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INDEPENDENCE VIA UNCORRELATEDNESS UNDER CERTAIN DEPENDENCE STRUCTURES[1]

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A characterization of independence via uncorrelatedness is shown to hold for the families satisfying positive and negative dependence conditions. For the associated random variables, the bounds on covariance functions obtained by Lebowitz (*Comm. Math. Phys.* **28** (1972), 313–321) readily yield such a characterization. An elementary proof for the same characterization is also given for a condition weaker than association, labeled as "strong positive (negative) orthant dependence." This condition is compared with the "linear positive dependence," under which Newman and Wright (*Ann. Probab.* **9** (1981), 671–675) obtained the characterization.

1. Introduction and summary. Among various notions of positive dependence, that of *association* has proved to be quite useful. Esary, Proschan and Walkup (1967), introduced this concept to obtain bounds related to coherent (co-ordinatewise increasing) functions occurring in the theory of reliability. The defining property for a random vector $X = (X_1, \cdots, X_n)$ (or more appropriately, its distribution) to be *associated* is that for f, g co-ordinatewise nondecreasing

$$\text{cov}[f(\mathbf{X}), g(\mathbf{X})] \geq 0. \tag{1.1}$$

In a completely different context, namely, the Ising model of statistical physics, Fortuin, Kasteleyn and Ginibre (1971) proved a similar inequality, which is well known now as the FKG inequality.

Since *association* represents a strong positive dependence, weaker concepts have been considered in the literature. As pointed out by Shaked (1982) many of these can be viewed as variations of the classes from which f, g are chosen and then (1.1) is imposed. There is also a negative analogue of the concept of *association*, (see Definition 2.3), for example, as in Joag-Dev and Proschan (1982).

The classes of multivariate distributions defined by these notions of positive or negative dependence for $(X_1, \cdots, X_n)$, invariably contain those where X_i are mutually independent. It is of some interest to see whether, in such classes, the simple condition of uncorrelatedness characterizes mutual independence.

Newman and Wright (1981) obtained certain bounds which implied such a characterization. In fact, their dependence condition, which can be described as "linear positive dependence" (see Definition 2.2), is weaker than the *association*. However, their proof is based on characteristic functions and is not elementary. An even more involved proof of the characterization for independence among the associated random variables was given by Wells (1977).

In this note, we point out a proof based on very elegant bounds (not so well known to probabilists and statisticians) obtained by Lebowitz (1972). These bounds readily provide the desired characterization of mutual independence for the class of *associated* or *negatively associated* random variables. Another notion of positive (negative) dependence weaker than association, called "strong positive (negative) orthant dependence" (see Definition 2.1), is shown to yield an elementary proof for the required characterization. It

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is shown that this concept of dependence is neither weaker nor stronger than linear positive dependence considered by Newmann and Wright (1981).

In what follows, a function defined on $R^n \to R$ will be said to be "increasing" ("decreasing") if it is co-ordinatewise nondecreasing (nonincreasing). The conditions of dependence, such as association etc. really apply to distributions, however, whenever convenient, we might express this by saying that X is *associated* or X_i are *associated.*

2. Concepts of dependence. Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a random vector with n real components. In the following definitions, A will denote an arbitrary proper subset of the index set 1, 2, $\cdots$, n, $\bar{A}$ its complement and $\mathbf{c} = (c_1, \cdots, c_n)$, a vector of constants.

DEFINITION 2.1. A vector $\mathbf{X}$ is said to be strongly positively orthant dependent (SPOD) if for every A and $\mathbf{c}$, the following three conditions hold.

$$P[\mathbf{X} \geq \mathbf{c}] \geq P[X_i \geq c_i, i \in A]P[X_j \geq c_j, j \in \bar{A}]. \tag{2.1}$$

$$P[\mathbf{X} \leq \mathbf{c}] \geq P[X_i \leq c_i, i \in A]P[X_j \leq c_j, j \in \bar{A}]. \tag{2.2}$$

and

$$P[X_i \geq c_i, i \in A, \ X_j \leq c_j, j \in \bar{A}] \leq P[X_i \geq c_i, i \in A]P[X_j \leq c_j, j \in \bar{A}]. \tag{2.3}$$

The vector $\mathbf{X}$ on the other hand, is said to be *strongly negatively orthant dependent* (SNOD) if the reverse inequalities between the left and right sides of (2.1), (2.2) and (2.3) hold for every $\mathbf{c}$.

REMARK. For $n = 2$, the inequalities (2.1), (2.2) and (2.3) are all equivalent and a pair of random variables satisfying such a dependence condition is said to be *positively quadrant dependent* (PQD). Its negative analog is known as NQD. Also note that the conditions (2.1) − (2.3) are variations of (1.1) where f, g are indicators of upper or lower orthants. The condition (2.1) clearly implies "positive upper orthant dependence" (PUOD) which requires

$$P[\mathbf{X} \geq c] \geq \textstyle\prod_{i=1}^{n} P[X_i \geq c_i].$$

In the same way (2.2) implies "positive lower orthant dependence", (PLOD). Thus SPOD implies PUOD and PLOD. A similar statement can be made for the negative dependence conditions.

DEFINITION 2.2. A vector $\mathbf{X}$ is said to be *linearly positively quadrant dependent* (LPQD) if for every pair of non-negative vectors $\mathbf{r}$, $\mathbf{s}$, and for every A, the pair $\sum_{i \in \bar{A}} r_i X_i$, $\sum_{j \in A} s_j X_j$ is PQD.

DEFINITION 2.3. A vector $\mathbf{X}$ is said to be *negatively associated* if for every pair of increasing functions f and g, and for every A,

$$\operatorname{cov}[f(X_i, i \in A), g(X_j, j \in \bar{A})] \leq 0. \tag{2.4}$$

REMARK. The concept LPQD is somewhat similar to one of those introduced by Shaked (1982) while that of negative association has been introduced by Joag-Dev and Proschan (1983).

LEMMA 1. *Neither of the two conditions* SPOD *and* LPQD *implies the other.*

PROOF. Throughout this proof $p(x_1x_2x_3)$ will denote $P[X_i = x_i, i = 1, 2, 3]$. Consider a vector (X_1, X_2, X_3) with the following joint distribution:

$$p(000) = p(121) = 3/14,$$
$$p(021) = p(100) = 2/14,$$
$$p(011) = p(020) = p(101) = p(110) = 1/14.$$

It can be checked that (X_1, X_2, X_3) is SPOD. The number of inequalities to be verified is reduced substantially by observing that (X_1, X_2, X_3) has the same distribution as $(1 - X_1, 2 - X_2, 1 - X_3)$. However,

$$3/14 = P[X_1 \geq 1, X_2 + X_3 \geq 2] < P[X_1 \geq 1]P[X_2 + X_3 \geq 2] = 1/4,$$

establishing that (X_1, X_2, X_3) is not LPQD.

To show that LPQD does not imply SPOD, again we consider a trivariate distribution. Note that LPQD condition is equivalent to

$$P[r_iX_i + r_jX_j \geq c_1 \mid X_k \geq c_2] \geq P[r_iX_i + r_jX_j \geq c_1] \tag{2.5}$$

holding for arbitrary $r_i > 0$ and each pair (c_1, c_2). Here, (i, j, k) are permutations of (1, 2, 3).

At this stage it should be noted that a bivariate distribution having four or six atoms on a lattice, as in the above example, has the following property. For every upper (lower) quadrant containing one or more atoms, there corresponds an upper (lower) half plane, defined by a line with a negative slope, which contains exactly the same atoms. For a trivariate distribution, the SPOD conditions (2.1) – (2.3) could be given the conditional form as in (2.5). In view of the above observation, failure of SPOD condition for some quadrant would imply the failure of (2.5). Thus to produce the required example, we are forced to look at the trivariate distributions where at least one set of the conditional bivariate distributions has nine or more atoms.

Consider the following distribution described by conditional distributions of (X_1, X_2) given X_3. The entries in the two tables are probabilities. Those on the left are multiplied by 72 and on the right by 48.

Given $X_3 = 0$

X_2 \ X_1	0	1	2
2	3	5	10
1	6	6	5
0	28	6	3

Given $X_3 = 1$

X_2 \ X_1	0	1	2
2	6	4	8
1	6	0	4
0	8	6	6

Further, $P[X_3 = 1] = 2/5$, $P[X_3 = 0] = 3/5$.

It can be verified (by a somewhat tedious process) that (X_1, X_2, X_3) is LPQD. Notice that (X_1, X_2, X_3) has the same distribution as (X_2, X_1, X_3) and this symmetry considerably reduces the number of verifications. However,

$$2/15 = P[X_i \geq 1, i = 1, 2, 3] < P[X_j \geq 1, j = 1, 2] \times P[X_3 \geq 1] = 7/50,$$

and hence (X_1, X_2, X_3) is not SPOD.

3. Characterization of independence. Let $\mathbf{X}$ be a random n-vector with real components X_i and Z_i be the indicators of the events $X_i \geq c_i$, where c_i are arbitrary constants. Let A, B be proper disjoint subsets of the index set $\{1, 2, \cdots, n\}$. Using the notation,

$$U(A) = \prod_{i \in A} Z_i, \quad V(A) = \sum_{i \in A} Z_i,$$

define

$$H(A, B) = \text{cov}[U(A), U(B)].$$

It follows that

$$\text{cov}[V(A), V(B)] = \textstyle\sum_{i \in A} \sum_{j \in B} H(i, j),$$

where $H(i, j) = H(\{i\}, \{j\})$.

Note that if $\mathbf{X}$ is *associated* then every pair (X_i, X_j) is PQD and hence $H(i, j) \geq 0$. Similarly, for *negatively associated* $\mathbf{X}$, the covariances $H(i, j) \leq 0$.

THEOREM 1 (*Lebowitz*). *If* $\mathbf{X}$ *is associated then*

$$0 \leq H(A, B) \leq \textstyle\sum_{i \in A} \sum_{j \in B} H(i, j). \tag{3.1}$$

For negatively associated $\mathbf{X}$ *the above inequalities are reversed.*

PROOF. Note that $V(A) - U(A)$, $U(A)$, $V(A)$ are increasing functions of Z_i. Further, due to inheritance of association of $\mathbf{Z}$ from $\mathbf{X}$, it follows that

$$\text{cov}[V(A) - U(A), V(B)] \geq 0, \tag{3.2}$$

$$\text{cov}[V(B) - U(B), U(A)] \geq 0 \tag{3.3}$$

so that

$$\text{cov}[U(A), U(B)] \leq \text{cov}[U(A), V(B)] \leq \text{cov}[V(A), V(B)]. \tag{3.4}$$

The first term in (3.4) is non-negative due to the *association* property of $\mathbf{Z}$ and thus (3.1) follows from (3.4). The assertion for *negatively associated* $\mathbf{X}$ follows by observing that the inequalities (3.2) – (3.4) are reversed.

COROLLARY. *If* $X_1, \cdots, X_n$ *are associated* (*negatively associated*) *and uncorrelated then* X_i *are mutually independent.*

PROOF. It was shown by Lehmann (1966) that if a pair of random variables is PQD (NQD) and uncorrelated then the random variables in the pair are independent. Thus it follows that X_i's are pairwise independent which in turn implies that $H(i, j) = 0$ for $i \neq j$. From Theorem 1, it follows that $H(A, B) = 0$, for every pair of disjoint sets. Since $c_1, \cdots, c_n$ which defined Z_i were chosen arbitrarily, it follows X_i's are independent.

THEOREM 2. *If* $(X_1, \cdots, X_n)$ *is* SPOD (*or* SNOD) *with* X_i, X_j *uncorrelated, then the* X_i *are mutually independent.*

PROOF. We use the same notation for Z_i as in Theorem 1. Recall that the assumption SPOD (SNOD) implies PQD (NQD) for every pair X_i, X_j, $i \neq j$ and hence uncorrelatedness would imply pairwise independence for X_i and hence for Z_i, $i = 1, \cdots, n$.

First the result for $n = 3$ will be established. The general result would then follow by induction. Let $p_i = P[Z_i = 1]$ and as in Section 2, $p(110)$ be the probability of $Z_1 = Z_2 = 1$, $Z_3 = 0$ etc.

From (2.2) and the pairwise independence, it follows that

$$p(101) \leq p_1(1 - p_2)p_3. \tag{3.5}$$

In general, a similar inequality holds whenever a triplet contains both 0 and 1. For example.

$$p(001) \leq (1 - p_1)(1 - p_2)p_3. \tag{3.6}$$

However, these have to be *equalities*, because if not, combining (3.5) and (3.6) it would follow that

$$P[Z_2 = 0, Z_3 = 1] < (1 - p_2)p_3, \tag{3.7}$$

violating the pairwise independence.

The only terms with possible reverse inequalities (apply (2.1) and (2.2)) are

$$p(111) \geq p_1 p_2 p_3 \tag{3.8}$$

and

$$p(000) \geq (1 - p_1)(1 - p_2)(1 - p_3). \tag{3.9}$$

But again these have to be equalities since the sum of the right and left sides of all these expressions has to be 1.

For the induction step, one may assume that every subset of cardinality $(k - 1)$ has random variables which are mutually independent. This will lead to inequalities similar to (3.5) and (3.6), for every k-tuple having both a 0 and a 1. The rest of the argument is identical.

The assertion with SNOD condition follows by reversing all the inequalities.

REMARK. For any notion of positive dependence which transmits those conditions to the indicators Z_i defined above, the characterization of independence will have to hold for these binary variables. If the inequalities such as (3.5) or (3.6) do not go in the same direction, one could assign probability mass such that all others are equalities while the mutual independence fails because of those terms. In this sense, the inequalities defining the positive (negative) dependence seem to be necessary.

Finally, consider the classical Bernstein example where a tetrahedron has 3 sides with 3 distinct colors and the fourth has stripes of all three. If X_i denotes the indicator of the presence of the ith color at the bottom of the tetrahedron (after a toss) then it is well known that the X_i's are pairwise independent but *not* mutually independent. It is interesting to note that the X_i's are (strictly) PUOD as well as NLOD. This illustrates that weak positive and negative dependence may hold at the same time, and in spite of the pairwise independence, the mutual independence might fail.

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