k d\alpha(n_k)) < \infty.$$

Hence for any $\eta > 0$ (6.5–iv) implies there exists k_0 such that

$$\textstyle\sum_{k \ge k_0} P(\max_{n \in I_k} q(V_n/\gamma(n)) \ge (2+2\eta)2A) \le \sum_{k \ge k_0} P\left(\max_{n \in I_k} q(V_n) \ge \left(1+\frac{\eta}{2}\right)2A\gamma(n_k)\right)$$

$$\le \textstyle\sum_{k \ge k_0} P(\max_{n \in I_k} q(V_n) \ge 2Ap_k\, d\alpha(n_k))$$

$$< \infty.$$

In the last inequality we have made use of the fact that $L_2\, d\alpha(n_k) \sim p_k$ as $k \to \infty$ and this follows from (6.5–v). Since $\eta > 0$ was arbitrary

$$\limsup_n q(V_n/\gamma(n)) \le 16e^2 \quad \text{w.p.1} \tag{6.20}$$

and it remains only to establish (6.19–ii).

Recalling $\{Z_j : j \ge 1\}$ as in (6.18) we choose $0 < \delta \le 1$ and apply Lemma 6.4 to $\{Z_j\}$.

To apply Lemma 6.4 we first identify c_1 and c_2 since $\gamma(t) = \alpha^{-1}\, d\alpha(t) \nearrow$. Now from (6.5–iv) we have $c_1 = 2 + \varepsilon$ for any $\varepsilon > 0$ and $c_2 = \sqrt{2(1-\varepsilon)}$ for any $\varepsilon > 0$. To see $c_2 = \sqrt{2(1-\varepsilon)}$ works for any $\varepsilon > 0$ note that (6.5–i, v) implies that for sufficiently large t

$$\frac{\gamma(2t)}{\gamma(t)} \sim \frac{d\alpha(2t)}{d\alpha(t)} \ge \frac{d(2(1-\varepsilon)\alpha(t))}{d(\alpha(t))}.$$

Now (2.61–iii) implies

$$d(2(1-\varepsilon)\alpha(t)) \ge \sqrt{2(1-\varepsilon)}\,\sqrt{\alpha(t)}\,\frac{d(\alpha(t))}{\sqrt{\alpha(t)}},$$

so combining these facts we see for all sufficiently large t

$$\frac{\gamma(2t)}{\gamma(t)} \ge \sqrt{2(1-\varepsilon)}.$$

Since $0 < \delta \le 1$ is fixed we easily have condition (a) of Lemma 6.4 by applying Lemma 6.6 since we are assuming X satisfies (2.63). To see that condition (c) in Lemma 6.4 applies recall that the $\{Z_j : j \ge 1\}$ are independent and symmetric, so for all sufficiently large k

$$\begin{aligned}
P(q(\textstyle\sum_{j\in I_k} Z_j I(q(Z_j) \le \Lambda(k)^{1/4}\gamma(n_k))) \ge \gamma(n_k))& \\
&\le 2P(q(\textstyle\sum_{j\in I_k} X_j) \ge \gamma(n_k)) \\
&= 2P(q(S_{n_{k-1}}) \ge \gamma(n_k)) \\
&\le 2P\left(q(S_{n_{k-1}}) \ge \frac{3}{2}\gamma(n_{k-1})\right) \quad \text{(by (6.5–iv))} \\
&\le 2P\left(q(S_{n_{k-1}}) \ge \frac{5}{4}d(n_k)\right) \quad \text{(by (6.5–vii))} \\
&\le \frac{1}{(4e^2)} < \frac{1}{24} \quad \text{(by (6.15))}.
\end{aligned}$$

Thus (c) of Lemma 6.4 holds with $\tau_0 = 1$, and once we establish (b) of Lemma 6.4 with $r = 2$ we will have

$$\limsup_n q(\textstyle\sum_{j=1}^n Z_j)/\gamma(n) \le \dfrac{(2+\varepsilon)\sqrt{2(1-\varepsilon)}}{\sqrt{2(1-\varepsilon)} - 1}\{24 + A\} \quad \text{w.p.1} \tag{6.21}$$

where $A = 4\delta$, $0 < \delta \le 1$. Since $\delta > 0$ and $\varepsilon > 0$ are arbitrary, (6.21) implies (6.19–ii).

Hence the theorem is proved once we verify

$$\textstyle\sum_k \Lambda^2(k) < \infty \tag{6.22}$$

where

$$\Lambda(k) = \textstyle\sum_{j\in I_k} E(q^2(Z_j)I(q(Z_j) \le \delta\gamma(n_k)))/\gamma^2(n_k)$$

and $0 < \delta \le 1$ is arbitrary. From (6.18) and that $\delta \le 1$ we see

$$\Lambda(k) = \textstyle\sum_{j\in I_k} E(q^2(X_j)I(d\alpha(n_k) < q(X_j) \le \gamma(n_k)))/\gamma^2(n_k).$$

To show (6.22) let X, Y be independent and identically distributed and set $\xi = q(X)$ and

$\eta = q(Y)$. Then there exists k_0 such that

$$\sum_{k\ge k_0} \Lambda^2(k) \le 2\sum_{k\ge k_0}\left(\frac{n_k}{\gamma^2(n_k)}\right)^2 E[\xi^2\eta^2 I(d\alpha(n_k)\le \xi\le \eta\le \gamma(n_k))]$$

$$= 2E\left\{\xi^2\eta^2\sum_{k\ge k_0}\left(\frac{n_k}{\gamma^2(n_k)}\right)^2 I(\xi\le\eta,\ \alpha^{-1}g\alpha(\eta)\le n_k\le \alpha^{-1}g(\xi))\right\}$$

$$\le 2\lambda^2 E\left\{\xi^2\eta^2\sum_{k\ge k_0}\left(\frac{\alpha^{-1}g\alpha(\eta)}{\eta^2}\right)^2 I(\xi\le\eta\le\alpha^{-1}(\xi),\right.$$
$$\left. L\alpha^{-1}g\alpha(\eta)\le k\log 2\le L\alpha^{-1}g(\xi))\right\}$$

by (6.5–vi)

$$\le \frac{2\lambda^2}{\log 2}E\left\{\frac{\xi^2\eta^2}{\eta^4}(\alpha^{-1}g\alpha(\eta))^2 I(\xi\le\eta\le\alpha^{-1}(\xi))[L(\alpha^{-1}g(\xi)/\alpha^{-1}g\alpha(\xi))]\right\}$$

$$\le \frac{2\lambda^2}{\log 2}E\left\{\xi^2\left(\frac{\alpha^{-1}g\alpha(\eta)}{\eta}\right)^2 I(\xi\le\eta\le\alpha^{-1}(\xi))L(m_1^2\lambda(L_2\xi)^2)\right\}$$

by (6.5–vi) for some finite constant m_1

with $t = \alpha^{-1}g(\xi)$ and $s = \alpha^{-1}g\alpha(\xi)$

$$\le \frac{2\lambda^3}{\log 2}E\{\alpha^{-1}g\alpha(\xi)\alpha^{-1}g\alpha(\eta)I(\xi\le\eta\le\alpha^{-1}(\xi))L(\lambda m_1^2(L_2\xi)^2)\}$$

by (6.5–vi)

$$\le 2m_2E[\alpha^{-1}g\alpha(\xi)\alpha^{-1}g\alpha(\eta)I(\xi\le\eta\le\alpha^{-1}(\xi))L_3\xi]$$

for some positive m_2

$$\le m_3E(L_3\xi(L_2\xi)^2g\alpha(\xi)g\alpha(\eta)I(\xi\le\eta\le\alpha^{-1}(\xi)))$$

for some positive m_3 by use of (6.5–v, i).

Now for s sufficiently large integration by parts and a standard application of Theorem 264 of Kestelman (1937) implies that

$$E(g\alpha(\eta)I(s\le\eta\le\alpha^{-1}(s)))\le P(\eta\ge s)g\alpha(s)+\int_{(s,\alpha^{-1}(s))}P(\eta>t)\,d(g\alpha)(t)$$

$$= P(\eta\ge s)g\alpha(s)+\int_s^{\alpha^{-1}(s)}P(\eta>t)(g\alpha)'(t)\,dt.$$

Recalling $g(u) = d^{-1}(u)$ we see that Lemma 6.3 and (2.61–i) imply that there exists a positive constant m_4 such that for all s sufficiently large

$$E(g\alpha(\eta)I(s\le\eta\le\alpha^{-1}(s)))\le m_4\left[\frac{g(\alpha(s))}{g(s)}+\int_s^{\alpha^{-1}(s)}\frac{g(\alpha(t))}{\alpha(t)}\frac{\alpha'(t)}{g(t)}\,dt\right].$$

Hence by (2.61–ii) for s sufficiently large there are positive constants m_5, m_6 such that

$$E(g(\alpha(\eta))I(s\le\eta\le\alpha^{-1}(s)))\le m_5\left[\frac{1}{(L_3s)^2(L_2s)}+\int_s^{\alpha^{-1}(s)}\frac{dt}{(L_3t)^2(L_2t)t}\right]\le m_6/(L_3s)(L_2s).$$

Using the independence of ξ and η and the previous inequalities we now have positive constants m_7 and m_8 such that

$$\textstyle\sum_{k\geq k_0}\Lambda^2(k) \leq m_7(1 + E[(L_2\xi)^2(L_3\xi)g(\alpha(\xi))/(L_3\xi L_2\xi)])$$
$$\leq m_8(E(\alpha^{-1}g\alpha(\xi)) + 1) < \infty.$$

Hence the theorem is proved.

PROOF OF COROLLARY 7. Let $X \in \mathrm{DA}(Z)$ and assume X is symmetric. Then there exists $\bar{d}(n) \nearrow \infty$ such that $\bar{d}(n+1)/\bar{d}(n) \to 1$, and

$$\mathscr{L}(S_n/\bar{d}(n)) \to \mathscr{L}(Z),$$

and, hence for all $f \in B^*$,

$$\mathscr{L}(f(S_n/\bar{d}(n))) \to N(\mathscr{L}(f(Z))).$$

If $0 < Ef^2(X) < \infty$ for some $f \in B^*$ such that $\mathscr{L}(f(Z)) \neq \delta_0$, then

$$\mathscr{L}(f(S_n/\sqrt{n})) \to N(0, E(f^2(X))),$$

and hence

$$\lim_n \bar{d}(n)/\sqrt{n} = c \quad (0 < c < \infty).$$

But then

$$\mathscr{L}\left(\frac{S_n}{\sqrt{n}}\right) \to \mathscr{L}(cZ)$$

and hence $Ef^2(X) < \infty$ for all $f \in B^*$. Taking $d(t) = \sqrt{t}$ we have $g(t) = t^2$ and (2.61) holds. Thus

$$\alpha^{-1}g\alpha(t) \sim t^2/L_2t$$

and

$$\alpha^{-1}\,d\alpha(t) \sim \sqrt{tL_2t}$$

so Theorem 4.1 of Goodman, Kuelbs, Zinn (1981) completes the proof in this case.

Hence we assume $Ef^2(X) = 0$ or ∞ for all $f \in B^*$ such that $\mathscr{L}(f(Z)) \neq \delta_0$. If $E(f^2(X)) = 0 \forall f \in B^*$, then $X = 0$ so $X \notin DA(Z)$ and hence we choose $f \in B^*$ such that

$$Ef^2(X) = \infty.$$

We now show the corollary holds with $d = K_{f(X)}$ and that $\bar{d}(n) \approx d(n)$. Let $\xi = f(X)$ and set

$$U(t) = E(\xi^2 I(|\xi| \leq t)) \qquad (0 \leq t < \infty),$$

and

$$V(t) = E(|\xi| I(|\xi| > t)) \quad (0 \leq t < \infty).$$

Then $\xi \in \mathrm{DA}(f(Z))$, so it is well known that $U(t)$ is slowly varying and

$$\lim_{t\to\infty}\frac{t^2P(|\xi| > t)}{U(t)} = 0. \tag{6.23}$$

Now $U(t)$ slowly varying at infinity implies

$$\lim_{t\to\infty}\frac{U(t)}{t^\varepsilon} = 0$$

for all $\varepsilon > 0$, and hence (6.23) implies

$$E|\xi|^\alpha < \infty$$

for $0 < \alpha < 2$. Further, by the "equivalence lemma" of Hahn and Klass (1980) or Feller (1971) we have

$$\lim_{t\to\infty} \frac{tV(t)}{U(t)} = 0. \tag{6.24}$$

Hence, if we define

$$S(t) = \int_0^t E(|\xi| I(|\xi| > u))\, du, \tag{6.25}$$

then

$$S(t) = E(|\xi|(|\xi| \wedge t)) = U(t) + tV(t) \sim U(t)$$

is slowly varying. By the representation theorem for slowly varying functions as given in Seneta (1976) we have

$$\frac{S(t)}{S(\alpha(t))} = \frac{a(t)\exp\left\{\int_1^t \frac{\varepsilon(s)}{s}\, ds\right\}}{a(\alpha(t))\exp\left\{\int_1^{\alpha(t)} \frac{\varepsilon(s)}{s}\, ds\right\}} \tag{6.26}$$

where

$$0 < \lim_{t\to\infty} a(t) = c < \infty \quad \text{and} \quad \lim_{s\to\infty} \varepsilon(s) = 0.$$

Hence for all $\delta > 0$ we have for all t sufficiently large that

$$1 \le \frac{S(t)}{S(\alpha(t))} \le 2\exp\left\{\int_{\alpha(t)}^t \frac{\varepsilon(s)}{s}\, ds\right\} \le 2\exp\{\delta L_3 t\} = 2(L_2 t)^\delta. \tag{6.27}$$

Hence, if we let

$$g(t) = t^2/S(t), \tag{6.28}$$

then for t sufficiently large

$$\frac{g(\alpha(t))}{g(t)} \le \frac{2}{(L_2 t)^{2-\delta}}.$$

Since $S(t) = \int_0^t E(|\xi| I(|\xi| > s))\, ds$ we have

$$S'(t) = E(|\xi| I(|\xi| > t)) = V(t) \ge 0$$

on the complement of the discontinuity points A of the decreasing function $E(|\xi| I(|\xi| > t))$.

Further,

$$g'(t) = \frac{2tS(t) - t^2 S'(t)}{S^2(t)} \le \frac{2t}{S(t)} = 2g(t)/t \tag{6.29}$$

on A^c. Further, since $S(t) \sim U(t)$ as $t \to \infty$, $S'(t) = V(t)$ on $t \in A^c$, and $tS'(t)/U(t) \to 0$ we have

$$\lim_{t\to\infty} \frac{tS'(t)}{S(t)} = 0. \tag{6.30}$$

Therefore, combining (6.28), (6.29), and (6.30) we have for all sufficiently large t that

$$\frac{g(t)}{t} = \frac{t}{S(t)} < \frac{3t}{2S(t)} < g'(t).$$

Also, $g(t)/t^2 = 1/S(t) \downarrow$ so we have (2.61–i, ii, iii).

Applying (2.20) of Lemma 1 we have

$$E|f(S_n)| \approx g^{-1}(n), \tag{6.31}$$

so defining $d(t) = g^{-1}(t)$ we have

$$\frac{E|f(S_n)|}{d(n)} \approx 1.$$

Now $\mathscr{L}(f(S_n)/\bar{d}(n)) \to \mathscr{L}(f(Z))$ and by A. de Acosta and E. Giné (1979), Theorem 6.1, for $1 \le \alpha < 2$

$$\sup_n E\left|\frac{f(S_n)}{\bar{d}(n)}\right|^\alpha < \infty.$$

Hence by uniform integrability

$$\lim_n E\left|f\left(\frac{S_n}{\bar{d}(n)}\right)\right| = E|f(Z)| > 0, \tag{6.32}$$

and by combining (6.31) and (6.32) we have $\bar{d}(n) \approx d(n)$. Therefore, $\{S_n/d(n)\}$ is tight so (2.61–iv) holds with $q(\cdot) = \|\cdot\|$.

Applying Theorem 5 we now have

$$\limsup_n \|S_n\|/\alpha^{-1}\, d\alpha(n) < \infty \quad \text{w.p.1}$$

iff

$$E(\alpha^{-1}g\alpha(\|X\|)) < \infty.$$

Further, by Klass (1976), Theorem 1.2, we have

$$0 < \limsup_n \frac{|f(S_n)|}{\alpha^{-1}\, d\alpha(n)} < \infty$$

since $\alpha^{-1}d\alpha(n) \approx (L_2n)\, d(n/L_2n)$ by Lemma 6.5–(v) and d as defined here is the K-function, K_f, of Klass. Hence

$$0 < \limsup \|S_n\|/\alpha^{-1}\, d\alpha(n) < \infty \quad \text{w.p.1}$$

iff

$$E(\alpha^{-1}g\alpha(\|X\|)) < \infty.$$

To complete the proof in the symmetric case it now suffices to show that $E(\alpha^{-1}g\alpha(\|X\|)) < \infty$ implies

$$\left\{\frac{S_n}{\alpha^{-1}\, d\alpha(n)}\right\}$$

is conditionally compact.

Since $\{S_n/d(n)\}$ is tight we choose a compact convex symmetric set D such that

$$\sup_n P\left(\frac{S_n}{d(n)} \notin D\right) \le \frac{1}{16e^2}.$$

Define the semi-norm $q = q_\delta (0 < \delta)$ on B by

$$q(x) = \inf\{t : x/t \in D^\delta\}.$$

Then q is equivalent to $\|\cdot\|$ (so $q(x) < \infty$ for all $x \in B$), and $E(\alpha^{-1}g\alpha(q(X))) < \infty$ by Lemma 6.6. Hence by applying Theorem 5 (since d and g satisfy (2.61) with $t_0 = 1$) we have for every $\delta > 0$ that $q = q_\delta$ is such that

$$\limsup_n q\left(\frac{S_n}{\alpha^{-1}\, d\alpha(n)}\right) \le 26\,[32e^2 + 96(2+\sqrt{2})] \quad \text{w.p.l.}$$

Hence $\{S_n/\alpha^{-1}\, d\alpha(n)\}$ is relatively compact with probability one and with limit set contained in the compact set $26[32e^2 + 96(2+\sqrt{2})]D$. Of course, the limit set is not $\{0\}$ as $\limsup_n \|S_n\|/\alpha^{-1}\, d\alpha(n) > 0$ w.p.l.

If X is not symmetric, then $X \in \mathrm{DA}(Z)$ implies there are shifts β_n and normalizations $\{\bar{d}(n)\}$ such that $\mathscr{L}((S_n - \beta_n)/\bar{d}(n)) \to \mathscr{L}(Z)$, and $\beta_n \sum_{j=1}^n E(XI(\|X\| \le \bar{d}(n))) = o(\bar{d}(n))$ by Corollary 2.12 of A. de Acosta, A. Araujo, and E. Giné (1978). Hence $\mathscr{L}((S_n - S_n')/\bar{d}(n)) \to \mathscr{L}(Z - Z')$ where X' and Z' are independent copies of X and Z, respectively. Since the normalizing functions we construct for $X - X'$ again we call them d and g, satisfy (2.61) and $d(n) \approx \bar{d}(n)$, the condition (2.65) for X implies (2.65) also holds for $X - X'$, and hence by the previous case

$$\left\{\frac{S_n - S_n'}{\alpha^{-1}\, d\alpha(n)}\right\}$$

is conditionally compact and has a non-zero cluster set with probability one. If C is a compact convex symmetric set which contains the cluster set of

$$\left\{\frac{S_n - S_n'}{\alpha^{-1}\, d\alpha(n)}\right\}$$

w.p.l., then by applying Lemma 2-xi we have for all $\rho > \limsup_n \|X_n\|/\alpha^{-1}\, d\alpha(n)$ that with probability one

$$\left\{\frac{S_n - nE(XI(\|X\| \le \rho\gamma(n)))}{\gamma(n)}\right\}$$

is relatively compact with cluster set in $2C$. Here, of course $\gamma(n) = \alpha^{-1}\, d\alpha(n)$. Since (2.65) holds, any $\rho > 0$ works (see Lemma 6.6).

Thus Corollary 7 is proved provided

$$\lim_n \frac{n}{\gamma_n} \|E(XI(\|X\| > \gamma_n))\| = 0.$$

That is, given the above limit is zero, then (2.64) holds as $2C$ is compact and hence if

$$C\left(\left\{\frac{S_n - nE(X)}{\gamma_n}\right\}\right) = \{0\},$$

then

$$\lim_n \frac{S_n - nE(X)}{\gamma_n} = 0 \quad \text{w.p.l.}$$

Thus

$$\lim_n \frac{S_n - S_n'}{\gamma_n} = 0 \quad \text{w.p.l}$$

which contradicts the fact that

$$C\left(\left\{\frac{S_n - S_n'}{\gamma_n}\right\}\right) \ne \{0\}.$$

Hence Corollary 7 will be proved if we show

$$\limsup_n \frac{n}{\gamma_n} \|E(XI(\|X\| > \gamma_n))\| = 0.$$

Now

$$\frac{n}{\gamma_n}\|E(XI(\|X\|>\gamma_n))\| \le \frac{n}{\gamma_n} E(\|X\|I(\|X\|>\gamma_n))$$

$$= \frac{n}{\gamma_n}\left[\gamma_n P(\|X\|>\gamma_n) + \int_{\gamma_n}^{\infty} P(\|X\|>t)\,dt\right],$$

and $\lim_n nP(\|X\|>\gamma_n) = \lim_n nP(\alpha^{-1}g\alpha(\|X\|)>n) = 0$ from (2.65), so it suffices to show that

$$\lim_n \frac{n}{\gamma_n}\int_{\gamma_n}^{\infty} P(\|X\|>t)\,dt = 0.$$

Using the properties of $g(t)$, its inverse $d(t)$, and $\alpha(t) = t/L_2t$ and $\alpha^{-1}(t)$, it follows that $\gamma(t)$ is strictly increasing and continuous for all t sufficiently large, and $\gamma'(t)$ exists except possibly at a countable number of points. Further, using the chain rule and (2.61–i) we get

$$d'(t) = \frac{1}{g'(d(t))} \approx \frac{d(t)}{g(d(t))} = \frac{d(t)}{t}$$

for t sufficiently large except for countably many t's. Again, using the chain rule and that

$$L\alpha^{-1}d\alpha(t) \approx Lt$$

we obtain for all except countably many t's that

$$\gamma'(t) \approx \frac{d(t/L_2t)}{t/L_2t}$$

as $t \to \infty$. Thus by applying Theorem 2.64 of Kestelman (1937) we have a constant $c<\infty$ such that

$$\frac{n}{\gamma_n}\int_{\gamma_n}^{\infty} P(\|X\|>t)\,dt = \frac{n}{\gamma_n}\sum_{j=n}^{\infty}\int_{\gamma_j}^{\gamma_{j+1}} P(\|X\|>t)\,dt \le \frac{n}{\gamma_n}\sum_{j=n}^{\infty} P(\|X\|>\gamma_j)(\gamma_{j+1}-\gamma_j)$$

$$= \frac{n}{\gamma_n}\sum_{j\ge n} P(\|X\|>\gamma_j)\int_j^{j+1}\gamma'(t)\,dt \le \frac{cn}{\gamma_n}\sum_{j\ge n} P(\|X\|>\gamma_j)\int_j^{j+1}\frac{d(t/L_2t)}{t/L_2t}\,dt.$$

Since $\dfrac{d(s)}{s}\downarrow$ we thus have

$$\frac{n}{\gamma_n}\int_{\gamma_n}^{\infty} P(\|X\|>t)\,dt \le \frac{cn}{\gamma_n}\frac{d(n/L_2n)}{n/L_2n}\sum_{j\ge n} P(\|X\|>\gamma_j) = \frac{cL_2n\,d(n/L_2n)}{\gamma_n}\sum_{j\ge n} P(\|X\|>\gamma_j).$$

Now $\gamma_n \approx L_2n\ d(n/L_2n)$ and $\sum_{j\ge n} P(\|X\|>\gamma_j) \to 0$ as $n\to\infty$ so we have the proof complete.

PROOF OF LEMMA 2. Fix $A<\infty$, let $T=\{n: |\gamma_n|\le A\}$, and assume T is infinite. Then, if $\lambda > M$ and $n\in T$

$$\frac{nP(q(X-X')>2\lambda A)}{1+nP(q(X-X')\ge 2\lambda A)} \le P(\max_{j\le n} q(X_j-X_j')>2\lambda A)$$

$$\le P(\max_{j\le n} q(S_j-S_j')>\lambda A)$$

$$\le 2P(q(S_n-S_n')>\lambda A).$$

Using (2.66–a) and $\lambda > M$ we have

$$\lim_{n\to\infty, n\in T} P(q(S_n - S_n') > \lambda A) = 0$$

and hence $nP(q(X - X') > 2\lambda A) = 0$. Thus T infinite implies

$$P(q(X - X') \le 2\lambda A) = 1$$

for all $\lambda > M$.

Now since we assumed X is not equal to a constant, it is the case that $q(X - X') \neq 0$ with positive probability. Further, since we assume the q-topology on B is separable there exists a q-continuous linear functional f on B such that

$$\sup_{q(x)\le 1} |f(x)| \le 1$$

and such that $f(X - X')$ is not degenerate at zero. Hence
$0 < E(f^2(X - X')) \le E(q^2(X - X')) \le 4\lambda^2 A^2 < \infty$ and the central limit theorem (on $\mathbb{R}^1$) applies. But this contradicts

$$\lim_{n\to\infty, n\in T} P(|\textstyle\sum_{j=1}^n f(X_j - X_j')| > \lambda A) \le \lim_{n\to\infty, n\in T} P(q(S_n - S_n') > \lambda A) = 0.$$

Hence T is finite and since A was arbitrary $\gamma_n \to \infty$ as claimed.

To prove (ii) observe that for all $\{c_n\} \subseteq B$

$$M \le \limsup_n \frac{q(S_n - c_n)}{\gamma_n} + \limsup_n q\frac{(S_n' - c_n)}{\gamma_n} \quad \text{w.p.1},$$

and since $\gamma_n \to \infty$, Kolmogorov's zero-one law implies each of the limit superiors is a constant. Now $\{(S_n - c_n)/\gamma_n\}$ and $\{(S_n' - c_n)/\gamma_n\}$ have the same distribution so (ii) holds.

The proof of (iii) and (iv) follows immediately from Fubini's Theorem.

To prove (v) observe that by (i) and (iii) there exists $\{c_n\} \subseteq B$ such that

$$M = \limsup_n \frac{q(S_n - c_n)}{\gamma_n} = \limsup_n \frac{q(S_n - X_1 - c_n)}{\gamma_n} = \limsup_n \frac{q(S_{n-1} - c_n)}{\gamma_n}.$$

Hence

$$\limsup_n \frac{q(X_n)}{\gamma_n} \le \limsup_n \frac{q(S_n - c_n)}{\gamma_n} + \limsup_n \frac{q(S_{n-1} - c_n)}{\gamma_n} = 2M.$$

To prove (vi) let

$$\begin{aligned}
X_j^\rho &= X_j I(q(X_j) \le \rho\gamma_n) - E(X_j I(q(X_j) \le \rho\gamma_n)) \\
\tilde{X}_j^\rho &= X_j I(q(X_j) \le \rho\gamma_n) - X_j' I(q(X_j') \le \rho\gamma_n) \\
S_n^\rho &= \textstyle\sum_{j=1}^n X_j^\rho \\
\tilde{S}_n^\rho &= \textstyle\sum_{j=1}^n \tilde{X}_j^\rho.
\end{aligned}$$

Then Jensen's inequality implies

$$Eq(S_n^\rho/\gamma_n) \le Eq(\tilde{S}_n^\rho/\gamma_n), \tag{6.33}$$

and for sufficiently large n Lemma 6.1a implies

$$Eq(\tilde{S}_n^\rho/\gamma_n) \le 24t_0 + 12\rho, \tag{6.34}$$

since for all $\gamma > 0$

$$\begin{aligned}
|P(q(\tilde{S}_n) > \lambda\gamma(n)) - P(q(\tilde{S}_n^\rho) > \lambda\gamma(n))| &\le 2P(\max_{1\le j\le n} q(X_j) > \rho\gamma(n)) \\
&\le 2nP(q(X) > \rho\gamma(n)) \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

where $\rho > \limsup_n q(X_n)/\gamma_n$ and $\gamma_n \nearrow$.

Combining (6.33), (6.34), we get (2.67)(vi) since

$$P(q({}^{\rho}S_n/\gamma_n) \geq \lambda) \leq P(q(S_n^{\rho}/\gamma_n) \geq \lambda) + nP(q(X) > \rho\gamma(n))$$
$$\leq \frac{1}{\lambda} E(q(S_n^{\rho}/\gamma_n)) + nP(q(x) > \rho\gamma(n)).$$

To prove (vii) we first assume that X is symmetric and then (b) implies $\{S_n/\gamma_n\}$ is q-conditionally compact with probability one. That is, (b) and Fubini's theorem implies there exists $(c_n) \subseteq B$ such that

$$\left\{\frac{S_n - c_n}{\gamma_n}\right\}$$

is q-conditionally compact with probability one. Hence $\{(-S_n + c_n)/\gamma_n\}$ is q-conditionally compact and hence by symmetry so is $\{(S_n + c_n)/\gamma_n\}$. Adding we have $\{2c_n/\gamma_n\}$ q-conditonally compact with probability one, and thus $\{Sn/\gamma n\}$ is q-conditionally compact with probability one.

Further, since B is separable in the q-topology, Lemma 1 of Kuelbs (1981) implies that there is a q-compact convex symmetric set $D \subseteq B$ such that

$$P(C_q(\{S_n/\gamma_n\}) \subseteq D) = 1$$

and for all $\delta > 0$

$$P\left(\left\{\frac{S_n}{\gamma_n}\right\} \in D^{\delta} \quad \text{eventually}\right) = 1.$$

Here, of course, D^{δ} is as defined in (2.67–x). Therefore,

$$P\left(\frac{S_n}{\gamma_n} \in D^{\delta}\right) \geq P\left(\frac{S_m}{\gamma_m} \in D^{\delta} \quad \text{for all} \quad m \geq n\right) \rightarrow_{n\to\infty} 1.$$

Now D q-compact implies that for all $\eta > 0$ there exists $f_1, \cdots, f_N$ q-continuous linear functionals such that

$$q(x) \leq \sup_{1\leq j\leq N} |f_j(x)| + 4\eta$$

for all $x \in D^{\eta}$. Hence

$$P\left(q\left(\frac{S_n}{\gamma_n}\right) > \varepsilon\right) \leq P\left(\frac{S_n}{\gamma_n} \notin D^{\varepsilon/8}\right) + P\left(q\left(\frac{S_n}{\gamma_n}\right) > \varepsilon, \frac{S_n}{\gamma_n} \in D^{\varepsilon/8}\right)$$
$$\leq o(1) + P\left(\sup_{j\leq N}\left|f_j\left(\frac{S_n}{\gamma_n}\right)\right| > \varepsilon/2\right)$$
$$\leq o(1) + \textstyle\sum_{j=1}^{N} P\left(\left|f_j\left(\frac{S_n}{\gamma_n}\right)\right| > \varepsilon/2\right)$$
$$= o(1)$$

by Kesten (1972), Lemma 4, page 728.

We now consider the general case. By the symmetric case

$$q\left(\frac{S_n - S_n'}{\gamma_n}\right) \rightarrow_{\text{prob.}} 0.$$

Therefore, for all $\rho > \limsup_n q(X_n)/\gamma_n$ and $\varepsilon > 0$ we have

$$P(q(\textstyle\sum_{j=1}^{n} [X_j I(q(X_j) \leq \rho\gamma_n) - X_j' I(q(X_j') \leq \rho\gamma_n)]) > \varepsilon\gamma_n)$$
(6.35)
$$\leq P\left(q\left(\frac{S_n - S_n'}{\gamma_n}\right) > \varepsilon\right) + 2nP(\|X\| > \rho\gamma(n)) \to 0 \quad \text{as} \quad n \to \infty.$$

Further, by Lemma 6.1–a we have

$$\text{(6.36)} \quad \limsup_n E\left(q^2\left(\frac{1}{\gamma_n}\sum_{j=1}^n [X_j I(q(X_j) \le \rho\gamma_n) - X_j' I(q(X_j') \le \rho\gamma_n)]\right)\right) \le 18(2\rho)^2.$$

Combining (6.35), (6.36) and standard arguments we thus have

$$\text{(6.37)} \quad \lim_n E\left(\frac{q(\sum_{j=1}^n [X_j I(q(X_j) \le \rho\gamma_n) - X_j' I(q(X_j') \le \rho\gamma_n)])}{\gamma_n}\right) = 0.$$

Using Fubini's Theorem and Jensen's inequality (6.37) implies that

$$\text{(6.38)} \quad \lim_n E\left(q\left(\sum_{j=1}^n \frac{[X_j I(q(X_j) \le \rho\gamma_n) - EXI(q(X) \le \rho\gamma_n)]}{\gamma_n}\right)\right) = 0.$$

Chebyshev's inequality, (6.38), and

$$nP(q(X_j) > \rho\gamma_n) \to_{n\to\infty} 0$$

now give (2.67–vii).

To prove (2.67–viii) recall that (iii) implies there exists $\{c_n\} \subseteq B$ such that

$$\text{(6.39)} \quad \limsup_n q\frac{(S_n - c_n)}{\gamma_n} = M \quad \text{w.p.l.}$$

Hence, if $\lambda > M$, for n sufficiently large we can find a sample point $\omega = \omega_n$ such that

$$q\left(\frac{S_n(\omega_n) - d_n}{\gamma_n}\right) \le A_\delta \quad \text{and} \quad q\left(\frac{S_n(\omega_n) - c_n}{\gamma_n}\right) < \lambda.$$

Hence for n sufficiently large we have

$$q\left(\frac{c_n - d_n}{\gamma_n}\right) \le q\left(\frac{S_n(\omega_n) - c_n}{\gamma_n}\right) + q\left(\frac{S_n(\omega_n) - d_n}{\gamma_n}\right) \le \lambda + A_\delta.$$

Thus

$$\limsup_n \frac{q(S_n - d_n)}{\gamma_n} \le \limsup_n \frac{q(S_n - c_n)}{\gamma_n} + \lambda + A_\sigma \le 2\lambda + a_\delta.$$

Since $\lambda > M$ was arbitrary we now have (viii).

To prove (ix), for each $\eta > 0$ we define the semi-norm $h(x) = h_\eta(x) = \inf\{t > 0: x/t \in D^\eta\}$. Then h is equivalent to q. By (iv) there exists $\{c_n\} \subseteq$ B such that for all $\eta > 0$

$$\limsup_n h\left(\frac{S_n - c_n}{\gamma_n}\right) \le 1 \quad \text{w.p.l.}$$

Hence by (viii) for all $\eta > 0$

$$\limsup_n h\left(\frac{S_n - d_n}{\gamma_n}\right) \le 2 + A_\delta \quad \text{w.p.l.}$$

Thus, for any $\lambda > 2 + A_\delta$

$$P\left(\frac{S_n - d_n}{\gamma_n} \in \lambda D^\eta \quad \text{eventually}\right) = 1.$$

Hence

$$\left\{\frac{S_n - d_n}{\gamma_n}\right\}$$

is q-relatively compact with cluster set contained in $(2 + A_\delta)D$.

To prove (x) note that if $\alpha > M$ then (a) implies

$$\lim_n P\left(q\left(\frac{S_n - S_n'}{\gamma_n}\right) > \alpha\right) = 0.$$

Hence (vi) implies that for all n sufficiently large and $\rho > \limsup_n q(X_n)/\gamma_n$ that

$$P(q(S_n - nE(XI(q(X) \le \rho\gamma_n))) \ge \lambda) \le \frac{24\alpha + 12\rho}{\lambda} + nP(q(X) > \rho\gamma_n). \tag{6.40}$$

Now fix $\lambda > 24\alpha + 12\rho$. Since $\lim_n nP(q(X) > \rho\gamma_n) = 0$ the right hand side of (6.40) is now less than $1 - \delta$ for some $\delta > 0$ provided n is sufficiently large. Applying (viii) we thus have

$$\limsup_n q\left(\frac{S_n - nE(XI(q(X) \le \rho\gamma_n))}{\gamma_n}\right) \le 2M + \lambda \quad \text{w.p.l}$$

where $\lambda > 24\alpha + 12\rho$. Since $\alpha > M$ was arbitrary we thus obtain

$$\limsup_n q\left(\frac{S_n - nE(Xq(X) \le \rho\gamma_n)}{\gamma_n}\right) \le 26M + 12\rho.$$

The proof of (xi) follows from (vii) and (ix). Hence Lemma 2 is proved.

7. Some examples and final remarks. In Goodman, Kuelbs, and Zinn (1981) necessary and sufficient conditions for the BLIL and CLIL (with classical normalizations) were found for Hilbert space valued random variables. For example, if X takes values in a separable Hilbert space H then $X \ne 0$,

$$E(f(X)) = 0, \quad E(f^2(X)) < \infty \quad (f \in H^*) \tag{7.1}$$

and

$$E(\|X\|^2/L_2\|X\|) < \infty \tag{7.2}$$

are necessary and sufficient for

$$0 < \limsup_n \frac{\|S_n\|}{\sqrt{2nL_2n}} < \infty \quad \text{w.p.1.} \tag{7.3}$$

It is easy to construct random variables X satisfying (7.1) and (7.2) such that $E\|X\|^2 = \infty$, so we assume X has these properties as well as being symmetric. Then (7.3) holds, and $X \notin \mathrm{DPA}(Z)$ for any Gaussian law on H. To see this last claim, note that if $X \in \mathrm{DPA}(Z)$ then X symmetric implies there are normalizing constants $\{\alpha_n\}$ and a subsequence $\{n'\}$ such that

$$\mathscr{L}(S_{n'}/\alpha_{n'}) \to \mathscr{L}(Z).$$

However, then $\mathscr{L}(f(S_{n'}/\alpha_{n'})) \to \mathscr{L}(f(Z))$ for all $f \in H^*$, and since Z is non-degenerate with $Ef^2(X) < \infty$ for all $f \in H^*$ we must have $\alpha_n' \approx \sqrt{n'}$. In fact,

$$E\left(f^2\left(\frac{S_{n'}}{\alpha_n}\right)\right) = \frac{n'E(f^2(X))}{\alpha_{n'}^2} \to Ef^2(Z)$$

for all $f \in H^*$. Hence $n'/\alpha_{n'}^2 \to c > 0$ (since Z is nondegenerate), and hence for n' sufficiently large

$$\infty > E(\|Z\|^2) = \textstyle\sum_{i=1}^{\infty} E(e_i^2(Z)) \ge \frac{c}{2} \sum_{j=1}^{\infty} E(e_i^2(X)) = \infty.$$

Thus we have a contradiction, so $X \notin \mathrm{DPA}(Z)$ for any Gaussian random variable Z with values in H.

On the other hand, since X satisfies (7.1) and (7.2) we have (7.3), so X has LIL behavior.

Applying the remark following Theorem 2 we thus have $\{d_k\} \nearrow$ and $\{n_k\} \nearrow \infty$ such that

(7.4) $$\left\{\frac{S_{n_k}}{d_k}\right\}$$

is stochastically bounded in H,

(7.5) $$E(\|\sum_{j=1}^{n_k} X_j I(\|X_j\| \le d_k)\|^2)/d_k^2 \to_{k\to\infty} 1.$$

Further, (7.5), $E\|X\|^2 = \infty$, and $Ef^2(X) < \infty$ for all $f \in H^*$ implies that

(7.6) $$\mathscr{L}\left(f\left(\frac{S_{n_k}}{d_k}\right)\right) \to \mathscr{L}(\delta_0) \quad (f \in H^*).$$

That is, (7.5) and $E\|X\|^2 = \infty$ implies

$$\frac{n_k}{d_k^2} \to_{k\to\infty} 0,$$

and hence the classical central limit theorem on the line implies (7.6). In fact, (7.5) implies the constants $\{d_k\}$ are such that $d_k \sim \sqrt{n_k}$ iff $E\|X\|^2 < \infty$.

The concept of a generalized domain of attraction has recently been considered in a number of articles (see, for example, Hahn and Klass, 1980) and is as follows. We say a B-valued random variable is in the generalized domain of attraction of a random variable Z if there are shifts $\{\delta_n\}$ and linear transformations $\{T_n\}$ such that

$$\mathscr{L}(T_n(S_n - \delta_n)) \to \mathscr{L}(Z).$$

Using the idea of a generalized domain of attraction and the method of proof in Theorem 1, it is possible to prove an analogous result for generalized domains of attraction. That is, it is easy to see that one could prove the following result.

THEOREM. *Let X be symmetric and in the generalized domain of attraction of the mean zero Gaussian random variable Z. Let K denote the unit ball of $H_{\mathscr{L}(Z)}$. Then, there exists a subsequence of integers $\{n_k\}$ and linear operators $\{\tilde{T}_k\}$ such that if*

$$T_n = \frac{\tilde{T}_k}{\sqrt{2Lk}} \quad n \in (n_{k-1}, n_k],$$

then

$$P(\{T_n(S_n)\} \twoheadrightarrow K) = 1.$$

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