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# SOJOURNS OF STATIONARY PROCESSES IN RARE SETS[1]

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Let $X(t)$, $t \geq 0$, be a stationary process assuming values in a measure space $B$. The family of measurable subsets $A_u$, $u > 0$ is called "rare" if $P(X(0) \in A_u) \to 0$ for $u \to \infty$. Put $L_t(u) = \text{mes}\{s: 0 \leq s \leq t, X(s) \in A_u\}$. Under specified conditions it is shown that there exists a function $v = v(u)$ and a nonincreasing function $-\Gamma'(x)$ such that $P(v(u)L_t(u) > x)/E(v(u)L_t(u)) \to -\Gamma'(x)$, $x > 0$, for $u \to \infty$ and fixed $t > 0$. If $u = u(t)$ varies appropriately with $t$, then, under suitable conditions, the random variable $v(u)L_t(u)$ has, for $t \to \infty$, a limiting distribution of the form of a compound Poisson distribution. The results are applied to Markov processes and Gaussian processes.

**1. Introduction and summary.** Let $X(t)$, $t \geq 0$, be a separable, measurable stationary stochastic process assuming values in a measure space $B$. The particular nature of the space is not of significance; however, for definiteness, we can take $B$ as the real line or a finite dimensional Euclidian space. Let $\{A_u, u > 0\}$ be a family of measurable subsets of $B$. The family is called *rare* if

$$\lim_{u\to\infty} P(X(0) \in A_u) = 0. \tag{1.1}$$

We define the sojourn time of $X(s)$, $0 \leq s \leq t$, in $A_u$ as

$$L_t(u) = \int_0^t I_{[X(s)\in A_u]}\, ds, \tag{1.2}$$

where $I_{[\ ]}$ is the indicator function.

In this paper we derive two limit theorems for $L_t(u)$. The first is the local sojourn theorem (Theorem 2.1). By stationarity and Fubini's theorem, it is easily seen that

$$EL_t(u) = tP(X(0) \in A_u); \tag{1.3}$$

hence $EL_t(u) \to 0$ for $u \to \infty$ if $\{A_u\}$ is rare. We will show that, under general conditions, there is a function $v = v(u)$ tending to $\infty$ with $u$ such that

$$\frac{P(v \cdot L_t(u) > x)}{E(v \cdot L_t(u))} \tag{1.4}$$

converges to a certain limit for each $x > 0$ and $t > 0$, and that the limit does not depend on $t$.

Our second major result, which is partially based on the first, is the global sojourn theorem (Theorem 4.1). We define $u = u(t)$ as a function of $t$ such that the denominator in (1.4), which, by (1.3), is equal to $tvP(X(0) \in A_u)$, converges to 1. Then we prove, under certain additional mixing conditions on the process, that the numerator in (1.4) converges to a limit, that is, the random variable $v \cdot L_t(u)$ has a limiting distribution. The Laplace-Stieltjes transform is identified, and is shown to represent an infinitely divisible distribution whose Lévy spectral measure is derived from the local sojourn limit.

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These results are generalizations of earlier ones about the sojourns of a stationary process above high levels. General processes were considered in [3], and more specific ones in [2], [4], [5] and [6]. The theory had been closely tied to that of the extreme values of independent random variables; and there $B$ was the real line and $A_u = (u, \infty)$. In this paper we will show that the theory of sojourns in rare sets is quite independent of the particular nature of the set. For example, $\{A_u\}$ may represent a nested family of closed sets shrinking to a fixed point. In this case the sojourns are identified in radio engineering as "fades."

The sojourns in rare sets are also related to the concept of local time. Indeed, if $A_u$ is defined as a neighborhood of a point $x$, and $\bigcap_{u>0} A_u = \{x\}$, then the local time of $X$ at $x$ can often be defined as the limit of $L_t(u)/\text{mes}(A_u)$ for $u \to \infty$. It is known that the local time does not usually exist when the dimensionality of the state space is large relative to that of the time parameter set. For example, it does not exist for Brownian motion in several dimensions. However, our results imply a distributional definition of local time even when the latter does not exist in the ordinary sense. Here the normalization of $L_t(u)$ is not a division by $\text{mes}(A_u)$ but by another function of $u$.

The conditions for the validity of the local sojourn theorem are slightly more general than those of [3] in the case considered there. However, the conditions for the global theorem are not strictly comparable to those in the earlier version but are certainly simpler to use. As an application we show that they are convenient to check for a certain class of Markov processes. We give, as an example, the verification of the conditions for an $M$-dimensional Ornstein-Uhlenbeck process, for $M \geq 3$. Then we show that our general results can be applied to examples of rare sets of stationary Gaussian processes which are not necessarily Markovian.

The conditions on the process $X(t)$ for the global sojourn limit in [3] consisted of local and global mixing conditions stated in a relatively complex form. The present conditions, stated in Sections 3 and 4 below, are relatively simple, and are analogous to the corresponding conditions $C(u_T)$ and $C'(u_T)$ of Leadbetter, Lindgren and Rootzen [7]. The main difference between our global mixing condition and their condition $C(u_T)$ is that ours is stated in terms of ratios of probabilities and theirs is stated in terms of differences.

In the appendix we furnish a correction to the statement of the local mixing condition in [3], Lemma 17.2. The corrected version is based on the current more general approach to the global theorem.

**2. The local sojourn limit theorem.** Our first result is a generalization of the sojourn limit theorem in [3], Theorem 3.1. Here we introduce the adjective "local" to signify that the time interval is fixed.

THEOREM 2.1. *Suppose that there is a continuous nonnegative function $v = v(u)$ such that*

$$\lim_{u\to\infty} v(u) = \infty, \tag{2.1}$$

*and*

$$\lim_{u\to\infty} v(u) P(X(0) \in A_u) = 0 \tag{2.2}$$

*and a stochastic process $Z(t)$, $t \geq 0$, in $B$ and a measurable subset $A$ of $B$ such that*

$$\lim_{u\to\infty} P(X(t/v) \in A_u,\ t \in T \mid X(0) \in A_u) = P(Z(t) \in A,\ t \in T), \tag{2.3}$$

*for all finite subsets $T$ of $(0, \infty)$. Assume also that*

$$\lim_{d\to\infty} \limsup_{u\to\infty} v \int_{d/v}^{1} P(X(s) \in A_u \mid X(0) \in A_u)\, ds = 0. \tag{2.4}$$

*Define*

$$\Gamma(x) = P\left\{\int_0^\infty I_{[Z(t)\in A]}\, dt > x\right\}, \quad x \geq 0; \tag{2.5}$$

*then*

$$\lim_{u\to\infty} \frac{\int_x^\infty P(v \cdot L_t(u) > y)\, dy}{tv(u)P(X(0) \in A_u)} = \Gamma(x) \tag{2.6}$$

*for each* $x > 0$ *in the continuity set of* $\Gamma$, *for each* $0 \le t \le 1$.

PROOF. The proof is practically the same as that of [3], Theorem 3.1, where $X(t)$ was taken to be real valued, and $A_u = (u, \infty)$. The only modification requiring a comment is the demonstration of the convergence of the conditional moments of the random variable in [3], formula (3.15), whose form here is

$$v(u) \cdot L_{d/v(u)}(u) = \int_0^d I_{[X(s/v)\in A_u]}\, ds.$$

The conditional moments given $X(0) \in A_u$ are now computed in the same way as before:

$$E[(v\cdot L_{d/v})^k | X(0) \in A_u]$$

$$= \int_0^d \cdots \int_0^d P(X(s_i/v) \in A_u, i = 1, \cdots, k | X(0) \in A_u)\, ds_1 \cdots ds_k$$

$$\to \int_0^d \cdots \int_0^d P(Z(s_i) \in A, i = 1, \cdots, k)\, ds_1 \cdots ds_k,$$

where the latter limit relation follows from (2.3).

We also have the following version of [3], Corollary 3.1:

$$\lim_{u\to\infty} \frac{P(vL_t(u) > x)}{tv\cdot P(X(0) \in A_u)} = -\Gamma'(x), \quad \text{a.e. } x > 0. \tag{2.7}$$

Throughout this work we will assume that

$$\int_0^\infty I_{[Z(t)\in A]}\, dt < \infty, \quad \text{almost surely}; \tag{2.8}$$

otherwise, we would have to make obvious but cumbersome modifications.

In applications, condition (2.3) can often be verified by using the construction in the earlier version, [3], Theorem 3.1. Suppose, for each $u$, that there is a continuous function $g_u$ mapping $B$ into $B$ and which maps $A_u$ one to one onto $A$. Then the condition (2.3) may be expressed as

$$\lim_{u\to\infty} P(g_u \circ X(t/v) \in A, t \in T | g_u \circ X(0) \in A) = P(Z(t) \in A, t \in T).$$

This condition certainly holds if the conditional finite dimensional distributions of the process $g_u \circ X(t/v)$, given $g_u \circ X(0) \in A$, converge to those of the process $Z(t)$, and $A$ is a continuity set of the distribution of $Z(t)$ for each $t$.

REMARK 1. The convergence of the conditional finite dimensional distributions may, in applications, be demonstrated by proving the convergence of the conditional joint densities. For arbitrary $0 < t_1 < \cdots < t_k$ and a point $(x_0, x_1, \cdots, x_k)$, where $x_0 \in A$ and $x_i \in B$, $i = 1, \cdots, k$, let $f_k(x_0, x_1, \cdots, x_k; u)$ be the joint density of $(g_u \circ X(0), g_u \circ X(t_1/v), \cdots, g_u \circ X(t_k/v))$, restricted to the set $A \times B^k$; then $f_k/P(g_u \circ X(0) \in A)$ is a density function on the latter set. If the function converges almost everywhere on this set to a density function, then, by Scheffe's Theorem [8], the corresponding distribution function converges.

REMARK 2. The function $g_u(x) = w(u) \cdot (x - u)$ was used in [3] to map $A_u = (u, \infty)$ onto $A = (0, \infty)$.

REMARK 3. We comment on the condition $v(u) \to \infty$, and its relation to (2.4). A much simpler version of Theorem 2.1 is valid for the case where $v$ is taken to be identically equal to 1. Indeed, in this case the theorem is true without the requirement (2.4). Thus, the reader might question the role of the condition $v \to \infty$. The answer is that the latter describes the fact that the events $X(t) \in A_u$ are infrequent for large $u$: For every $t > 0$,

$$\lim_{u\to\infty} P(X(t) \in A_u \mid X(0) \in A_u) = 0.$$

If $v$ were bounded, then the condition (2.3) would allow $P(X(t) \in A_u \mid X(0) \in A_u)$ to be bounded away from 0. The infrequency of $A_u$ is used to get the particular limiting distribution of $L_t$ for $t \to \infty$ in our application to the global sojourn limit theorem. Indeed, it is an essential element of the proof of the compound Poisson limit theorem in [2] upon which Theorem 4.1 below is based.

REMARK 4. In [3] we used the earlier version of Theorem 2.1 to obtain a general lower bound on $P(\sup(X(s): 0 \le s \le t) > u)$ for large $u$ and fixed $t$. In the same way we can now obtain a bound for $P(X(s) \in A_u$ for some $0 \le s \le t)$. According to the same calculations in [3], Section 11, we have

$$\liminf_{u\to\infty} \frac{P(X(s) \in A_u, \text{ for some } 0 \le s \le t)}{tvP(X(0) \in A_u)} \ge -\Gamma'(0).$$

**3. An extension of the local sojourn limit theorem under mixing.** We now introduce a mixing condition on the events $\{X(t) \in A_u\}$ for time points $t$ which are mutually separated by sufficiently large intervals. This condition provides the asymptotic independence which is needed for the derivation of our limit theorem.

Suppose that the function $P(X(0) \in A_u)$ is continuous for all sufficiently large $u$; then (2.2) implies that the equation

$$t \cdot v(u)P(X(0) \in A_u) = 1 \tag{3.1}$$

has, for all sufficiently large $t$, a solution $u$. Let $u = u(t)$ stand for the largest solution. In what follows $u$ is understood to be a function of $t$ even though the argument is suppressed; and $v(u)$ and $P(X(0) \in A_u)$ are corresponding functions of $t$.

For any finite subset $S$, we put $\#(S)$ = cardinality of $S$; and for any pair of subsets $S_1$ and $S_2$, we put $d(S_1, S_2) = \inf(|s_1 - s_2|; s_1 \in S_1, s_2 \in S_2)$. Consider the mixing condition: *For all positive integers m and N,*

$$\lim_{t\to\infty}\sup_{\#(S_1)\le N, \#(S_2)\le N, d(S_1,S_2)\ge \frac{t}{m}} \left| \frac{P(X(s) \in A_u, s \in S_1; X(s) \in A_u, s \in S_2)}{P(X(s) \in A_u, s \in S_1)P(X(s) \in A_u, s \in S_2)} - 1 \right| = 0. \tag{3.2}$$

The condition above implies an extension from pairs of sets $S_1$ and $S_2$ to any finite collection: *For all positive integers m, N and k,*

$$\lim_{t\to\infty}\sup_{\max(\#(S_i), i=1,\dots,k)\le N, d(S_i,S_j)\ge \frac{t}{m}, i\ne j} \left| \frac{P(\cap_{i=1}^k \{X(s) \in A_u, s \in S_i\})}{\prod_{i=1}^k P(X(s) \in A_u, s \in S_i)} - 1 \right| = 0. \tag{3.3}$$

In preparation for the proof of the main result of this section, we present some elementary preliminary results.

LEMMA 3.1. *For every $x > 0$, $t > 0$,*

$$P(L_t(u) > x, X(t) \in A_u) = P(L_t(u) > x, X(0) \in A_u). \tag{3.4}$$

PROOF. The proof of [3], Theorem 3.1, was based on the identity

$$\int_x^\infty P(L_t(u) > y)\,dy = \int_0^t P(L_s(u) > x, X(s) \in A_u)\,ds. \tag{3.5}$$

The argument preceding the statement of that theorem shows that $X(s)$ on the right hand side of (3.5) can be replaced by $X(0)$; hence,

$$\int_0^t P(L_s(u) > x, X(s) \in A_u)\,ds = \int_0^t P(L_s(u) > x, X(0) \in A_u)\,ds,$$

and the claim (3.4) follows by differentiation.

LEMMA 3.2. *For every* $0 < t_1 < t_2$

$$\int_{t_1}^{t_2} P(X(s) \in A_u \,|\, X(0) \in A_u)\,ds = \int_0^{t_2-t_1} P(X(s) \in A_u \,|\, X(t_2) \in A_u)\,ds.$$

PROOF. Since $X(0)$ and $X(s)$ have the same marginal distributions, we have

$$P(X(s) \in A_u \,|\, X(0) \in A_u) = P(X(0) \in A_u \,|\, X(s) \in A_u),$$

and the latter, by stationarity, is equal to

$$P(X(t-s) \in A_u \,|\, X(t) \in A_u).$$

Thus, by integration,

$$\int_{t_1}^{t_2} P(X(s) \in A_u \,|\, X(0) \in A_u)\,ds = \int_{t_1}^{t_2} P(X(t_2 - s) \in A_u \,|\, X(t_2) \in A_u)\,ds,$$

and the latter, by a change of variable, is equal to

$$\int_0^{t_2-t_1} P(X(s) \in A_u \,|\, X(t_2) \in A_u)\,ds.$$

THEOREM 3.1. *Under the conditions of Theorem* 2.1 *and the condition* (3.2), *if* $J_i$ *is an interval of unit length,* $i = 1, \cdots, k$, *and we define*

$$L_{J_i} = \int_{J_i} I_{[X(s) \in A_u]}\,ds,$$

*then, for every* $k \geq 1$ *and* $m \geq 1$,

$\lim_{t\to\infty} \sup_{d(J_i,J_j) \geq t/m}$

$$\left| \frac{\int_{x_1}^\infty \cdots \int_{x_k}^\infty P(v \cdot L_{J_i} > y_i, i = 1, \cdots, k)\,dy_k \cdots dy_1}{v^k P^k(X(0) \in A_u)} - \prod_{i=1}^k \Gamma(x_i) \right| = 0 \tag{3.6}$$

*for all* $x_i$ *in the continuity set of* $\Gamma$, $i = 1, \cdots, k$.

PROOF. We adapt the relevant part of the argument in [3], Section 18. Put

$$L_s^i = L^i_{J_i \cap [0,s]}.$$

Then, by an extension of the reasoning behind (3.5), we have

$$\int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} P(vL_{J_i} > y_i,\ i=1,\cdots,k)\,dy_k\cdots dy_1 \tag{3.7}$$
$$= v^k \int_{J_1}\cdots\int_{J_k} P(vL^i_{s_i} > x_i,\ X(s_i)\in A_u,\ i=1,\cdots,k)\,ds_k\cdots ds_1,$$

for every set of disjoint intervals $J_1, \cdots, J_k$.

Next we show that the sojourns $L^i_s$ may be clipped to a small interval of length $d/v$ with right endpoint $s_i$, where $d$ is some fixed but large number. Indeed, by Markov's inequality, for every $\varepsilon > 0$,

$$v^k \int_{J_1}\cdots\int_{J_k} P(vL^i_{s_i-d/v} > \varepsilon,\ \text{for some}\ 1\le i\le k,$$
$$X(s_i)\in A_u,\ \text{for all}\ 1\le i\le k)\,ds_k\cdots ds_1$$
$$\le \frac{v^k}{\varepsilon}\int_{J_1}\cdots\int_{J_k} \textstyle\sum_{i=1}^k E\{v\cdot L^i_{s_i-d/v}\prod_{j=1}^k I_{[X(s_j)\in A_u]}\}\,ds_k\cdots ds_1,$$

which, by Fubini's theorem, is equal to

$$\frac{v^k}{\varepsilon}\int_{J_1}\cdots\int_{J_k}\textstyle\sum_{i=1}^k v\displaystyle\int_{J_i\cap[0,s_i-d/v]} P(X(s)\in A_u,\ X(s_j)\in A_u, \tag{3.8}$$
$$j=1,\cdots,k)\,ds\,ds_k\cdots ds_1.$$

By (3.3), if $d(J_i, J_j) \ge t/m$ for all $i \ne j$, then

$$P(X(s)\in A_u, X(s_j)\in A_u, j=1,\cdots,k) \sim P^k(X(0)\in A_u)P(X(s)\in A_u \mid X(s_i)\in A_u)$$

for $t\to\infty$ uniformly for $s\in J_i$, for $i=1,\cdots,k$. Hence, the expression (3.8) is approximately equal to

$$\varepsilon^{-1}[v\cdot P(X(0)\in A_u)]k \textstyle\sum_{i=1}^k v\displaystyle\int_{J_i}\int_{J_i\cap[0,s_i-d/v]} P(X(s)\in A_u\mid X(s_i)\in A_u)\,ds\,ds_i,$$

which, by stationarity, is at most equal to

$$k\varepsilon^{-1}[v\cdot P(X(0)\in A_u)]^k v\int_0^{1-d/v} P(X(s)\in A_u\mid X(1)\in A_u)\,ds.$$

By Lemma 3.2, the latter is equal to

$$k\varepsilon^{-1}[v\cdot P(X(0)\in A_u)]^k\, v\int_{d/v}^{1} P(X(s)\in A_u\mid X(0)\in A_u)\,ds.$$

By condition (2.4) this expression is of smaller order of magnitude than $k\varepsilon^{-1}[v\cdot P(X(0)\in A_u)]^k$ for $t\to\infty$ and then $d\to\infty$. Therefore when the members of (3.7) are divided by $[v\cdot P(X(0)\in A_u)]^k$, the right hand member is approximately equal to

$$\frac{\int_{J_1}\cdots\int_{J_k} P(v\cdot(L^i_{s_i} - L^i_{s_i-d/v}) > x_i,\ X(s_i)\in A_u,\ i=1,\cdots,k)\,ds_k\cdots ds_1}{P^k(X(0)\in A_u)} \tag{3.9}$$

because $\varepsilon > 0$ is arbitrary.

In order to complete the proof we will show that the expression (3.9) converges to $\prod_{i=1}^k \Gamma(x_i)$ under the uniform limiting operation indicated in (3.6). First we note that the expression (3.9) may be written as the product of the factor

$$\frac{P(X(s_i) \in A_u,\ i = 1, \cdots, k)}{P^k(X(0) \in A_u)}$$

which, by (3.3), converges to 1 for $t \to \infty$, and the factor

$$\int_{J_1} \cdots \int_{J_k} P\{v\cdot(L^i_{s_i} - L^i_{s_i - d/v}) > x_i,\ i = 1, \cdots, k \mid X(s_i) \in A_u,\ i = 1, \cdots, k\}\, ds_k \cdots ds_1. \tag{3.10}$$

Next we will show that the integrand in (3.10) converges to $\prod_{i=1}^k \Gamma(x_i)$ for $t \to \infty$ and then $d \to \infty$. For this purpose it suffices to show that the random variables $v\cdot\ (L^i_{s_i} - L^i_{s_i - d/v})$, $i = 1, \cdots k$ are conditionally, given $X(s_i) \in A_u$, $i = 1, \cdots, k$, asymptotically independent, so that the multiple integral (3.10) factors into a product of single integrals to which Condition (3.3) and Theorem 2.1 are applied.

For arbitrary positive integers $p_1, \cdots, p_k$ let us estimate

$$E\{\textstyle\prod_{i=1}^k (v\cdot L^i_{s_i} - L^i_{s_i - d/v})^{p_i} \mid X(s_i) \in A_u,\ i = 1, \cdots, k\}. \tag{3.11}$$

Expand each factor in the product as a $p_i$-fold multiple integral:

$$(v\cdot L^i_{s_i} - L^i_{s_i - d/v})^{p_i} = \int_{s_i - d/v}^{s_i} \cdots \int_{s_i - d/v}^{s_i} I_{[X(\sigma_{ih}) \in A_u, h=1,\cdots p_i]}\, d\sigma_{i1} \cdots d\sigma_{ip_i},$$

and then write the expected value of the product over $i = 1, \cdots, k$ of these integrals as a multiple integral of

$$P(X(\sigma_{ih}) \in A_u,\ h = 1, \cdots, p_i,\ i = 1, \cdots, k \mid X(s_i) \in A_u,\ i = 1, \cdots, k).$$

By (3.3) this is asymptotically equal to

$$\textstyle\prod_{i=1}^k P(X(\sigma_{ih}) \in A_u,\ h = 1, \cdots, p_i \mid X(s_i) \in A_u).$$

Integrating this over the intervals $[s_i - d/v,\ s_i]$ we obtain the product of moments

$$\textstyle\prod_{i=1}^k E[(v\cdot L^i_{s_i} - L^i_{s_i - d/v})^{p_i} \mid X(s_i) \in A_u],$$

which, by (3.4), is equal to

$$\textstyle\prod_{i=1}^k E[(v\cdot L^i_{s_i} - L^i_{s_i - d/v})^{p_i} \mid X(s_i - d/v) \in A_u],$$

which, by stationarity and the moment computation in the proof of Theorem 2.1, converges, for $t \to \infty$, to

$$\textstyle\prod_{i=1}^k E\left(\int_0^d I_{[Z(t) \in A]}\, dt\right)^{p_i}.$$

Therefore, the random variables $v\cdot L^i_{s_i} - L_{s_i - d/v}$ in (3.10) are conditionally asymptotically independent, and so, by the comments following (3.10), the latter integral converges to $\prod_{i=1}^k \Gamma(x_i)$. Here the application of the moment convergence theorem is justified by the net convergence theory given in [3].

**4. The global sojourn limit theorem.** In this section we prove our main result, namely, that the random variable $v(u)\cdot L_t(u)$ has, for $t \to \infty$, a limiting distribution of an explicit form. The proof follows the plan of the corresponding proof in [3], Sections 16–18. The random variable $vL_t$ is represented as the sum of nonnegative random variables from a stationary array. Suppose for simplicity as in [3] that $t$ assumes only positive integer values $n$. (The general case follows by a simple extension). Put

$$X_{n,j} = v\cdot(L_j(u) - L_{j-1}(u)), \quad j = 1, \cdots, n; \tag{4.1}$$

then

$$vL_t(u) = \textstyle\sum_{j=1}^{n} X_{n,j}. \tag{4.2}$$

For the convenience of the reader, we quote the "compound Poisson limit theorem" for stationary sums. The following four assumptions are made:

4.I There is a nonincreasing function $H(x)$ such that

$$\lim_{x\to\infty} H(x) = 0, \quad \int_0^\infty x\, dH(x) > -\infty, \tag{4.3}$$

$$\lim_{n\to\infty} n \int_0^y x\, dP(X_{n,1} > x) = \int_0^y x\, dH(x) \tag{4.4}$$

at every continuity point $y$ of the limiting function; furthermore, the relation above also holds for $y = \infty$:

$$\lim_{n\to\infty} nEX_{n,1} = -\int_0^\infty x\, dH(x). \tag{4.5}$$

4.II. The "local mixing" condition holds:

$$\lim_{k\to\infty} \limsup_{n\to\infty} [k \textstyle\sum_{1\le i<j\le[n/k]} EX_{n,i}X_{n,j}] = 0. \tag{4.6}$$

4.III. The following "global mixing" condition holds for the $k$-dimensional joint distributions for each fixed $k \ge 2$, and each $q$, $0 < q < 1$:

$$\lim_{n\to\infty} \sup_{1\le j_1<\cdots<j_k\le n, \min(j_{h+1}-j_h, 1<h<k)>qn} \left| \frac{P(X_{n,j_1} > x_1, \cdots, X_{n,j_k} > x_k)}{P(X_{n,j_1} > x_1) \cdots P(X_{n,j_k} > x_k)} - 1 \right| = 0, \tag{4.7}$$

for every $k$-tuple of $x$'s such that $H(x_i) > 0$ and $x_i$ is a point of continuity of $H$, $i = 1, \cdots, k$.

4.IV. The following global mixing condition holds for the joint moments of order 1, for each $k \ge 2$, each $h$, $1 < h < k$, and each $q$, $0 < q < 1$:

$$\lim_{n\to\infty} \sup_{1\le j_1<\cdots<j_k\le n, j_h-j_{h-1}>qn} \left| \frac{EX_{n,j_1}\cdots X_{n,j_k}}{(EX_{n,j_1}\cdots X_{n,j_{h-1}})E(X_{n,j_h}\cdots X_{n,j_k})} - 1 \right| = 0. \tag{4.8}$$

The following is the result of [2]:

THEOREM 4.A. *Under Assumptions* 4.I *through* 4.IV *above, the distribution of* $\sum_{j=1}^{n} X_{n,j}$ *converges, for* $n \to \infty$ *to the distribution with the Laplace-Stieltjes transform,*

$$\Omega(s) = \exp\left[\int_0^\infty (1 - e^{-sx})\, dH(x)\right]. \tag{4.9}$$

Now we introduce a new local mixing condition for the stationary process $X$. The index

$u$ is again assumed to be related to $t$ through the equation (3.1). *For every* $\delta > 0$,

$$\lim_{k\to\infty} \limsup_{t\to\infty} \frac{1}{t} \int_{\delta}^{t/k} \frac{P(X(0) \in A_u, X(s) \in A_u)}{P^2(X(0) \in A_u)}\, ds = 0. \tag{4.10}$$

Our main result is:

THEOREM 4.1. *If* $X(t)$, $t \geq 0$, *satisfies the conditions of Theorem* 2.1, *and also* (3.2) *and* (4.10), *then* $v \cdot L_t(u)$ *has, for* $t \to \infty$, *a limiting distribution with the Laplace-Stieltjes transform* (4.9) *with* $H(x) = -\Gamma'(x)$.

PROOF. The outline of the proof is similar to that in [3]. We show that the assumptions of the theorem imply that random variables $X_{n,j}$ in (4.1) satisfy the four conditions of Theorem 4.A.

The conditions of Theorem 2.1 imply that the array $\{X_{n,j}\}$ satisfies Assumption I of Theorem 4.A. The proof is exactly the same as that of [3], Lemma 17.1.

Theorem 3.1 implies that the array satisfies (4.7); indeed, this follows from the multivariate version of [3], Lemma 2.1. Therefore, the validity of Assumption 4.III for the array is a consequence of Theorem 2.1 and the mixing condition (3.2).

The mixing condition (3.2) also implies that the array satisfies the condition of Assumption 4.IV. Indeed,

$$EX_{n,i_1} \cdots X_{n,i_k} = v^k \int_{i_1-1}^{i_1} \cdots \int_{i_k-1}^{i_k} P(X(s_i) \in A_u,\, i = 1, \cdots, k)\, ds_k \cdots ds_1,$$

and so the asymptotic factorization in (3.2) implies the corresponding factorization (4.8).

We have shown that the array satisfies the conditions of Assumptions 4.I, 4.III and 4.IV. It remains to be shown that Assumption 4.II holds. Since the local mixing condition (4.10) is fundamentally different from the corresponding one in [3], we will give the full proof. (Comments about the earlier condition are given in Section 9.)

The condition (4.10) does not imply that the random variables $X_{n,j}$ satisfy 4.II; however, it implies that a suitably modified version of these random variables satisfies 4.II. Then we show that Theorem 4.A holds for modified array, and, finally that the distribution of the sum of the modified array is close to that of the sum of the original array.

For arbitrary $\delta$, $0 < \delta < 1$, we modify the random variable $X_{n,j}$ in (4.1) by removing a subinterval of length $\delta$ from the right end of the interval $[j-1, j]$, and define

$$X'_{n,j} = v \cdot (L_{j-\delta} - L_{j-1}).$$

It follows from (3.1) that

$$E\left| \textstyle\sum_{j=1}^{n} X_{n,j} - \sum_{j=1}^{n} X'_{n,j} \right| = vt\delta P(X(0) \in A_u) = \delta. \tag{4.11}$$

If (4.10) is assumed, then (4.6) holds for the array $\{X'_{n,j}\}$. Indeed, the stationarity of the array implies

$$k \textstyle\sum_{i,j=1, i\neq j}^{[n/k]} EX'_{n,i}X'_{n,j} \leq n \sum_{j=2}^{[n/k]} EX'_{n,1}X'_{n,j},$$

where the latter, by Fubini's theorem, is

$$nv^2 \textstyle\sum_{j=2}^{[n/k]} \displaystyle\int_0^{1-\delta} \int_{j-1}^{j-\delta} P(X(s) \in A_u, X(s') \in A_u)\, ds\, ds',$$

which is at most equal to

$$nv^2 \int_0^{1-\delta} \int_1^{[n/k]} P(X(s) \in A_u, X(s') \in A_u)\, ds\, ds'.$$

By stationarity, this is equal to

$$nv^2 \int_0^{1-\delta} \int_1^{[n/k]} P(X(s-s') \in A_u, X(0) \in A_u)\, ds\, ds'$$

which, by a change of variable of integration, becomes

$$nv^2 \int_0^{1-\delta} \int_{1-s'}^{[n/k]-s'} P(X(s) \in A_u, X(0) \in A_u)\, ds\, ds'.$$

The latter is at most equal to

$$nv^2 \int_\delta^{n/k} P(X(s) \in A_u, X(0) \in A_u)\, ds,$$

which, by the identification of $n$ as $t$ and the relation (3.1), is equal to the expression under the limit sign in (4.10). Thus we have shown that the array $\{X'_{n,j}\}$ satisfies (4.6).

Now the conditions on $X$ assumed in our Theorem imply not only the validity of Assumptions 4.I, 4.III, and 4.IV for the array $\{X_{n,j}\}$, but also their validity for the array $\{X'_{n,j}\}$. The only modification is that $-\Gamma'(x)$ is replaced by $-(1-\delta)\Gamma'(x)$. These assertions can be verified by checking the proofs given above for the array $\{X_{n,j}\}$. In order to complete the proof of the theorem, we note that we have just demonstrated that the array $\{X'_{n,j}\}$ also satisfies the condition of 4.II, for every $\delta > 0$. Therefore, by Theorem 4.A, $\sum X'_{n,j}$ has a limiting distribution with the transform

$$\Omega(s) = \exp\left\{-(1-\delta) \int_0^\infty (1-e^{-sx})\, d\Gamma'(x)\right\}.$$

Furthermore, by (4.11), the $L_1$-distance between the sums $\sum X_{n,j}$ and $\sum X'_{n,j}$ is equal to $\delta$ for all $n$. Since $\delta$ is arbitrary, it follows that $\sum X_{n,j}$ also has a limiting distribution, and that the limiting transform is given by (4.9) with $H(x) = -\Gamma'(x)$.

REMARK. $u$ was defined as a function of $t$ through (3.1). The proof of Theorem 4.1 requires only that $u = u(t)$ satisfy

$$\lim_{t\to\infty} t \cdot v(u) P(X(0) \in A_u) = 1, \tag{4.12}$$

so that our results are more generally valid.

**5. Application to a Markov process.** Let $X(t)$, $t \geq 0$, be a stationary Markov process in $R$ having a transition density with respect to Lebesgue measure. Let the density be of the form $p(t; x, y)$, representing the conditional density of $X(t)$ at $y$, given $X(0) = x$. Suppose also that the stationary marginal distribution has a density $f(x)$.

The hypothesis of the following theorem is stated in terms of the family of mappings $(g_u)$ discussed in Section 2.1.

THEOREM 5.1. *Let the sets $A$ and $A_u$, and the function $g_u(x)$ be defined as in Section 2.1. Let $(g_u^{-1}(x))'$ be the modulus of Jacobian of $g_u^{-1}$. If there is a density function $h(x)$ with support in $A$, a Markov transition density $q(t; x, y)$ such that*

$$\lim_{u\to\infty} \frac{f(g_u^{-1}(x))(g_u^{-1}(x))'}{\int_A f(g_u^{-1}(y))(g_u^{-1}(y))'\, dy} = h(x) \tag{5.1}$$

*almost everywhere on $A$, and a function $v = v(u)$ such that*

$$\lim_{u\to\infty} p(t/v; g_u^{-1}(x), g_u^{-1}(y))(g_u^{-1}(y))' = q(t; x, y), \tag{5.2}$$

*for almost all x, y, for each $t \geq 0$, then Condition* (2.3) *of Theorem* 2.1 *holds. More exactly, there is a Markov process $Z(t)$ in $B$ with the initial distribution over $A$ with density $h$, and with transition density function $q$ such that* (2.3) *holds.*

PROOF. According to Remark 1 of Section 2, it suffices to prove the convergence of the conditional finite dimensional densities of the process $g_u \circ X(t/v)$, given $g_u \circ X(0) \in A$. For arbitrary $0 < t_1 \cdots < t_k$ and $x_0$ in $A$ and $(x_1, \cdots, x_k)$ in $B^k$, the conditional joint density of $(g_u \circ X(0), g_u \circ X(t_1/v), \cdots, g_u \circ X(t_k/v))$ at the point $(x_0, x_1, \cdots, x_k)$ in $A \times B^k$, given $g_u \circ X(0) \in A$ is, by the Markov property, equal to the product of

$$\frac{f(g_u^{-1}(x_0))(g_u^{-1}(x_0))'}{\int_A f(g_u^{-1}(y))(g_u^{-1}(y))'\,dy} \tag{5.3}$$

and

$$\textstyle\prod_{i=1}^k p\left(\frac{t_i - t_{i-1}}{v}; g_u^{-1}(x_{i-1}), g_u^{-1}(x_i)\right)(g_u^{-1}(x_i))', \tag{5.4}$$

where $t_0 = 0$. According to (5.1) and (5.2), the factors (5.3) and (5.4) converge to the limits $h(x_0)$ and

$$\textstyle\prod_{i=1}^k q(t_i - t_{i-1}; x_{i-1}, x_i).$$

This proves that the conditional joint density converges to the joint density of the Markov process $Z(t)$ with the indicated stationary distribution and transition density.

THEOREM 5.2. *If the Markov process satisfies the condition*

$$\lim_{t\to\infty}\sup_{x,y\in A_u, s\geq t/m}\left|\frac{p(s; x, y)}{f(y)} - 1\right| = 0 \tag{5.5}$$

*for every $m \geq 1$, then it necessarily satisfies condition* (3.2).

PROOF. For arbitrary sets $S_1$ and $S_2$ of cardinality at most $N$, with $s < s'$ for all $s \in S_1$, $s' \in S_2$, put $s_1 = \max S_1$ and $s_2 = \min S_2$. By the Markov property, the probability

$$P(X(s) \in A_u, s \in S_1 \cup S_2)$$

is equal to

$$\int_{A_u}\int_{A_u} P(X(s) \in A_u, s \in S_1 - \{s_1\} \mid X(s_1) = x, X(s_2) = y)$$
$$\cdot P(X(s) \in A_u, s \in S_2 - \{s_2\} \mid X(s_1) = x, X(s_2) = y)$$
$$f(x)p(s_2 - s_1; x, y)\,dx\,dy.$$

According to condition (5.5), if $s_2 - s_1 \geq t/m$ and $t$ is sufficiently large, then the function $p(s_2 - s_1; x, y)$ in the integrand above may be replaced by $f(y)$. The resulting integral factors into the product $P(X(s) \in A_u, s \in S_1)P(X(s) \in A_u, s \in S_2)$.

Theorems 5.1 and 5.2 are useful for the Markov process because the conditions (2.3) and (3.2), which are stated in terms of finite dimensional distributions of arbitrary order, are replaced by conditions involving only the marginal density and the transition density. The conditions (2.4) and (4.10) are stated in terms of the bivariate distributions alone, and these can be expressed simply in terms of the marginal and transition densities. The results of this section can be applied not only to a Markov process but also to one which is embeddable in a Markov process because the dimension of $B$ is arbitrary. Indeed, suppose that $(X(t), Y(t))$ is a Markov process in $B \times B$, and that $(A_u)$ is rare for $X(t)$. Then $(A_u \times B)$ is rare for $(X(t), Y(t))$, and the sojourns of $X(t)$ in $A_u$ are identical with the sojourns of $(X(t), Y(t))$ in $A_u \times B$. Thus the distribution results for the sojourns of the Markov process $(X(t), Y(t))$ are equally valid for the non-Markov process $X(t)$. However, the conditions of the theorems have to be shown to hold for the augmented Markov process.

Finally we remark that Theorem 5.1 is better than [3], Theorem 9.1 even in the particular case $A_u = (u, \infty)$ because the hypothesis of reversibility has been dropped.

**6. The sojourn of a several dimensional Gaussian Markov process in a small cube.** Let $X_1(t)$, $t \geq 0$, be a real Gaussian process with mean 0 and covariance function $EX_1(s)X_1(t) = e^{-|t-s|}$. This is the well known Ornstein-Uhlenbeck process: It is Markovian with the standard normal stationary density $\phi(x)$ and the transition density

$$\frac{1}{(1-e^{-2t})^{1/2}}\phi\left(\frac{y-xe^{-t}}{(1-e^{-2t})^{1/2}}\right).$$

Let $X(t)$ be the vector process obtained by taking $M$ independent copies of $X_1$:

$$X(t) = (X_1(t), \cdots, X_M(t)).$$

For $u > 0$, define the $M$-dimensional cube $A_u = [-u^{-1}, u^{-1}]^M$. If $x$ and $y$ are vectors with components $x_i$ and $y_i$, $i = 1, \cdots, M$, respectively, then $X(t)$ has the stationary density

$$f(x) = \textstyle\prod_{i=1}^M \phi(x_i) \tag{6.1}$$

and the transition density

$$p(t; x, y) = (1-e^{-2t})^{-M/2} \textstyle\prod_{i=1}^M \phi\left(\frac{y_i - x_i e^{-t}}{(1-e^{-2t})^{1/2}}\right). \tag{6.2}$$

Define

$$v(u) = u^2, \tag{6.3}$$

and put $A = [-1, 1]^M$ and $g_u(x) = ux$, where $x$ is an $M$-component vector and $u > 0$.

THEOREM 6.1. *If $M \geq 3$, then $X(t)$ satisfies the conditions in the hypothesis of Theorem 2.1, and where $Z(t)$ is a Brownian motion in $R^M$ with variance parameter 2, $A$ is the cube $[-1, 1]^M$, and $Z(0)$ is uniformly distributed on $A$.*

PROOF. By the elementary properties of jointly normally distributed random variables, we have

$$E(uX_1(t/u^2) \mid uX_1(0) = x) = xe^{-t/u^2} \to x,$$
$$\mathrm{Var}(uX_1(t/u^2) \mid uX_1(0) = x) = u^2(1-e^{-2t/u^2}) \to 2t,$$
$$\mathrm{Var}(u[X(t/u^2) - X(s/u^2)] \mid uX_1(0) = x) \to 2|t-s|.$$

Furthermore, it is easily seen that the conditional density of $uX_1(0)$, given $u|X_1(0)| \leq 1$, converges to the uniform density on $[-1, 1]$. Therefore, by the independence of the components $X_i(t)$, $i = 1, \cdots, M$, conditions (5.1) and (5.2) hold with $h$ uniform on $A$, and

$$q(t; x, y) = \textstyle\prod_{i=1}^M (2t)^{-1/2} \phi\left(\frac{y_i - x_i}{(2t)^{1/2}}\right). \tag{6.4}$$

Next we verify (2.4). The expression following the limit sign assumes the form

$$u^2 \int_{d/u^2}^{1} \left\{\frac{P(|X_1(0)| \leq u^{-1}, |X_1(s)| \leq u^{-1})}{P(|X_1(0)| \leq u^{-1})}\right\}^M ds. \tag{6.5}$$

By the form of the bivariate normal density the numerator within the braces above is at most equal to

$$[(\pi/2)u^2(1-e^{-2s})^{1/2}]^{-1}$$

and the denominator is, for large $u$, asymptotically equal to $(2/\pi)^{1/2}/u$. Therefore, the expression (6.5) is asymptotically at most equal to

$$\left(\frac{2}{\pi}\right)^{M/2} u^2 \int_{d/u^2}^{1} [u^2(1-e^{-2s})]^{-M/2}\, ds,$$

which, after the change of variable $s = t/u^2$, is seen to converge for $u \to \infty$, to

$$\left(\frac{2}{\pi}\right)^{M/2} \int_{d}^{\infty} (2t)^{-M/2}\, dt.$$

This is finite for $M \geq 3$, and converges to 0 for $d \to \infty$. This completes the proof.

Theorem 2.1 implies the following result for $X(t)$: For every $x > 0$, and $M \geq 3$,

$$\lim_{u\to\infty} \frac{P(u^2\text{mes}\{s: 0 \leq s \leq t, |X_i(s)| \leq u^{-1}, i = 1, \cdots, M\} > x)}{u^{2-M}(2/\pi)^{M/2}} \tag{6.6}$$
$$= -\frac{d}{dx} P(\text{mes}\{s: s \geq 0, Z(s) \in [-1, 1]^M\} > x).$$

Although $X$ does not have a local time, the formula above gives an asymptotic estimate of the time spent in the cube.

Next we consider the verification of the conditions of Theorem 4.1. First we choose the explicit form of a function $u(t)$ which satisfies (4.12), namely,

$$u(t) = (2/\pi)^{(M/2(M-2))} t^{1/(M-2)}. \tag{6.7}$$

THEOREM 6.2. *If* $u(t)$ *is defined by* (6.7), *and* $M \geq 3$, *then the conditions of Theorem* 4.1 *are satisfied.*

PROOF. Let us first verify (4.10). By the same analysis as that for (6.5), we find that the expression following the limit sign in (4.10) is asymptotically at most equal to

$$\frac{1}{t} \int_{\delta}^{t/k} (1-e^{-2s})^{-M/2}\, ds,$$

which converges to $1/k$ for $t \to \infty$. This confirms the condition (4.10).

Next we verify (5.5). The ratio $p(s; x, y)/f(y)$ is identical with the ratio of the joint density of $X(0)$ and $X(s)$ to the product of the marginal densities, and so takes the form of the product of functions

$$(1-e^{-2s})^{-1/2}\exp\left\{-\frac{x_i^2 e^{-2s} - 2x_i y_i e^{-s} + y_i^2 e^{-2s}}{2(1-e^{-2s})}\right\} i = 1, \cdots, M.$$

Each of these factors converges to 1 for $s \to \infty$, uniformly for all $x_i$ and $y_i$ in compact sets, and so (5.5) holds.

**7. Application to Gaussian processes: necessary conditions.** Let $X(t)$, $t \geq 0$, be a real stationary Gaussian process with mean 0, variance 1 and continuous covariance function $r(t)$. In this section we show that the conditions of Theorems 2.1 and 4.1 imply corresponding conditions on $r(t)$. Then we show that in several cases the latter conditions are actually sufficient for the conclusions of the corresponding theorems.

Let $\phi(x, y; \rho)$ be the standard bivariate normal density function with correlation coefficient $\rho$; then $\phi$ satisfies the well known differential equation,

$$\frac{\partial\phi}{\partial\rho} = \frac{\partial^2\phi}{\partial x \partial y}. \tag{7.1}$$

It is also well known that

$$\phi(x, y; 0) = \phi(x)\phi(y),$$

where $\phi(x)$ is the standard normal density. Thus, the fundamental theorem of calculus, together with (7.1), implies

$$\phi(x, y; \rho) - \phi(x)\phi(y) = \int_0^\rho \frac{\partial^2\phi}{\partial x \partial y}(x, y; z)\, dz. \tag{7.2}$$

It is also well known that

$$\phi(x, y; \rho) \to \delta(x - y)\phi(x), \quad \text{for} \quad \rho \to 1,$$

where $\delta$ is the delta function at 0; hence, we have the formal relation

$$\delta(x - y)\phi(x) - \phi(x, y; \rho) = \int_\rho^1 \frac{\partial^2\phi}{\partial x \partial y}(x, y; z)\, dz. \tag{7.3}$$

Define

$$\Psi_u(\rho) = \int_{A_u}\int_{A_u} \frac{\partial^2\phi}{\partial x \partial y}(x, y; \rho)\, dx\, dy. \tag{7.4}$$

By integration of each member of (7.2) over $A_u \times A_u$, we obtain

$$P(X(0) \in A_u, X(s) \in A_u) - P^2(X(0) \in A_u) = \int_0^{r(s)} \Psi_u(z)\, dz. \tag{7.5}$$

Similarly, integration in (7.3) yields

$$P(X(0) \in A_u) - P(X(0) \in A_u, X(s) \in A_u) = \int_{r(s)}^1 \Psi_u(z)\, dz. \tag{7.6}$$

The latter implies

$$P(X(s) \in A_u \mid X(0) \in A_u) = 1 - \frac{\int_{r(s)}^1 \Psi_u(z)\, dz}{P(X(0) \in A_u)}.$$

If in the relation above we put $s = t/v(u)$, and fix $t$, and let $u \to \infty$, then condition (2.3) requires

$$1 - \frac{\int_{r(t/v)}^1 \Psi_u(z)\, dz}{P(X(0) \in A_u)} \to P(Z(t) \in A),$$

or equivalently,

$$\lim_{u\to\infty} \frac{\int_{r(t/v)}^1 \Psi_u(z)\, dz}{\int_{A_u} \phi(x)\, dx} = P(Z(t) \notin A), \tag{7.7}$$

for all $t \geq 0$.

Formula (7.5) and the assumption (2.2) imply that the condition (2.4) for Gaussian processes is equivalent to

$$\lim_{d\to\infty}\limsup_{u\to\infty}\frac{v\displaystyle\int_{d/v}^{1}\int_{0}^{r(s)}\Psi_u(z)\,dz\,ds}{\displaystyle\int_{A_u}\phi(x)\,dx}=0, \tag{7.8}$$

and that condition (4.10) is equivalent to

$$\lim_{k\to\infty}\limsup_{t\to\infty}\frac{\displaystyle\int_{\delta}^{t/k}\int_{0}^{r(s)}\Psi_u(z)\,dz\,ds}{t\left(\displaystyle\int_{A_u}\phi(x)\,dx\right)^2}=0. \tag{7.9}$$

The condition (3.2), with $S_1 = \{0\}$ and $S_2 = \{t/m\}$, implies

$$\lim_{t\to\infty}\frac{P(X(0)\in A_u,\ X(t/m)\in A_u)}{P^2(X(0)\in A_u)}=1.$$

By (7.5), this is equivalent to

$$\lim_{t\to\infty}\frac{\displaystyle\int_{0}^{r(t/m)}\Psi_u(z)\,dz}{\left(\displaystyle\int_{A_u}\phi(x)\,dx\right)^2}=0. \tag{7.10}$$

EXAMPLE 7.1. Suppose $A_u = (u, \infty)$; then $\Psi_u(z) = \phi(u, u; z)$, and the latter, by the definition, is equal to

$$\frac{1}{2\pi(1-z^2)^{1/2}}\exp\left(-\frac{u^2}{1+z}\right). \tag{7.11}$$

We have the well known relation

$$\int_u^\infty \phi(x)\,dx \sim \frac{\phi(u)}{u},\quad \text{for}\quad u\to\infty. \tag{7.12}$$

Under the mapping $g_u(x) = u(x-u)$, we have $A = (0, \infty)$. The condition (7.7) becomes

$$\lim_{u\to\infty}\frac{\displaystyle\int_{r(t/v)}^{1}(1-z^2)^{-1/2}\exp\left(-\frac{u^2}{1+z}\right)dz}{\sqrt{2\pi}\,\exp(-u^2/2)/u}=P(Z(t)\le 0).$$

After the change of variable $y = u^2(1-z)$, this relation becomes

$$\lim_{u\to\infty}\frac{1}{2\sqrt{\pi}}\int_0^{u^2(1-r(t/v))}y^{-1/2}e^{-y/4}\,dy=P(Z(t)\le 0). \tag{7.13}$$

This is equivalent to the existence of

$$\lim_{u\to\infty}u^2(1-r(t/v)) \tag{7.14}$$

for every $t \ge 0$. If the latter is continuous, and is not identically equal to 0, then $1 - r(t)$ is necessarily regularly varying of index $\alpha > 0$. Since $r$ is a covariance function, it is required also that $\alpha \le 2$.

Let $v = v(u)$ be any function satisfying

$$\lim_{u\to\infty} u^2(1 - r(1/v(u))) = 1; \tag{7.15}$$

then, as is shown in [2], (2.3) holds, and $Z(t)$ is of the form $U(t) - t^\alpha$ where $U(t)$ is Gaussian with $EU(t) \equiv 0$, $EU^2(0) = 0$ and $E(U(t) - U(s))^2 = 2\,|\,t - s\,|^\alpha$.

By calculations similar to those in [3], Section 7, and to those leading to (7.13) above, it can be shown that condition (7.8) is satisfied.

If $u$ is defined by (3.1) or (4.12), then as in [2],

$$u^2 \sim 2 \log t, \quad \text{for} \quad t \to \infty. \tag{7.16}$$

The ratio in (7.10) is asymptotic to

$$u^2 \int_0^{r(t/m)} (1 - z^2)^{-1/2} \exp\left[\frac{u^2 z}{1 + z}\right] dz,$$

which, by the substitution $y = u^2 z$, is equal to

$$\int_0^{u^2 r(t/m)} (1 - yu^{-2}) \exp\left[y \Big/ \left(1 + \frac{y}{u^2}\right)\right] dy.$$

It can be shown by elementary arguments that this has the limit 0 if and only if $u^2 r(t/m) \to 0$. By (7.16), the latter is equivalent to

$$\lim_{t\to\infty} r(t) \log t = 0. \tag{7.17}$$

As shown in [2], this is a sufficient mixing condition for the conclusion of Theorem 4.1.

The condition (7.17) also implies (7.9). As in the preceding calculation, the ratio in (7.9) is asymptotic to

$$\frac{1}{t} \int_\delta^{t/k} u^2 \int_0^{r(s)} (1 - z^2)^{-1/2} \exp\left[\frac{u^2 z}{1 + z}\right] dz\, ds. \tag{7.18}$$

For every $\delta > 0$, $r(s)$ is bounded away from 1 on $s \geq \delta$; this is a familiar property of nonperiodic correlation functions. Therefore, the factor $(1 - z^2)^{-1/2}$ in the integrand in (7.18) is bounded away from 0 and $\infty$; thus, in proving that (7.18) converges to 0, it is sufficient to consider the integral without that factor:

$$\frac{1}{t} \int_\delta^{t/k} u^2 \int_0^{r(s)} \exp\left[\frac{u^2 z}{1 + z}\right] dz\, ds. \tag{7.19}$$

For every $\varepsilon > 0$, the set $\{s : |\,r(s)\,| > \varepsilon\}$ is bounded. Furthermore $z/(1 + z)$ is strictly less than ½ if $z$ is bounded away from 1. Therefore, the portion of the integral in (7.18) over the set $\{s : |\,r(s)\,| > \varepsilon\}$ is at most equal to a constant multiple of

$$u^2 e^{u^2 \theta},$$

for some $\theta$, $0 < \theta < ½$, which, by (7.16), is $o(t)$ for $t \to \infty$. Therefore, the corresponding portion of (7.19), after division by $t$, converges to 0. Thus, in estimating (7.19), it suffices to assume that $|\,r(s)\,| \leq \varepsilon$, for arbitrary $\varepsilon > 0$.

Now let us estimate the portion of (7.19) corresponding to the domain of integration $0 \leq s \leq t^c$, where the number $0 < c < 1$ will be specified below. This expression is at most equal to

$$t^c \frac{u^2}{t} \varepsilon \exp\left[\frac{u^2 \varepsilon}{1 + \varepsilon}\right],$$

which, by (7.16), is of the order

$$u^2 t^{2\varepsilon/(1+\varepsilon)+c-1},$$

which tends to 0 if $c < (1 - \varepsilon)/(1 + \varepsilon)$.

Finally, we estimate the portion of (7.19) corresponding to the domain of integration $t^c \le s \le t/k$,

$$\frac{1}{t}\int_{t^c}^{t/k} u^2 \int_0^{r(s)} \exp\left[\frac{u^2 z}{1+z}\right] dz\, ds,$$

which, since $|r(s)| \le \varepsilon$, is at most

$$\frac{1}{t}\int_{t^c}^{t/k} u^2 \int_0^{|r(s)|} e^{u^2 z(1-\varepsilon)^{-1}}\, dz\, ds = (1-\varepsilon)\frac{1}{t}\int_{t^c}^{t/k} (e^{u^2|r(s)|(1-\varepsilon)^{-1}} - 1)\, ds.$$

The latter tends to 0 for $t \to \infty$ under (7.17).

EXAMPLE 7.2. For an integer $M > 0$, consider a vector stationary Gaussian process $X(t)$ whose components $X_i(t)$, $i = 1, \cdots, M$, are independent copies of a real stationary process with covariance function $r(t)$. This is a generalization of the process considered in Section 6. Let $A_u$ be the $M$-dimensional cube defined there; and with $A$ and $g_u$ similarly defined. Put

$$\Psi_u(z) = 2[\phi(u^{-1}, u^{-1}; z) - \phi(u^{-1}, u^{-1}, -z)]$$
$$= \frac{1}{\pi}(1-z^2)^{-1/2}\left[\exp\left(\frac{-u^{-2}}{1+z}\right) - \exp\left(\frac{-u^{-2}}{1-z}\right)\right].$$

By definition,

$$\text{(7.20)} \qquad P(X(0) \in A_u) = P^M(|X_1(0)| \le u^{-1}),$$
$$P(X(0) \in A_u, X(s) \in A_u) = P^M(|X_1(0)| \le u^{-1}, |X_1(s)| \le u^{-1});$$

hence, by the reasoning leading to (7.7), the condition (2.3) requires that (7.7) holds in the form

$$\sqrt{2\pi}\lim_{u\to\infty} u \int_{r(t/v)}^{1} (\phi(u^{-1}, u^{-1}; z) - \phi(u^{-1}, u^{-1}; -z))\, dz = P(Z(t) \notin A).$$

By the change of variable $y = u^2(1-z)$, the relation above is seen to be equivalent to

$$\sqrt{2\pi}\lim_{u\to\infty} \int_0^{u^2(1-r(t/v))} (2y)^{-1/2}(1 - e^{-1/y})\, dy = P(Z(t) \notin A).$$

The limit above exists if and only if (7.14) exists, which by the reasoning in Example 7.1 above, implies that $1 - r(t)$ is regularly varying, and that $v(u)$ satisfies (7.15). It also follows, that this condition is sufficient for (2.3) and that the process $Z(t)$ is identical with the Gaussian process $U(t)$ defined above; the reasoning is nearly the same as that for the corresponding conclusion in Example 7.1.

Next we show that if (7.15) holds, then condition (2.4) holds for $M > 2/\alpha$. The reasoning is the same as that following (6.5) except that the general covariance $r(t)$ is used in the place of the Markov covariance $e^{-|t|}$. The expression following the limit sign in (2.4) assumes the form (6.5) with $v$ in place of $u^2$, and is asymptotically at most equal to

$$\left(\frac{2}{\pi}\right)^{M/2} v \int_{d/v}^{1} [u^2(1-r^2(s))]^{-M/2}\, ds.$$

After the change of variable $s = t/v$, and the application of the regular variation of $1 - r(s)$, it is seen that the expression above converges for $u \to \infty$ to

$$\left(\frac{1}{\pi}\right)^{M/2} \int_d^{\infty} t^{-\alpha M/2}\, dt,$$

which converges to 0 for $d \to \infty$, for $M > 2/\alpha$.

The condition (4.12) in this example becomes

$$t \cdot v(u)\left(\frac{2}{\pi u^2}\right)^{M/2} \to 1, \quad \text{for} \quad t \to \infty. \tag{7.21}$$

If $1 - r(t) \sim |t|^\alpha$ for $t \to 0$, then $v(u) \sim u^{2/\alpha}$, and (7.21) holds if $u(t) = \mathit{constant} \cdot t^{\alpha/(M\alpha-2)}$. This tends to $\infty$ with $t$ under the current condition $M > 2/\alpha$.

By virtue of (7.20), the condition (3.2) implies that (7.10) holds. By the specific form of $\Psi_u$, (7.10) implies the simple mixing condition $r(t) \to 0$ for $t \to \infty$. We have not yet determined whether this is also sufficient for (3.2).

The condition (4.10) also follows from the existence of (7.14) and the regular variation of $1 - r$. Indeed, by the same reasoning as in the proof of Theorem 6.2, the expression following the limit sign in (4.10) is asymptotically at most equal to

$$\frac{1}{t}\int_\delta^{t/k} (1 - r^2(s))^{-M/2}\, ds,$$

which converges to $1/k$ for $t \to \infty$.

**8. Application to Gaussian processes: sufficient conditions.** In Section 7, we found that the necessary conditions for (2.3), (2.4) and (4.10) were actually sufficient in the examples considered. However, we were not able to show that the necessary condition (7.10) for (3.2) is sufficient for the latter; instead we referred to the earlier version of our limit theorem in [3] for Example 7.1.

The complex conditions on the finite dimensional distributions of a general, not necessarily Gaussian process, given in [3] for $A_u = (u, \infty)$, were, in fact, suggested by the calculations in the Gaussian case. The present condition (3.2) was introduced for the purposes of simplification and applications to more general families $(A_u)$. However, (3.2) does not, without some stretching, easily apply to the Gaussian case.

Let us show how (3.2) can be adapted to the latter case. Examination of the proofs of Theorems 3.1 and 4.1 show that (3.2) can be weakened to permit more convenient calculations. It is sufficient that (3.2) hold for sets $S_1$ and $S_2$ of the particular nature used in the proofs of the theorems; $S_1 \cup S_2$ consists of $k$ clusters of points, where the points within each cluster are separated by small prescribed distances, and where the clusters are mutually separated by distances of magnitude of the order $t$. Such detailed conditions for Gaussian process have already been verified in [2]. Thus our present condition (3.2) is at least closely related to the previous sufficient conditions.

We now propose a second approach to the proof of Theorem 4.1 through (3.2) in the Gaussian case. It was actually introduced earlier in [1]. Though the latter work contained some errors which were noted in [2], the method of the calculation of the distribution of $vL_t(u)$ for $t \to \infty$ is still valid, and, in fact, can be applied to an arbitrary family $(A_u)$. The idea of the method is this. For a given large $t$, we construct a stationary Gaussian process $X_t(s)$, $s \geq 0$, on the same probability space as $X(s)$, $s \geq 0$. By the construction, the distributions of $X_t(s)$ are close to those of $X(s)$. However, $X_t(s)$ has the property that its covariance function vanishes outside an interval of length $o(t)$ for $t \to \infty$. Thus, for each $m \geq 1$, the ratio of probabilities under the limit sign is, for the process $X_t(s)$, equal to 1 for all sufficiently large $t$. Thus $X_t(\cdot)$ has the required global mixing properties needed for Theorem 3.1. Since the distributions of $X_t$ are close to those of $X$, it is to be expected, in examples under consideration, that the conditions on $r$ which are sufficient for (2.3), (2.4) and (4.10) for the validity of the theorems on the sojourns of $X$, will also be sufficient for the sojourns of $X_t$. Finally we show that the sojourns of $X$ are asymptotically equivalent to those of $X_t$.

The details of the construction of $X_t$ are similar to those in [1]. Let $f(\lambda)$ be the spectral density function of $X$, and let $b(t)$, $-\infty < t < \infty$, be a non-negative function with compact support and $\int_{-\infty}^{\infty} b^2(t)\, dt = 1$. Then

$$\rho(t) = \int_{-\infty}^{\infty} b(t+s)b(s)\, ds$$

is a covariance function with compact support and $\rho(0) = 1$. Let $q(\lambda)$ be its density:

$$q(\lambda) = \frac{1}{2\pi} \left| \int_{\infty}^{\infty} e^{i\lambda s} b(s)\, ds \right|^2 .$$

Let $w(t)$ be an increasing positive function such that $w(t) \to \infty$, $w(t) = o(t)$, for $t \to \infty$; and define

(8.1) $$(q_t * f)(\lambda) = \int_{\infty}^{\infty} f\left(\lambda + \frac{y}{w(t)}\right) q(y)\, dy.$$

Let

(8.2) $$X(s) = \int_{-\infty}^{\infty} e^{i\lambda s} f^{1/2}(\lambda)\, U(d\lambda)$$

be the spectral representation of $X$ with respect to the Brownian motion $U$; and define

(8.3) $$X_t(s) = \int_{-\infty}^{\infty} e^{i\lambda s} (q_t * f)^{1/2}(\lambda)\, U(d\lambda).$$

The latter is stationary with mean 0 and covariance function $r(s)\rho(s/w(t))$. The support of the latter function is in an interval of length $o(t)$, for $t \to \infty$, and the function converges everywhere to $r(s)$.

Put

$$L_t^*(u) = \int_0^t I_{[X_t(s) \in A_u]}\, ds.$$

If we can show, by the methods used in Theorem 4.1, that $vL_t^*(u)$ has a limiting distribution, then it will follow that $vL_t(u)$ has the same limiting distribution if it can be established that

$$\lim_{t\to\infty} vE\, |L_t(u) - L_t^*(u)| = 0.$$

By applying Fubini's theorem, one sees that it suffices to prove that

$$\lim_{t\to\infty} vtP(X(0) \in A_u, X_t(0) \notin A_u) = \lim_{t\to\infty} vtP(X(0) \notin A_u, X_t(0) \in A_u) = 0.$$

By (3.1) and the identity of the distributions of $X(0)$ and $X_t(0)$, the relations above are equivalent to

(8.4) $$\lim_{t\to\infty} vtP(X(0) \in A_u, X_t(0) \in A_u) = 1.$$

Put $\eta(t) = EX(0)X_t(0)$; then, by (8.2) and (8.3), we have

(8.5) $$\eta(t) = \int_{-\infty}^{\infty} [f(\lambda)(q_t * f)(\lambda)]^{1/2}\, d\lambda.$$

Then formula (7.6) and the relation (3.1) imply that (8.4) holds if and only if

(8.6) $$\lim_{t\to\infty} vt \int_{\eta(t)}^{1} \Psi_u(z)\, dz = 0.$$

The latter condition holds if $\eta(t)$ tends to 1 sufficiently rapidly for $t \to \infty$. This is related to the smoothness of $f$. Indeed, $1 - \eta(t)$ is equal to

$$\frac{1}{2}\int_{-\infty}^{\infty} \sqrt{f}(\sqrt{f} - \sqrt{q_t * f})\, d\lambda + \frac{1}{2}\int_{-\infty}^{\infty} \sqrt{q_t * f}(\sqrt{q_t * f} - \sqrt{f})\, d\lambda,$$

or, equivalently,

$$\frac{1}{2}\int_{-\infty}^{\infty} (\sqrt{f} - \sqrt{q_t * f})^2\, d\lambda.$$

By (8.1) the latter tends to 0 for $t \to \infty$. The rate of the convergence to 0 depends on the smoothness of $f$. Such an estimate was given in the particular case $A_u = (u, \infty)$ considered in [1]. We have not been able to formulate this rate of convergence directly in terms of $r(t)$, but only in terms of the spectral density itself.

We expect to apply the method in this section to the problem in Example 7.2 in a future publication.

**9. Corrections of previous results.** In [3], Lemma 17.2, it was incorrectly stated that the condition

$$\limsup_{t\to\infty} \int_0^1 \frac{P(X(0) > u, X(ts) > u)}{P^2(X(0) > u)}\, ds < \infty \tag{9.1}$$

is sufficient for the local mixing condition 4.II of Theorem 4A. The proof is correct except for the last sentence, which is marred by the fact that $u$ is a function of $t$. The conclusion of the previous sentence implies that the lemma is correct if (9.1) is replaced by

$$\lim_{k\to\infty}\limsup_{t\to\infty} \frac{1}{t}\int_0^{t/k} \frac{P(X(0) > u, X(s) > u)}{P^2(X(0) > u)}\, ds = 0. \tag{9.2}$$

The condition (4.10), which is much weaker, should now be used in the place of (9.2).

In [3], page 8, line 13, the sentence should be "Since $L_s - L_{d/v}$ is nonnegative...." Also $L_s - L_{d/v}$ should be replaced by $L_{d/v}$ in formula (3.12).

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