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ON THE MAXIMUM OF A RANDOM WALK WITH SMALL NEGATIVE DRIFT

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Let $X_1, X_2, \cdots$ be i.i.d. mean zero random variables. Let $S_n = X_1 + \cdots + X_n$ and $M_\varepsilon = \sup_{n\geq 1}(S_n - n\varepsilon)^+$ for $\varepsilon > 0$. Suppose $\sigma^2 = E(X_1)^2$ is positive and finite. Then $EM_\varepsilon < \infty$ and $2\varepsilon\sigma^{-2}EM_\varepsilon$ converges to 1 as $\varepsilon \searrow 0^+$. In this paper we obtain an approximation of the discrepancy $1 - 2\varepsilon\sigma^{-2}EM_\varepsilon$ as $\varepsilon \searrow 0^+$. To do so we derive a first order approximation of $P(M_\varepsilon < y)$ which is uniform in y as $\varepsilon \searrow 0^+$ and asymptotically exact for y on $[y_\varepsilon, \infty)$ provided $y_\varepsilon \to \infty$. Approximation of $P(M_\varepsilon < y)$ necessitates a digression into renewal theory. We derive an approximation of the expected time $E\tau_y$ required by a sum $T_n = Y_1 + \cdots + Y_n$ of i.i.d. non-negative random variables to reach or exceed y. The bounds obtained are of particular interest when $EX = \infty$ and are best possible in a rather strong sense.

0. Introduction. Throughout this paper let $X, X_1, X_2, \cdots$ be i.i.d. mean zero random variables and also let $S_n = X_1 + \cdots + X_n$, $X_\varepsilon = X - \varepsilon$, and $M_\varepsilon = \sup_{n\geq 1}(S_n - n\varepsilon)^+$. It is well known (cf. Kiefer-Wolfowitz, 1956) that the condition $E(X^+)^2 < \infty$ (henceforth always assumed to hold) is necessary and sufficient for EM_ε to be finite. Kingman (1962) has shown that when $0 < \sigma^2 \equiv EX^2 < \infty$,

$$EM_\varepsilon = (2\varepsilon)^{-1}\{EX_\varepsilon^2 - E((X_\varepsilon^- - M_\varepsilon)^+)^2\}. \tag{0.1}$$

Since $M_\varepsilon \to_{\text{a.s.}} \infty$ as $\varepsilon \searrow 0^+$ and $EX_\varepsilon^2 = \sigma^2 + \varepsilon^2$ it is obvious that whenever $0 < \sigma^2 < \infty$, we obtain the asymptotic formula

$$2\varepsilon\sigma^{-2}EM_\varepsilon \to 1 \quad \text{as} \quad \varepsilon \searrow 0^+. \tag{0.2}$$

In this paper we derive a better asymptotic approximation of EM_ε as ε tends to zero. To do so, we find an asymptotically accurate expression for the error term $E((X_\varepsilon^- - M_\varepsilon)^+)^2$, together with simple approximations thereto. This necessitates approximation of $P(M_\varepsilon < y)$ for all $y > 0$. Previous attention seems to have focused on the tail probability $P(M_\varepsilon > y)$ rather than $P(M_\varepsilon < y)$, with good approximations presented only for $y \geq \delta/\varepsilon$ and X-distributions having a moment generating function in some neighborhood of the origin (see Cramer (1954), Borovkov (1962), Feller (1966, page 393), von Bahr (1974), Siegmund (1975a,b; 1978), Woodroofe (1978)).

As an associated matter, but one of independent interest, we introduce an elementary but apparently new result in renewal theory. It gives a best-possible approximation of the expected time taken by a sum of non-negative (possibly infinite mean) i.i.d. random variables to cross a horizontal boundary.

These results may have some application to insurance risk theory, storage theory, scheduling, and queueing theory (cf. Feller, 1966, page 180). For illustration, consider the problem of an insurance company contemplating issuance of a new policy. Let C denote

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the magnitude of a random claim and suppose the distribution of claims is (would that it were only possible) known. Clearly, the insurance company must charge more than EC dollars per policy or face certain bankruptcy. Suppose for simplicity that each claim is settled before the next policy is sold. Letting $X_\varepsilon = C - EC - \varepsilon$, M_ε represents the amount of capital the company would require for its policy to become profitable. Hence the interest in EM_ε for various $\varepsilon > 0$.

The paper is organized as follows: Section 1 contains a simple derivation of two results, (0.1), and the relationship between EM_ε and the first ladder height. Though these results are perhaps well known, they serve to make the paper almost self-contained. Section 2 develops a uniformly accurate approximation of $P(M_\varepsilon < y)$, valid for y on any collection of intervals $[y_\varepsilon, \infty)$ provided $y_\varepsilon \to \infty$ as $\varepsilon \searrow 0^+$. The derivation of this result depends upon proof that (at least if $EX^2 < \infty$) the first ladder height of $S_n - n\varepsilon$ converges in L^1 to the first ladder height of S_n. Section 3 presents our digression into renewal theory. Section 4 introduces approximations of $P(M_\varepsilon < y)$, valid for low-order y. These approximations are partially dependent on Section 3. Section 5 applies the results of Sections 2–4 to both asymptotic evaluation and asymptotic approximation of the error term $E((X_\varepsilon^- - M_\varepsilon)^+)^2$ and, more generally, $E((X_\varepsilon^- - M_\varepsilon)^+)^\alpha$ for $\alpha > 1$. In particular, whenever $EX^2 < \infty$, it is shown that $L \equiv \lim_{\varepsilon \to 0^+} \varepsilon^{-1} E((X_\varepsilon^- - M_\varepsilon)^+)^2 < \infty$ iff $E(X^-)^3 < \infty$. Moreover, a relatively simple asymptotic expression is found for $E((X_\varepsilon^- - M_\varepsilon)^+)^2$ when $L = \infty$, i.e. when $E(X^-)^3 = \infty$. Note that $x^+ \equiv \max\{x, 0\}$ and $x^- \equiv \max\{-x, 0\}$.

1. Classical results.

We present a slight extension of Kingman's result (1962).

THEOREM 1.1. *Given the basic hypotheses of this paper,*

$$EM_\varepsilon = (2\varepsilon)^{-1}\{E(X_\varepsilon^+)^2 + E\{(X_\varepsilon^-)^2 - ((X_\varepsilon^- - M_\varepsilon)^+)^2\}\}. \tag{1.1}$$

If in fact $EX^2 \equiv \sigma^2 < \infty$ *then*

$$EM_\varepsilon = (2\varepsilon)^{-1}\{\sigma^2 + \varepsilon^2 - E((X_\varepsilon^- - M_\varepsilon)^+)^2\}. \tag{1.2}$$

PROOF. Write $M_\varepsilon^2 - ((X_\varepsilon + M_\varepsilon)^+)^2$ as

$$M_\varepsilon^2 - (X_\varepsilon + M_\varepsilon)^2 + ((X_\varepsilon + M_\varepsilon)^-)^2$$
$$= -(X_\varepsilon^+)^2 - 2X_\varepsilon M_\varepsilon - (X_\varepsilon^-)^2(I(X_\varepsilon^- \le M_\varepsilon) + I(X_\varepsilon^- > M_\varepsilon)) + ((X_\varepsilon^- - M_\varepsilon)^+)^2.$$

Observe that $E(X_\varepsilon^+)^2 < \infty$, and so by Kiefer and Wolfowitz (1956) $E|X_\varepsilon M_\varepsilon| = E|X_\varepsilon| EM_\varepsilon < \infty$, $E(X_\varepsilon^-)^2 I(X_\varepsilon^- \le M_\varepsilon) \le EX_\varepsilon^- M_\varepsilon = EX_\varepsilon^- EM_\varepsilon < \infty$, and

$$0 \le E\{(X_\varepsilon^-)^2 - (X_\varepsilon^- - M_\varepsilon)^2\} I(X_\varepsilon^- > M_\varepsilon) \le E2X_\varepsilon^- M_\varepsilon I(X_\varepsilon^- > M_\varepsilon)$$
$$\le 2EX_\varepsilon^- M_\varepsilon = 2EX_\varepsilon^- EM_\varepsilon < \infty.$$

Hence $M_\varepsilon^2 - ((X_\varepsilon + M_\varepsilon)^+)^2$ is Lebesgue integrable. But since M_ε and $(X_\varepsilon + M_\varepsilon)^+$ have the same distribution, the integral must be zero. Hence

$$-2EM_\varepsilon EX_\varepsilon = E(X_\varepsilon^+)^2 + E(X_\varepsilon^-)^2 I(X_\varepsilon^- \le M_\varepsilon) + E\{(X_\varepsilon^-)^2 - (X_\varepsilon^- - M_\varepsilon)^2\} I(M_\varepsilon < X_\varepsilon^-).$$

Since $EX_\varepsilon = E(X - \varepsilon) = -\varepsilon$, (1.1) follows and (1.2) is a trivial further consequence. □

REMARK 1.2. Y.S. Chow and T.L. Lai (1978) have pointed out that by considering $M_\varepsilon^{k+1} - ((X_\varepsilon + M_\varepsilon)^+)^{k+1} = M_\varepsilon^{k+1} - (X_\varepsilon + M_\varepsilon)^{k+1} + (-(X_\varepsilon + M_\varepsilon)^-)^{k+1}$, Kingman's techniques can be used to generate an expression for EM_ε^k in terms of $EM_\varepsilon, \cdots, EM_\varepsilon^{k-1}, EX^2, EX^3, \cdots, EX^{k+1}$, and $E((X_\varepsilon^- - M_\varepsilon)^+)^{k+1}$. Such an expression makes EM_ε^k amenable to highly accurate approximation as $\varepsilon \searrow 0^+$. The Chow-Lai idea can be sharpened using our method of proving (1.1). This is useful if $E(X^-)^j = \infty$ for some $2 < j \le k+1$. Thereby, we obtain,

$$EM_\varepsilon^k = \{(k+1)\varepsilon\}^{-1}\{\sum_{j=2}^{i}\binom{k+1}{j}EM_\varepsilon^{k+1-j}EX_\varepsilon^j$$

$$+\sum_{j=i+1}^{k+1}\binom{k+1}{j}EM_\varepsilon^{k+1-j}E((X_\varepsilon)^+)^j \tag{1.3}$$

$$+\sum_{j=i+1}^{k+1}\binom{k+1}{j}EM_\varepsilon^{k+1-j}(-X_\varepsilon^-)^j I(X_\varepsilon^- \le M_\varepsilon)$$

$$-\sum_{j=0}^{i}\binom{k+1}{j}EM_\varepsilon^{k+1-j}(-X_\varepsilon^-)^j I(M_\varepsilon < X_\varepsilon^-)\}$$

provided $E(X^+)^{k+1} < \infty$ and $E(X^-)^i < \infty$ for some $1 \le i \le k+1$.

When $i = k+1$ we obtain the Chow-Lai result

$$EM_\varepsilon^k = \{(k+1)\varepsilon\}^{-1}\{\sum_{j=2}^{k+1}\binom{k+1}{j}EM_\varepsilon^{k+1-j}EX_\varepsilon^j + (-1)^k E((X_\varepsilon^- - M_\varepsilon)^+)^{k+1}\}. \tag{1.4}$$

We introduce some additional notation which will be used throughout the sequel. For $\varepsilon \ge 0$ let

$$\tau_+(\varepsilon) = \begin{cases} 1^{\text{st}}\, n \ge 1\colon\ S_n - n\varepsilon > 0 & \text{if such } n \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

and

$$\tau_-(\varepsilon) = 1^{\text{st}}\, n \ge 1\colon\ S_n - n\varepsilon \le 0.$$

Let $\tau_+ = \tau_+(0)$, $\tau_- = \tau_-(0)$.

THEOREM 1.3. *For* $\varepsilon > 0$,

$$E(S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon))I(\tau_+(\varepsilon) < \infty) = P(M_\varepsilon = 0)EM_\varepsilon. \tag{1.5}$$

PROOF. Let $Y_{1\varepsilon}, Y_{2\varepsilon}, \cdots$ be i.i.d. random variables, where $Y_{1\varepsilon} = S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon)$, conditional on the event $\tau_+(\varepsilon) < \infty$. Let L_ε be a geometric random variable independent of $\{Y_{n\varepsilon}\}$ such that for integers $n \ge 0$, $P(L_\varepsilon = n) = p_\varepsilon(1-p_\varepsilon)^n$, where $p_\varepsilon = P(M_\varepsilon = 0)$. Then M_ε has the same distribution as $\sum_{j=1}^{L_\varepsilon} Y_{j\varepsilon}$, a fact also noted and utilized by Siegmund (1978, Theorem 1). Hence

$$EM_\varepsilon = EL_\varepsilon EY_{1\varepsilon} = (1-p_\varepsilon)EY_{1\varepsilon}/p_\varepsilon = E(S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon))I(\tau_+(\varepsilon) < \infty)/P(M_\varepsilon = 0). \qquad \square$$

2. Uniform asymptotic approximation of $P(M_\varepsilon < y)$ for large y as $\varepsilon \searrow 0^+$.

LEMMA 2.1. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *mean zero random variables with variance* $0 < \sigma^2 < \infty$. *Let* $S_n = X_1 + \cdots + X_n$. *Let*

$$\tau_+ = 1^{\text{st}}\, n\colon S_n > 0, \quad \tau_- = 1^{\text{st}}\, n\colon S_n \le 0.$$

Then

$$ES_{\tau_+}ES_{\tau_-}^- = \sigma^2/2. \tag{2.1}$$

PROOF. From Chung (1968, page 262, Theorem 8.4.6) we obtain

$$ES_{\tau_+} = (\sigma/\sqrt{2})\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}(.5 - P(S_n > 0))\right\}$$

and

$$ES_{\tau_-}^- = (\sigma/\sqrt{2})\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}(.5 - P(S_n \le 0))\right\}.$$

Multiplying these two expressions yields (2.1).

LEMMA 2.2. *Under the conditions of Lemma* 2.1, *let* $M_\varepsilon = \sup_{n\ge 1}(S_n - n\varepsilon)^+$,

$$\tau_+(\varepsilon) = \begin{cases} 1^{\text{st}}\ n:\ S_n - n\varepsilon > 0 \text{ if such } n < \infty \text{ exists} \\ \infty \quad \text{otherwise} \end{cases}$$

$$\tau_-(\varepsilon) = 1^{\text{st}}\ n:\ S_n - n\varepsilon \le 0.$$

Then

$$\lim_{\varepsilon\searrow 0^+} E(S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon))I(\tau_+(\varepsilon) < \infty) = ES_{\tau_+}, \tag{2.2}$$

$$\lim_{\varepsilon\searrow 0^+} E(S_{\tau_-(\varepsilon)} - \varepsilon\tau_-(\varepsilon))^- = ES_{\tau_-}^-, \tag{2.3}$$

and

$$\lim_{\varepsilon\searrow 0^+} \varepsilon^{-1}P(M_\varepsilon = 0) = 2\sigma^{-2}ES_{\tau_+}. \tag{2.4}$$

PROOF. We recall the following facts.
(i) $EM_\varepsilon \sim \sigma^2/2\varepsilon$ (see (0.2))
(ii) $s_+(\varepsilon) \equiv E(S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon))I(\tau_+(\varepsilon) < \infty) = P(M_\varepsilon = 0)EM_\varepsilon$ (see (1.5))
(iii) $P(M_\varepsilon = 0) = 1/E\tau_-(\varepsilon)$ (see Feller, 1966)
(iv) $\varepsilon E\tau_-(\varepsilon) = E(S_{\tau_-(\varepsilon)} - \varepsilon\tau_-(\varepsilon))^- \equiv s_-(\varepsilon)$ (Wald's equation)
Using (i) and (ii) it is obvious that (2.4) follows from (2.2).
Hence we focus on (2.2) and (2.3). Algebraic manipulation of (i) – (iv) shows

$$s_+(\varepsilon)s_-(\varepsilon) \sim \sigma^2/2. \tag{2.5}$$

Clearly,

$$(S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon))I(\tau_+(\varepsilon) < \infty) \to_{\text{a.s.}} S_{\tau_+}$$

and

$$(S_{\tau_-(\varepsilon)} - \varepsilon\tau_-(\varepsilon))^- \to_{\text{a.s.}} S_{\tau_-}^-.$$

By Fatou's lemma,

$$ES_{\tau_+} \le \liminf_{\varepsilon\searrow 0^+} s_+(\varepsilon)$$

and

$$ES_{\tau_-}^- \le \liminf_{\varepsilon\searrow 0^+} s_-(\varepsilon).$$

In view of (2.1) and (2.5) we must have

$$\limsup_{\varepsilon\searrow 0^+} s_+(\varepsilon) \le ES_{\tau_+}$$

and

$$\limsup_{\varepsilon\searrow 0^+} s_-(\varepsilon) \le ES_{\tau_-}^-,$$

which proves (2.2) and (2.3).□

COROLLARY 2.3. *For any sequence* $\{y_\varepsilon\}$ *such that* $y_\varepsilon \to \infty$ *as* $\varepsilon \searrow 0^+$,

$$\lim_{\varepsilon\searrow 0^+} E(S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon))I(S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon) > y_\varepsilon,\ \tau_+(\varepsilon) < \infty) = 0. \tag{2.6}$$

PROOF. According to Theorem 4.5.4 on page 90 of Chung (1968), (2.2) implies that

$\{(S_{\tau_+(\varepsilon_n)} - \varepsilon_n \tau_+(\varepsilon_n))I(\tau_+(\varepsilon_n) < \infty)\}$ is uniformly integrable for every $\{\varepsilon_n\}$ such that $\varepsilon_n \searrow 0$. Now (2.6) is immediate. □

LEMMA 2.4. *Our assumptions and definitions are the same as those of Lemmas* 2.1 *and* 2.2 *Let* $Y_{1\varepsilon}, Y_{2\varepsilon}, \cdots$ *be* i.i.d. *random variables, where* $Y_{1\varepsilon}$ *is distributed as* $S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon)$, *given* $\tau_+(\varepsilon) < \infty$. *Let* $S_{n\varepsilon} = Y_{1\varepsilon} + \cdots + Y_{n\varepsilon}$. *Then for any* $y_\varepsilon \to \infty$ *as* $\varepsilon \to 0^+$ *there exists* $\delta_\varepsilon \to 0^+$ *as* $\varepsilon \to 0^+$ *such that*

$$\lim_{\varepsilon \searrow 0^+} \sup_{\{n,y: n \geq (1+\delta_\varepsilon)y/ES_{\tau_+}, y \geq y_\varepsilon\}} P(S_{n\varepsilon} \leq y) = 0 \tag{2.7}$$

and

$$\lim_{\varepsilon \searrow 0^+} \inf_{\{n,y: 1 \leq n \leq (1-\delta_\varepsilon)y/ES_{\tau_+}, y \geq y_\varepsilon\}} P(S_{n\varepsilon} \leq y) = 1. \tag{2.8}$$

PROOF. Note that it suffices to prove these statements for arbitrary fixed $\delta > 0$. Since $Y_{j\varepsilon}$ is positive, $P(S_{n\varepsilon} \leq y)$ is non-increasing in n. Therefore (2.7) and (2.8) are dependent for their validity on the two values of n closest to $(1 \pm \delta)y/ES_{\tau_+}$. Clearly, then, both (2.7) and (2.8) are consequences of (and in fact equivalent to) the statement that for any $n_\varepsilon \to \infty$ as $\varepsilon \to 0$,

$$\frac{S_{n_\varepsilon,\varepsilon}}{n_\varepsilon} \to ES_{\tau_+} \text{ in probability.} \tag{2.9}$$

To prove (2.9), use the basic technique of the Weak Law of Large Numbers, noting that since $P(\tau_+(\varepsilon) < \infty) \to 1$, (2.2) and (2.6) imply that $EY_{1\varepsilon}I(Y_{1\varepsilon} \leq \sqrt{n_\varepsilon}\,) \to ES_{\tau_+}$ and $EY_{1\varepsilon}I(Y_{1\varepsilon} > \sqrt{n_\varepsilon}) \to 0$ as $\varepsilon \to 0^+$. □

REMARK 2.5 Lemma 2.4 holds whenever
(i) $EX = 0$,
(ii) $E(X^+)^2 < \infty$, and
(iii) $\lim_{\varepsilon \to 0^+} E(S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon))I(\tau_+(\varepsilon) < \infty) = ES_{\tau_+}$.
We conjecture that (iii) is always a consequence of (i) and (ii). □

The lemmas proved heretofore pave the way for our first result on asymptotic approximation.

THEOREM 2.6. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *mean zero random variables with finite, positive variance* σ^2. *Let* $M_\varepsilon = \sup_{n \geq 1}(S_n - n\varepsilon)^+$, *where* $S_n = X_1 + \cdots + X_n$. *Let* $y_\varepsilon \to \infty$ *as* $\varepsilon \searrow 0$. *Then*

$$\lim_{\varepsilon \searrow 0^+} \sup_{y \geq y_\varepsilon} \left| \frac{P(M_\varepsilon < y)}{1 - \exp(-2\varepsilon\sigma^{-2}y)} - 1 \right| = 0. \tag{2.10}$$

REMARK 2.7. It is instructive to compare the result above with Brownian motion. Let $B(t)$ denote standard Brownian motion with mean zero and variance parameter 1. Then for *all* positive y, ε and σ,

$$P(\sup_{t \geq 0}\{\sigma B(t) - \varepsilon t\} < y) = 1 - \exp(-2\varepsilon y\sigma^{-2}). \tag{2.11}$$

PROOF. Let $p_\varepsilon, L_\varepsilon, Y_{1\varepsilon}, Y_{2\varepsilon}, \cdots$ be defined exactly as in the proof of Theorem 1.3. Recall that

$$M_\varepsilon =_{\mathscr{D}} \textstyle\sum_{j=1}^{L_\varepsilon} Y_{j\varepsilon}.$$

Hence

$$P(M_\varepsilon < y) = \textstyle\sum_{n=0}^\infty P(L_\varepsilon = n, \sum_{j=1}^n Y_{j\varepsilon} < y) = \sum_{n=0}^\infty p_\varepsilon(1 - p_\varepsilon)^n P(S_{n\varepsilon} < y). \tag{2.12}$$

We intend to split this sum into three pieces. According to Lemma 2.4, there exists δ_ε

tending to zero as $\varepsilon \to 0$ such that for all $y \ge y_\varepsilon$,

$$|P(S_{n\varepsilon} < y) - 1| \le \delta_\varepsilon \quad \text{for} \quad 1 \le n \le y(1-\delta_\varepsilon)/ES_{\tau+}$$

and

$$P(S_{n\varepsilon} < y) \le \delta_\varepsilon \quad \text{for} \quad n \ge y(1+\delta_\varepsilon)/ES_{\tau+}.$$

Let $g_\varepsilon(y) = y(1-\delta_\varepsilon)/ES_{\tau_+}$ and $h_\varepsilon(y) = y(1+\delta_\varepsilon)/ES_{\tau_+}$. For any $r > 1$, the middle sum is (defined as)

$$\begin{aligned}\sum_{n=[g_\varepsilon(y)+1]}^{rg_\varepsilon(y)} (1-p_\varepsilon)^n P(S_{n\varepsilon} < y) &= \sum_{g_\varepsilon(y)<n\le h_\varepsilon(y)} (\,) + \sum_{h_\varepsilon(y)<n\le rg_\varepsilon(y)} (\,) \\ &\le (h_\varepsilon(y) - g_\varepsilon(y))(1-p_\varepsilon)^{g_\varepsilon(y)} + (r-1)g_\varepsilon(y)(1-p_\varepsilon)^{g_\varepsilon(y)}\delta_\varepsilon \\ &\le 2r\delta_\varepsilon g_\varepsilon(y)(1-p_\varepsilon)^{g_\varepsilon(y)} \quad \text{for} \quad \varepsilon > 0 \text{ sufficiently small} \\ &\le 2r\delta_\varepsilon(1-(1-p_\varepsilon)^{g_\varepsilon(y)})/p_\varepsilon.\end{aligned}$$

The initial sum is (defined as)

$$\sum_{n=0}^{[g_\varepsilon(y)]} (1-p_\varepsilon)^n P(S_{n\varepsilon} < y) \sim \sum_{j=0}^{[g_\varepsilon(y)]} (1-p_\varepsilon)^n \sim p_\varepsilon^{-1}(1-(1-p_\varepsilon)^{g_\varepsilon(y)})$$

as $\varepsilon \searrow 0$ uniformly in $y \ge y_\varepsilon$. δ_ε may be chosen to tend to zero so slowly that $E(Y_{1\varepsilon} \mid Y_{1\varepsilon} < y) \ge (1-\delta_\varepsilon)ES_{\tau_+}$ for $y \ge y_\varepsilon$. Incorporating Lemma 2.9 (to follow) the third sum is (defined as)

$$\begin{aligned}\sum_{n=[rg_\varepsilon(y)+1]}^{\infty} (1-p_\varepsilon)^n P(S_{n\varepsilon} < y) &\le \sum_{n=[rg_\varepsilon(y)+1]}^{\infty} e(1-p_\varepsilon)^n \exp\{-(1-e^{-1})ny^{-1}ES_{\tau+}(1-\delta_\varepsilon)\} \\ &\le e(q_{\varepsilon,y})^{rg_\varepsilon(y)}(1-q_{\varepsilon,y})^{-1}\end{aligned}$$

where

$$q_{\varepsilon,y} = (1-p_\varepsilon)\exp\{-(1-e^{-1})y^{-1}ES_{\tau+}(1-\delta_\varepsilon)\} = (1-p_\varepsilon)\exp\{-(1-e^{-1})/g_\varepsilon(y)\}.$$

Fix $0 < \delta << 1$. We can choose $r > 1$ sufficiently large, depending only on δ, so that for all sufficiently small $\varepsilon > 0$ and all $y \ge y_\varepsilon$,

$$e(q_{\varepsilon,y})^{rg_\varepsilon(y)}(1-q_{\varepsilon,y})^{-1} \le \delta(1-(1-p_\varepsilon)^{g_\varepsilon(y)})/p_\varepsilon. \tag{2.13}$$

Letting

$$R(\varepsilon, y) \equiv \sum_{n=0}^{\infty} (1-p_\varepsilon)^n P(S_{n\varepsilon} < y) p_\varepsilon (1-(1-p_\varepsilon)^{g_\varepsilon(y)})^{-1},$$

the preceding arguments show that

$$1 \le \liminf_{\varepsilon\searrow 0}(\inf_{\{y\ge y_\varepsilon\}} R(\varepsilon, y)) \le \limsup_{\varepsilon\searrow 0}(\sup_{\{y\ge y_\varepsilon\}} R(\varepsilon, y)) \le 1 + \delta.$$

Since $\delta > 0$ is arbitrary, the limit exists and equals 1. We now examine $1-(1-p_\varepsilon)^{g_\varepsilon(y)}$. Due to (2.4), $p_\varepsilon g_\varepsilon(y) \sim 2\varepsilon\sigma^{-2}y$. Hence $1-(1-p_\varepsilon)^{g_\varepsilon(y)} \sim 1-\exp(-2\varepsilon\sigma^{-2}y)$ as $\varepsilon \to 0$ uniformly in $y \ge y_\varepsilon$. This concludes the proof modulo Lemma 2.9. □

REMARK 2.8. By similar reasoning one may also show that as $\varepsilon \searrow 0^+$

$$p_\varepsilon \sum_{n=0}^{\infty} (1-p_\varepsilon)^n P(S_{T_n} < y)(1-\exp(-2\varepsilon y\sigma^{-2}))^{-1} \to 1 \text{ uniformly in } y \ge y_\varepsilon, \tag{2.14}$$

where $S_0 \equiv 0 \equiv T_0$ and for $n \ge 1$, $T_n = 1^{\text{st}}\ k$: $S_k > S_{T_{n-1}}$. (Note: $S_k \equiv X_1 + \cdots + X_k$.) Introduce a geometric random variable $\bar{L}_\varepsilon$, independent of $X_1, X_2, \cdots$ such that $P(\bar{L}_\varepsilon = 0) = 2\varepsilon\sigma^{-2}ES_{\tau_+} \equiv \bar{p}_\varepsilon \sim p_\varepsilon \equiv P(M_\varepsilon = 0)$. It follows that as $\varepsilon \searrow 0^+$,

$$P(M_\varepsilon < y)/P(S_{T_{\bar{L}_\varepsilon}} < y) \to 1 \quad \text{uniformly in} \quad y \ge y_\varepsilon. \tag{2.15}$$

Being in fact a consequence of Lemma 2.4, (2.15) holds whenever $s_+(\varepsilon) \equiv E(S_{\tau_+(\varepsilon)} - \varepsilon\tau_+(\varepsilon))I(\tau_+(\varepsilon) < \infty) \to ES_{\tau_+} < \infty$, provided we take $P(\bar{L}_\varepsilon = 0) \sim P(M_\varepsilon = 0)$ as $\varepsilon \to 0^+$.

Moreover, Theorem 2.6 then becomes

(2.16) $$\lim_{\varepsilon\searrow 0^+}\sup_{y\ge y_\varepsilon}|\,1 - P(M_\varepsilon < y)/(1 - \exp\{-yP(M_\varepsilon = 0)/ES_{\tau+}\})\,| = 0.$$

LEMMA 2.9. *Let $Y_1, Y_2, \cdots$ be* i.i.d. *non-negative random variables. Then*

(2.17) $$P(Y_1 + \cdots + Y_n < y) < \exp(1 - nP(Y_1 \ge y) - (1 - e^{-1})ny^{-1}E(Y_1 \mid Y_1 < y)).$$

PROOF. Suppose first that $P(Y_i < y) = 1$. Then for any $t > 0$,

$$\begin{aligned} P(Y_1 + \cdots + Y_n < y) &= P(\exp(ty - t\textstyle\sum_{j=1}^n Y_j) > 1) < E\exp(ty - t\sum_{j=1}^n Y_j) \\ &= (\exp ty)(E\exp(-tY_1))^n \le \exp\{ty + n\,E(\exp(-tY_1) - 1)\} \\ &\le \exp\left\{ty + n\left(\frac{e^{-ty}-1}{ty}\right)EtY_1\right\}\left(\text{since}\,\frac{e^{-tY_1}-1}{tY_1} \le \frac{e^{-ty}-1}{ty}\right). \end{aligned}$$

For simplicity we put $t = 1/y$. This gives (2.17). In the general case we have

$$\begin{aligned} P(Y_1 + \cdots + Y_n < y) &= P(\cap_{j=1}^n\{Y_j < y\})P(\textstyle\sum_{j=1}^n (Y_j \mid Y_j < y) < y) \\ &\le \exp(-nP(Y \ge y) + 1 + n(e^{-1} - 1)E\{(Y_1/y) \mid Y_1 < y\}). \quad \square \end{aligned}$$

Our next task is to approximate the distribution function of M_ε over bounded or slowly growing intervals. This necessitates a temporary digression into renewal theory.

3. A little renewal theory.

THEOREM 3.1. *Let $U, U_1, U_2, \cdots$ be* i.i.d. *non-negative random variables with $0 < EU \le \infty$. Let $t_y = 1^{\text{st}}\ n\colon \sum_{j=1}^n U_j \ge y$. Then*

(3.1) $$(1 + P(U < y)) \vee (y/E(U \wedge y)) \le Et_y \le \frac{y(1 + P(U < y))}{E(U \wedge y)}.$$

Moreover, if $\lim_{y\to\infty}\dfrac{yP(U \ge y)}{E(U \wedge y)} = 0$ *or* 1, *then*

(3.2) $$\lim_{y\to\infty} Et_yE(U \wedge y)/y = 1.$$

NOTE. From (3.1) we have $1 \le Et_yE(U \wedge y)/y \le 2$. Observe that (3.2) generalizes the standard result $Et_y/y \to 1/EU$.

PROOF. From our vantage point, the key idea is the observation that

(3.3) $$t_y = 1^{\text{st}}\ n\colon \textstyle\sum_{j=1}^n (U_j \wedge y) \ge y.$$

The main feature of this reformulation is that to treat possibly infinite-mean variables we create suitably bounded ones. In so doing we obtain a bound on the overshoot $(E\sum_{j=1}^{t_y}(U_j \wedge y)) - y$. By Wald's equation,

(3.4) $$E\textstyle\sum_{j=1}^{t_y}(U_j \wedge y) = Et_yE(U \wedge y).$$

Approximating $E\sum_{j=1}^{t_y}(U_j \wedge y)$,

$$\begin{aligned} y \le E\textstyle\sum_{j=1}^{t_y}(U_j \wedge y) &\le E(yI(t_y = 1) + 2yI(t_y > 1)) \\ &= yE(1 + I(t_y > 1)) = y(1 + P(U < y)). \end{aligned}$$

Combine these bounds with (3.4) to obtain all of (3.1) except half of the L.H.S. To deduce the remainder, note that

$$Et_y \ge E(I(t_y = 1) + 2I(t_y \ge 2)) = E(1 + I(t_y > 1)) = 1 + P(U < y).$$

Turning to the asymptotic situation, suppose $yP(U \geq y)/E(U \wedge y) \to 1$ as $y \to \infty$. Let $\bar{t}_y = $ 1st $n: U_n \geq y$. Then $t_y \leq \bar{t}_y$ so that, using (3.1),

$$1 \leq Et_yE(U \wedge y)/y \leq E\bar{t}_yE(U \wedge y)/y = E(U \wedge y)/yP(U \geq y) \to 1 \quad \text{as} \quad y \to \infty.$$

This proves (3.2) in one case.

Now assume $\lim_{y\to\infty} yP(U \geq y)/E(U \wedge y) = 0$. Then for every $\varepsilon > 0$,

$$\lim_{y\to\infty}(yP(U \geq \varepsilon y)/EU \wedge y) \leq \varepsilon^{-1}\lim_{y\to\infty}(\varepsilon yP(U \geq \varepsilon y)/EU \wedge \varepsilon y) = 0.$$

Hence there exists $a_y \to \infty$ such that

$$a_y/y \to 0 \text{ and } \lim_{y\to\infty} yP(U \geq a_y)/E(U \wedge y) = 0.$$

Let $\bar{\bar{t}}_y = 1^{\text{st}}\ n: U_n \geq a_y$. Then

$$I(t_y < \bar{\bar{t}}_y) \textstyle\sum_{j=1}^{t_y} (U_j \wedge y)/y \leq 1 + a_y/y$$

and $I(t_y \geq \bar{\bar{t}}_y) \sum_{j=1}^{t_y} (U_j \wedge y)/y \leq 2 \sum_{j=1}^{t_y} I(U_j \geq a_y)$. Employing these inequalities,

$$\begin{aligned}
1 &\leq Et_yE(U \wedge y)/y && \text{(by (3.1))}\\
&= E(\textstyle\sum_{j=1}^{t_y} (U_j \wedge y)/y)(I(t_y \leq \bar{\bar{t}}_y) + I(t_y \geq \bar{\bar{t}}_y)) && \text{(by Wald)}\\
&\leq 1 + a_y/y + 2E\textstyle\sum_{j=1}^{t_y} I(U_j \geq a_y)\\
&= 1 + a_y/y + 2Et_yP(U \geq a_y) && \text{(by Wald)}\\
&\leq 1 + a_y/y + 4(y/E(U \wedge y))P(U \geq a_y) && \text{(by (3.1))}\\
&\to 1 \quad \text{as} \quad y \to \infty, \qquad \text{proving (3.2).}
\end{aligned}$$

REMARK 3.2. The upper-bound in (3.1) can be somewhat improved. Using Lorden's (1970) result on excess over the boundary, $E\sum_{j=1}^{t_y} (U_j \wedge y) \leq y + E(U \wedge y)^2/E(U \wedge y)$. Hence we have the additional bound

$$Et_y \leq (y/E(U \wedge y))(1 + E(U \wedge y)^2/yE(U \wedge y)). \tag{3.5}$$

REMARK 3.3. The bounds in (3.1) are optimal as $y \searrow 0$ whenever $P(U > 0) = 1$. When $y \to \infty$ the bounds are also best possible. Specifically, there exist distributions for which

$$1 = \liminf_{y\to\infty} Et_yE(U \wedge y)/y < \limsup_{y\to\infty} Et_yE(U \wedge y)/y = 2. \tag{3.6}$$

Furthermore, one can show that whenever the R.H.S. of (3.6) obtains, so does the L.H.S. As an example, let $y_n = e^{n!}$, $P(U = y_n) \equiv 1/en!$, and $z_n = y_n + \sqrt{y_n}$. Typically t_{y_n} is the *first* $k: U_k = y_n$ and t_{z_n} is the *second* $k: U_k = y_n$. With this in mind it is easily seen that

$$Et_{y_n} \sim 1/P(U = y_n) \sim y_n/E(U \wedge y_n)$$

and

$$Et_{z_n} \sim 2/P(U = y_n) \sim 2y_n/E(U \wedge y_n) \sim 2z_n/E(U \wedge z_n). \qquad \square$$

4. Upper and lower asymptotic bounds for $P(M_\varepsilon < y)$ for "small" y as $\varepsilon \searrow 0^+$.

THEOREM 4.1. *Let $X_1, X_2, \cdots$ be* i.i.d. *mean zero random variables such that $0 < E(X_1^+)^2 < \infty$. Let $M_\varepsilon = \sup_{n\geq 1}(S_n - n\varepsilon)^+$, where $S_n = X_1 + \cdots + X_n$. Let $\tau_+ = 1^{\text{st}}\ n: S_n > 0$. Then there exists $y_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ such that*

$$\begin{aligned}
1 &= \liminf_{\varepsilon\to 0^+}\inf_{0<y\leq y_\varepsilon} \frac{P(M_\varepsilon < y)E(S_{\tau_+} \wedge y)}{yP(M_\varepsilon = 0)}\\
&\leq \limsup_{\varepsilon\to 0^+}\sup_{0<y\leq y_\varepsilon} \frac{P(M_\varepsilon < y)E(S_{\tau_+} \wedge y)}{yP(M_\varepsilon = 0)} \leq 2.
\end{aligned} \tag{4.1}$$

PROOF. Let $0 < y < \infty$ be a continuity point of the n-fold convolution of S_{τ_+} (with itself) for every $n \geq 1$. We need to determine the order of magnitude of $\sum_{n=0}^{\infty} (1-p_\varepsilon)^n P(S_{n\varepsilon} < y) = P(M_\varepsilon < y)/P(M_\varepsilon = 0)$ as $\varepsilon \to 0^+$, where $S_{n\varepsilon} = Y_{1\varepsilon} + \cdots + Y_{n\varepsilon}$ and $Y_{j\varepsilon}$ and p_ε are defined as usual (see the proof of Theorem 1.3). Let

(4.2) $$\tau_{y\varepsilon} = 1^{\text{st}}\, n : S_{n\varepsilon} \geq y.$$

Equivalently,

(4.3) $$\tau_{y\varepsilon} = 1^{\text{st}}\, n : \textstyle\sum_{j=1}^{n} (Y_{j\varepsilon} \wedge y) \geq y.$$

We also have

(4.4) $$\tau_{y\varepsilon} = \textstyle\sum_{n=0}^{\infty} I(S_{n\varepsilon} < y).$$

Invoking Wald's equation,

(4.5) $$E\tau_{y\varepsilon} E(Y_{1\varepsilon} \wedge y) = E \textstyle\sum_{j=1}^{\tau_{y\varepsilon}} (Y_{j\varepsilon} \wedge y) \qquad \text{(which holds even for } \varepsilon = 0)$$

Clearly, $Y_{1\varepsilon} \wedge y \to_{\text{a.s.}} S_{\tau_+} \wedge y$, $\tau_{y\varepsilon} \to_{\text{a.s.}} \tau_{y0}$ and $\sum_{j=1}^{\tau_{y\varepsilon}} (Y_{j\varepsilon} \wedge y) \to_{\text{a.s.}} \sum_{j=1}^{\tau_{y0}} (Y_{j0} \wedge y)$, where Y_{10}, $Y_{20}, \cdots$ are i.i.d., $Y_{10} = S_{\tau_+}$, and $\tau_{y0} = 1^{\text{st}}\, n : \sum_{j=1}^{n} (Y_{j0} \wedge y) \geq y$. By the bounded convergence theorem,

$$\lim_{\varepsilon \to 0^+} E(Y_{1\varepsilon} \wedge y) = E(S_{\tau_+} \wedge y) \quad \text{and}$$

$$\lim_{\varepsilon \to 0^+} E \textstyle\sum_{j=1}^{\tau_{y\varepsilon}} (Y_{j\varepsilon} \wedge y) = E \sum_{j=1}^{\tau_{y0}} (Y_{j0} \wedge y) \quad (\leq 2y).$$

Since both limits are positive as well as finite, (4.5) shows that $\lim_{\varepsilon \to 0^+} E\tau_{y\varepsilon}$ exists, is finite, and equals $E\tau_{y0}$. Using representation (4.4) of $\tau_{y\varepsilon}$ for $\varepsilon \geq 0$ and Fatou's lemma,

$$\begin{aligned} E\tau_{y0} &= \textstyle\sum_{n=0}^{\infty} P(S_{n0} < y) = \sum_{n=0}^{\infty} \liminf_{\varepsilon \to 0^+} (1-p_\varepsilon)^n P(S_{n\varepsilon} < y) \\ &\leq \liminf_{\varepsilon \to 0^+} \textstyle\sum_{n=0}^{\infty} (1-p_\varepsilon)^n P(S_{n\varepsilon} < y) \leq \limsup_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} P(S_{n\varepsilon} < y) \\ &= \limsup_{\varepsilon \to 0^+} E\tau_{y\varepsilon} = E\tau_{y0}. \end{aligned}$$

Hence

(4.6) $$\lim_{\varepsilon \to 0^+} \frac{P(M_\varepsilon < y)}{P(M_\varepsilon = 0) E\tau_{y0}} = 1.$$

Let $g(y) = y/E(S_{\tau_+} \wedge y) = 1/E\{(S_{\tau_+}/y) \wedge 1\}$ and $R_\varepsilon(y) = P(M_\varepsilon < y)/P(M_\varepsilon = 0)g(y)$. Note that $\lim_{y \to 0^+} R_\varepsilon(y) = \lim_{y \to 0^+} 1/g(y) = P(S_{\tau_+} > 0) = 1$. Hence set $R_\varepsilon(0) = 1$. Using (3.1) in conjunction with (4.6),

(4.7) $$1 \leq \liminf_{\varepsilon \to 0^+} R_\varepsilon(y) \leq \limsup_{\varepsilon \to 0^+} R_\varepsilon(y) \leq 2$$

for a dense set of $y \geq 0$. Since $P(M_\varepsilon < y)$ is non-decreasing and $g(y)$ is continuous (4.7) holds for all $y \geq 0$. Furthermore, a theorem of Dini shows that the convergence of $h_\varepsilon(y) \equiv (1 - R_\varepsilon(y))^+ + (R_\varepsilon(y) - 2)^+$ to zero as $\varepsilon \to 0^+$ occurs uniformly on compact intervals $[0, y]$. By a simple argument it follows that there exist $y_\varepsilon \to \infty$ such that $\lim_{\varepsilon \to 0^+} \sup_{0 \leq y \leq y_\varepsilon} h_\varepsilon(y) = 0$. □

REMARK 4.2. Inequality (4.1) remains valid for any $\bar{y}_\varepsilon \geq y_\varepsilon$ such that $\varepsilon \bar{y}_\varepsilon \to 0$. To see this note that by (2.4) and (2.10),

$$\begin{aligned} &\lim_{\varepsilon \searrow 0^+} \sup_{y_\varepsilon \leq y \leq \bar{y}_\varepsilon} \left| \frac{P(M_\varepsilon < y) E(S_{\tau_+} \wedge y)}{y P(M_\varepsilon = 0)} - 1 \right| \\ &= \lim_{\varepsilon \searrow 0^+} \sup_{y_\varepsilon \leq y \leq \bar{y}_\varepsilon} \left| \frac{P(M_\varepsilon < y)}{y 2\varepsilon\sigma^{-2}} - 1 \right| = \lim_{\varepsilon \searrow 0^+} \sup_{y_\varepsilon \leq y \leq \bar{y}_\varepsilon} \left| \frac{P(M_\varepsilon < y)}{1 - \exp\{-2\varepsilon\sigma^{-2} y\}} - 1 \right| = 0. \end{aligned}$$

Note also that both (4.1) and (2.10) continue to hold if $P(M_\varepsilon \leq y)$ is substituted for $P(M_\varepsilon \leq y)$.

5. Approximation of certain expectations. We require the following lemma. Its proof involves fairly standard use of compactness and will be omitted.

LEMMA 5.1. *Let $\{f_\varepsilon(\cdot):\varepsilon \geq 0\}$ be a collection of non-decreasing functions such that $\lim_{\varepsilon\searrow 0^+} f_\varepsilon(0+) = f_0(0+) > 0$ and $\lim_{\varepsilon\searrow 0^+} f_\varepsilon(u) = f_0(u)$ for almost all $u > 0$. Then there exists $y_\varepsilon \to \infty$ such that*

$$\lim_{\varepsilon\searrow 0^+}\sup_{0<y\leq y_\varepsilon}\left|\frac{\int_0^y f_\varepsilon(u)\,du}{\int_0^y f_0(u)\,du}-1\right| = 0. \tag{5.1}$$

THEOREM 5.2. *Let $X, X_1, X_2, \cdots$ be* i.i.d. *mean zero random variables with finite variance $\sigma^2 > 0$. Let $S_n = X_1 + \cdots + X_n$. Let $T_0 = S_0 = 0$ and for integers $n \geq 1$ let $T_n = 1^{\text{st}}\, k: S_k > S_{T_{n-1}}$. Let $\tau_m = 1^{\text{st}}\, n: S_{T_n} \geq m$ for real $m \geq 0$. Let $M_\varepsilon = \sup_{n\geq 1}(S_n - n\varepsilon)^+$. Fix $\alpha > 1$. If $E(X^-)^{\alpha+1} < \infty$ then*

$$\begin{aligned}\lim_{\varepsilon\searrow 0^+}\frac{E(((X-\varepsilon)^- - M_\varepsilon)^+)^\alpha}{\varepsilon} &= 2\alpha\sigma^{-2}ES_{T_1}E\int_0^{X^-}(X^- - m)^{\alpha-1}E\tau_m\,dm \\ &= 2\sigma^{-2}ES_{T_1}\textstyle\sum_{n=0}^\infty E((X^- - S_{T_n})^+)^\alpha < \infty.\end{aligned} \tag{5.2}$$

If $E(X^-)^{\alpha+1} = \infty$ and $E(X^-)^\alpha < \infty$ then

$$\lim_{\varepsilon\searrow 0^+}\frac{E(((X-\varepsilon)^- - M_\varepsilon)^+)^\alpha}{E((X^- - Y_\varepsilon)^+)^\alpha} = 1, \tag{5.3}$$

where Y_ε is independent of X and $P(Y_\varepsilon \leq y) = 1 - \exp\{-2\varepsilon y\sigma^{-2}\}$.

To approximate the limit in (5.2) one may use the simple inequalities $m/E(S_{T_1}\wedge m) \leq E\tau_m \leq 2m/E(S_{T_1}\wedge m)$ (see Theorem 3.1).

PROOF. We begin by showing that $(X-\varepsilon)^-$ may be replaced by X^-. We may assume $E(X^-)^\alpha < \infty$.

$$\begin{aligned}&\lim_{\varepsilon\searrow 0^+}\frac{E(((X-\varepsilon)^- - M_\varepsilon)^+)^\alpha - E((X^- - M_\varepsilon)^+)^\alpha}{\varepsilon}\\ &\quad\leq \lim_{\varepsilon\searrow 0^+}\frac{E(((X-\varepsilon)^- - M_\varepsilon)^+)^\alpha I(0<X<\varepsilon)}{\varepsilon}\\ &\qquad+\lim_{\varepsilon\searrow 0^+}\frac{E\{((\varepsilon + X^- - M_\varepsilon)^+)^\alpha - ((X^- - M_\varepsilon)^+)^\alpha\}I(X^-\geq 0)}{\varepsilon}\\ &\quad\leq 0 + \lim_{\varepsilon\searrow 0^+} E\alpha((\varepsilon + X^- - M_\varepsilon)^+)^{\alpha-1}I(X^-\geq 0)\\ &\quad= 0 \quad \text{since} \quad M_\varepsilon \to \infty \text{ a.s.} \quad \text{and} \quad E(X^-)^{\alpha-1} < \infty.\end{aligned}$$

Conditioning on X^- and integrating by parts twice we find that

$$E((X^- - M_\varepsilon)^+)^\alpha = \alpha(\alpha-1)E\int_0^{X^-}(X^- - m)^{\alpha-2}\int_0^m P(M_\varepsilon \leq u)\,du\,dm.$$

In view of (4.6), we may apply Lemma 5.1, obtaining the existence of $y_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ such that

$$\lim_{\varepsilon\searrow 0^+}\sup_{0<m\le y_\varepsilon}\left|\frac{\int_0^m P(M_\varepsilon<u)\,du}{P(M_\varepsilon=0)\int_0^m E\tau_u\,du}-1\right|=0.$$

Recalling that $P(M_\varepsilon = 0) \sim 2\varepsilon\sigma^{-2}ES_{T_1}$ and that for any random variable V, $\int_0^m P(V\le u)\,du = \int_0^m P(V<u)\,du$,

$$\lim_{\varepsilon\searrow 0^+}\frac{E((X-\varepsilon)^- - M_\varepsilon)^{+\alpha}}{P(M_\varepsilon=0)}$$

$$=\lim_{\varepsilon\searrow 0^+}\frac{E((X^- - M_\varepsilon)^+)^\alpha}{P(M_\varepsilon=0)} \tag{5.4}$$

$$=\lim_{\varepsilon\searrow 0^+}\alpha(\alpha-1)E\int_0^{X^-}((X^- - m)^{\alpha-2}\int_0^{m\wedge y_\varepsilon}E\tau_u\,du)\,dm$$

$$+\lim_{\varepsilon\searrow 0^+}\alpha(\alpha-1)E\int_0^{X^-}(X^- - m)^{\alpha-2}\int_{m\wedge y_\varepsilon}^m\frac{P(M_\varepsilon<u)}{P(M_\varepsilon=0)}\,du\,dm.$$

We focus on the second quantity first (unless X^- is bounded, whence only the first term counts). According to Theorem 2.6,

$$\lim_{\varepsilon\searrow 0^+}\frac{E\int_0^{X^-}(X^- - m)^{\alpha-2}\int_{m\wedge y_\varepsilon}^m P(M_\varepsilon<u)\,du\,dm}{E\int_0^{X^-}(X^- - m)^{\alpha-2}\int_{m\wedge y_\varepsilon}^m(1-\exp\{-2\varepsilon\sigma^{-2}u\})\,du\,dm}=1. \tag{5.5}$$

Now if $E(X^-)^{\alpha+1}<\infty$ then

$$E\int_0^{X^-}(X^- - m)^{\alpha-2}\int_0^m\frac{1-\exp\{-2\varepsilon\sigma^{-2}u\}}{\varepsilon}\,du\,dm\le E\int_0^{X^-}(X^- - m)^{\alpha-2}\int_0^m 2\sigma^{-2}u\,du\,dm$$

$$=E\int_0^{X^-}(X^- - m)^{\alpha-2}m^2\sigma^{-2}\,dm$$

$$=2E\int_0^{X^-}\frac{(X^- - m)^{\alpha-1}}{\alpha-1}m\sigma^{-2}\,dm$$

(integrating by parts)

$$=2E\int_0^{X^-}\frac{(X^- - m)^{\alpha}}{(\alpha-1)\alpha}\sigma^{-2}\,dm$$

(integrating by parts again)

$$=2E\frac{(X^-)^{\alpha+1}\sigma^{-2}}{(\alpha-1)\alpha(\alpha+1)}<\infty.$$

Hence by dominated convergence,

$$\lim_{\varepsilon \searrow 0^+} E \int_0^{X^-} (X^- - m)^{\alpha-2} \int_{m \wedge y_\varepsilon}^m \frac{1 - \exp(-2\varepsilon\sigma^{-2}u)}{P(M_\varepsilon = 0)} \, du \, dm = 0.$$

Since $E\tau_u \le \dfrac{2u}{ES_{T_1} \wedge u}$ it also follows that

$$\lim_{\varepsilon \searrow 0^+} E \int_0^{X^-} (X^- - m)^{\alpha-2} \int_{m \wedge y_\varepsilon}^m E\tau_u \, du \, dm = 0$$

whenever $E(X^-)^{\alpha+1} < \infty$.

Thus $E(X^-)^{\alpha+1} < \infty$ implies

$$\lim_{\varepsilon \searrow 0^+} \frac{E(((X - \varepsilon)^- - M_\varepsilon)^+)^\alpha}{P(M_\varepsilon = 0)} = \alpha(\alpha - 1)E \int_0^{X^-} (X^- - m)^{\alpha-2} \int_0^m E\tau_u \, du \, dm$$

$$= \alpha E \int_0^{X^-} (X^- - m)^{\alpha-1} E\tau_m \, dm,$$

from which half of (5.2) follows. To obtain the other half note that

$$\sum_{n=0}^\infty E((X^- - S_{T_n})^+)^\alpha = E \sum_{n=0}^\infty E\{((X^- - S_{T_n})^+)^\alpha \mid X^-\}$$

$$= E \sum_{n=0}^\infty \int_0^{X^-} \alpha(X^- - m)^{\alpha-1} P(S_{T_n} \le m) \, dm$$

(integrating by parts)

$$= E \sum_{n=0}^\infty \int_0^{X^-} \alpha(X^- - m)^{\alpha-1} P(S_{T_n} < m) \, dm$$

$$= \alpha E \int_0^{X^-} (X^- - m)^{\alpha-1} E\left(\sum_{n=0}^\infty I(S_{T_n} < m)\right) dm$$

$$= \alpha E \int_0^{X^-} (X^- - m)^{\alpha-1} E\tau_m \, dm.$$

Henceforth assume $E(X^-)^{\alpha-1} = \infty$. Take $x_\varepsilon \to \infty$ such that $\varepsilon x_\varepsilon \to 0$ as $\varepsilon \searrow 0^+$. We may assume y_ε tends to infinity so slowly that $(y_\varepsilon)^2 = o(E(X^-)^{\alpha+1} I(X^- \le x_\varepsilon))$. Then a bit of adding and subtracting plus use of (5.4) and (5.5) gives

$$\lim_{\varepsilon \searrow 0^+} \left| \frac{E((X^- - M_\varepsilon)^+)^\alpha}{\alpha(\alpha - 1)E \int_0^{X^-} (X^- - m)^{\alpha-2} \int_0^m (1 - \exp(-2\varepsilon\sigma^{-2}u)) \, du \, dm} - 1 \right|$$

$$\le \lim_{\varepsilon \searrow 0^+} \frac{P(M_\varepsilon = 0)E \int_0^{X^-} (X^- - m)^{\alpha-2} \int_0^{m \wedge y_\varepsilon} E\tau_u \, du \, dm}{E \int_0^{X^-} (X^- - m)^{\alpha-2} \int_0^m (1 - \exp(-2\varepsilon\sigma^{-2}u)) \, du \, dm}$$

$$+ \lim_{\varepsilon \searrow 0^+} \frac{E \int_0^{X^-} (X^- - m)^{\alpha-2} \int_0^{m \wedge y_\varepsilon} (1 - \exp(-2\varepsilon\sigma^{-2}u)) \, du \, dm}{E \int_0^{X^-} (X^- - m)^{\alpha-2} \int_0^m (1 - \exp(-2\varepsilon\sigma^{-2}u)) \, du \, dm}.$$

Since

$$\int_0^{m\wedge y_\varepsilon} E\tau_u\,du \le \int_0^{y_\varepsilon} E\tau_u\,du \le \int_0^{y_\varepsilon} \frac{2u}{E(S_{T_1}\wedge u)}\,du \sim (y_\varepsilon)^2/ES_{T_1}$$

and

$$\int_0^{m\wedge y_\varepsilon} (1-\exp(-2\varepsilon\sigma^{-2}u))\,du \le \int_0^{y_\varepsilon} 2\varepsilon\sigma^{-2}u\,du \le \varepsilon\sigma^{-2}(y_\varepsilon)^2,$$

we find that the sum of the first two limits above is at most

$$\lim_{\varepsilon\searrow 0^+} \frac{3\varepsilon\sigma^{-2}(y_\varepsilon)^2 E\dfrac{(X^-)^{\alpha-1}}{\alpha-1}}{E\left(\displaystyle\int_0^{X^-}(X^- - m)^{\alpha-2}\int_0^m (1-\exp(-2\varepsilon\sigma^{-2}u))\,du\,dm\right)I(X^-\le x_\varepsilon)}$$

$$= \lim_{\varepsilon\searrow 0^+} \frac{3(y_\varepsilon)^2E(X^-)^{\alpha-1}/(\alpha-1)}{E\displaystyle\int_0^{X^-}(X^- - m)^{\alpha-2}m^2\,dm\,I(X^-\le x_\varepsilon)} = \lim_{\varepsilon\searrow 0^+} \frac{3(y_\varepsilon)^2\alpha(\alpha+1)E(X^-)^{\alpha-1}}{E(X^-)^{\alpha+1}I(X^-\le x_\varepsilon)} = 0.$$

Now (5.3) follows easily upon integrating twice by parts in the reverse direction. □

REMARK 5.3. Conclusion (5.3) can be maintained with

(5.6) $$P(Y_\varepsilon \le y) = 1-\exp\{-yP(M_\varepsilon = 0)/ES_{T_1}\}$$

whenever $\alpha > 1$, $E(X^+)^2 < \infty$, $E(X^-)^\alpha < \infty$, $E(X^-)^{\alpha+1} = \infty$, and (using previous notation) $\lim_{\varepsilon\searrow 0^+} s_+(\varepsilon) = ES_{T_1}$ (see (2.16) and Remark 2.5).

The limiting value in (5.2) can be identified explicitly whenever S_{T_1} is constant a.s. We compute this limit for the case $\alpha = 2$, since it is easy, and also germane to approximating EM_ε.

COROLLARY 5.4. *Given the conditions of Theorem* 5.2, *suppose X assumes integer values less than or equal to one. Assume also that $E(X^-)^3 < \infty$. Then*

(5.7) $$\lim_{\varepsilon\searrow 0} \frac{E(((X-\varepsilon)^- - M_\varepsilon)^+)^2}{\varepsilon} = \frac{EX^-(X^-+1)(2X^-+1)}{3EX^2}.$$

PROOF. Clearly $S_{T_n} = n$ a.s. Hence

$$\sum_{n=0}^\infty E((X^- - S_{T_n})^+)^2 = \sum_{k=0}^\infty \sum_{n=0}^k (k-n)^2P(X^-=k) = \sum_{k=0}^\infty\sum_{n=0}^k n^2P(X^-=k)$$

$$= \sum_{k=0}^\infty \frac{k(k+1)(2k+1)}{6}P(X^-=k) = E\frac{X^-(X^-+1)(2X^-+1)}{6}.$$

Now apply Theorem 5.2. □

When $E(X^-)^{\alpha+1} = \infty$, the asymptotic growth rate of $E(((X-\varepsilon)^- - M_\varepsilon)^+)^\alpha$ can (in principle) be computed explicitly by means of (5.3). The result assumes a particularly simple form, however, for the distributions considered below.

COROLLARY 5.5. *Given the conditions of Theorem* 5.2, *suppose $E(X^-)^{\alpha+1}I(X^-\le y)$ is a slowly varying function increasing to infinity. Then*

(5.8) $$\lim_{\varepsilon\searrow 0^+} \frac{E(((X-\varepsilon)^- - M_\varepsilon)^+)^\alpha}{2\varepsilon\sigma^{-2}E(X^-)^{\alpha+1}I(X^-\le 1/\varepsilon)} = (\alpha+1)^{-1}.$$

PROOF. To begin with, we assert that

$$\lim_{y\to\infty} \frac{yE(X^-)^\alpha I(X^- > y)}{E(X^-)^{\alpha+1} I(X^- \le y)} = 0. \tag{5.9}$$

Fix $0 < c - 1 \ll 1$. There exists y_0 such that for $y \ge y_0$,

$$E(X^-)^{\alpha+1} I(X^- \le 2y) \le cE(X^-)^{\alpha+1} I(X^- \le y).$$

For such y,

$$\begin{aligned} &yE(X^-)^\alpha I(X^- > y) \\ &\quad = \textstyle\sum_{n=0}^\infty Ey(X^-)^\alpha I(2^n y < X^- \le 2^{n+1}y) \le \sum_{n=0}^\infty E2^{-n}(X^-)^{\alpha+1} I(2^n y < X^- \le 2^{n+1}y) \\ &\quad \le \textstyle\sum_{n=0}^\infty 2^{-n}(c-1)E(X^-)^{\alpha+1} I(X^- \le 2^n y) \le \sum_{n=0}^\infty (c/2)^n (c-1)E(X^-)^{\alpha+1} I(X^- \le y) \\ &\quad = (2(c-1)/(2-c))E(X^-)^{\alpha+1} I(X^- \le y). \end{aligned}$$

Send y to infinity and $c \to 1$ to obtain (5.9).

Proceeding, let Y_ε be as in Theorem 5.2. Observe that (5.8) will follow immediately from Theorem 5.2 provided we prove that $E((X^- - Y_\varepsilon)^+)^\alpha \sim b_\varepsilon$ as $\varepsilon \searrow 0^+$, where

$$b_\varepsilon \equiv (2\varepsilon\sigma^{-2}/(\alpha+1))E(X^-)^{\alpha+1} I(\varepsilon X^- \le 1).$$

Take $\delta_\varepsilon \to 0$ such that

$$E(X^-)^{\alpha+1} I(\varepsilon X^- \le \delta_\varepsilon)/E(X^-)^{\alpha+1} I(\varepsilon X^- \le 1) \to 1 \quad \text{as} \quad \varepsilon \searrow 0^+.$$

We approximate $E((X^- - Y_\varepsilon)^+)^\alpha$ on three intervals. First,

$$\begin{aligned} E((X^- - Y_\varepsilon)^+)^\alpha I(\varepsilon X^- \le \delta_\varepsilon) &= E\int_0^{X^- I(\varepsilon X^- \le \delta_\varepsilon)} (X^- - y)^\alpha 2\varepsilon\sigma^{-2} \exp\{-2\varepsilon\sigma^{-2} y\}\, dy \\ &\sim E\int_0^{X^- I(\varepsilon X^- \le \delta_\varepsilon)} (X^- - y)^\alpha 2\varepsilon\sigma^{-2}\, dy \\ &= (2\varepsilon\sigma^{-2}/(\alpha+1))E(X^-)^{\alpha+1} I(\varepsilon X^- \le \delta^\varepsilon) \\ &\sim b_\varepsilon \quad \text{as} \quad \varepsilon \searrow 0^+. \end{aligned}$$

Second,

$$\begin{aligned} E((X^- - Y_\varepsilon)^+)^\alpha I(\delta_\varepsilon < \varepsilon X^- \le 1) &\le E\left(\int_0^{X^-} (X^- - y)^\alpha 2\varepsilon\sigma^{-2}\, dy\, I(\delta_\varepsilon < \varepsilon X^- \le 1)\right) \\ &= (2\varepsilon\sigma^{-2}/(\alpha+1))E(X^-)^{\alpha+1} I(\delta_\varepsilon < \varepsilon X^- \le 1) \\ &= o(b_\varepsilon). \end{aligned}$$

Third,

$$\begin{aligned} E((X^- - Y_\varepsilon)^+)^\alpha I(\varepsilon X^- > 1) &\le E(X^-)^\alpha I(\varepsilon X^- > 1) \\ &= o(\varepsilon E(X^-)^{\alpha+1} I(\varepsilon X^- \le 1)) \quad \text{by (5.9)} \\ &= o(b_\varepsilon) \quad \text{as} \quad \varepsilon \searrow 0^+ \end{aligned}$$

Indeed, $E((X^- - Y_\varepsilon)^+)^\alpha / b_\varepsilon \to 1$ as $\varepsilon \searrow 0^+$. □

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REFERENCES

VON BAHR, B. (1974). Ruin probabilities expressed in terms of ladder height distributions. *Scandinavian Actuarial J.* 190–204.

BOROVKOV, A. A. (1962). New limit theorems in boundary problems for sums of independent terms. *Selected Trans. Math. Statist. Probab.* **5** 315–372.

CHOW, Y. S. and LAI, T. L. (1978). On the maximal excess in boundary crossings of random walks related to fluctuation theory and laws of large numbers. Columbia University, preprint.

CHUNG, K. L. (1968). *A Course in Probability Theory.* Harcourt, New York.

CRAMÉR, H. (1954). On some questions connected with mathematical risk. *Univ. Calif. Publ. Statist.* **2** 5, 99–125.

FELLER W. (1966). *An Introduction to Probability Theory and Its Applications, Vol. II.* Wiley, New York.

KIEFER, J. and WOLFOWITZ, J. (1956). On the characteristics of the general queueing process, with applications to random walk. *Ann. Math. Statist.* **27** 147–161.

KINGMAN, J. F. C. (1962). Some inequalities for the queue GI/G/1. *Biometrika* **49** 315–324.

LORDEN, G. (1970). On excess over the boundary. *Ann. Math. Statist.* **41** 2, 520–527.

SIEGMUND, D. (1975a). Error probabilities and average sample number of the sequential probability ratio test. *J. Roy. Statist. Soc. Ser. B* **37** 394–401.

SIEGMUND, D. (1975b). The time until ruin in collective risk theory. *Mitt. Verein. Schweiz. Versich.-Math.* **75** 157–166.

SIEGMUND, D. (1978). Corrected diffusion approximations in certain random walk problems. Tech. Rept. 118, Statistics Dept., Stanford University.

WOODROOFE, M. (1978). Repeated likelihood ratio tests, University of Michigan Technical Report.

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