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CORRELATED RANDOM WALKS

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We consider random walks on lattices with finite memory and a finite number of possible steps. Using a local limit theorem, we generalize Polya's theorem to such walks, describe how to compute tail probabilities when the number of steps is large, and obtain asymptotic estimates for the average number of points visited.

We study what we call a *finitary correlated random walk* (fcrw) defined as follows. There is a *finite* set $\mathbf{S}$ of *states* together with a step probability $p: Z^d \times \mathbf{S} \times \mathbf{S} \to \mathbf{R}$ having finite support. We interpret $p(\mathbf{k}, s \mid t)$ as the probability of moving to state s and taking a step $\mathbf{k}$ given that the fcrw is in state t. The probability that an fcrw starting in state s_0, takes steps $\mathbf{k}_1, \cdots, \mathbf{k}_n$ and ends in state s_n is

$$\Pr(\mathbf{k}_1, \cdots, \mathbf{k}_n; s_n \mid s_0) = \textstyle\sum_{s_1, \ldots, s_{n-1} \in S} \prod_{i=1}^{n} p(\mathbf{k}_i, s_i \mid s_{i-1}).$$

The probability that an n step fcrw starting in state s ends in state t at location $\mathbf{k}$ is

$$\Pr(\mathbf{k}; n, t \mid s) = \textstyle\sum \Pr(\mathbf{k}_1, \cdots, \mathbf{k}_n; t \mid s),$$

the sum ranging over all $\mathbf{k}_i$ such that $\mathbf{k}_1 + \cdots + \mathbf{k}_n = \mathbf{k}$. Since p has finite support, this sum is finite. The probability $p_n(\mathbf{k})$ that an n-step fcrw ends in position $\mathbf{k}$ is

$$p_n(\mathbf{k}) = \textstyle\sum_{s,t} \Pr(\mathbf{k}; n, t \mid s) q(s),$$

where q is the *starting probability*.

We impose two conditions on an fcrw. The first ensures that it is essentially d-dimensional and the second that all states are recurrent.

CONDITION A. For some n and some $i, j \in \mathbf{S}$, the vector space over R spanned by differences of those $\mathbf{k}$ with $\Pr(\mathbf{k}; n, j \mid i) \neq 0$ is $\mathbf{R}^d$.

CONDITION B. The directed graph with vertex set S and edge (i, j) when $p(\mathbf{k}, j \mid i) = 0$ for some $\mathbf{k}$ is strongly connected.

An fcrw is *drift free* if the expected value of $\mathbf{k}$ is $o(n)$; i.e., $\sum \mathbf{k}\, p_n(\mathbf{k}) = o(n)$.

We can view a fcrw as a Markov chain $\{(X_n, Y_n), n \geq 0\}$ where $X_n \in \mathbf{S}$, $Y_n \in$

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$\mathbb{Z}^d$, and

$$P\{(X_n, Y_n)|(X_{n-1}, Y_{n-1})\} = p(Y_n, X_n | X_{n-1}).$$

We are interested in $Y_1 + \cdots + Y_n$, the location after n steps of an fcrw starting at the origin. The process $\{X_n, n \geq 0\}$ is also a Markov chain and Condition B says that it is irreducible.

Now consider some examples of fcrw's, beginning with (uncorrelated) random walks on $\mathbf{Z}^d$. These have $|S| = 1$ and $p(\mathbf{k}, s | s) = \Pr(\mathbf{k})$, the probability of a step of $\mathbf{k}$. If $\Pr(\mathbf{k}) = 1/2d$ when $\mathbf{k} \in \mathbf{Z}^d$ is a unit vector and zero otherwise, we have the classical nearest neighbor walk. If $\Pr(\mathbf{k}) = 2^{-d}$ when $\mathbf{k} = (\pm 1, \cdots, \pm 1)$ and zero otherwise, we have a random walk on the generalized body centered cubic lattice. If $d = 2$ and $\Pr(\mathbf{k}) = 1/6$ for $\mathbf{k} = (\pm 1, 0)$, $(0, \pm 1)$ and $\pm(1, 1)$, we have the triangular lattice.

The hexagonal lattice can be obtained by introducing two states 0 and 1 in the last example and defining $p(\mathbf{k}, s | t) = 1/3$ when $s \neq t$ and $(-1)^t \mathbf{k} = (1, 0)$, $(0, 1)$, or $(-1, -1)$.

Consider a random walk with $\Pr(\mathbf{k}) = 1/m$ when $\mathbf{k} \in \mathbf{S}$ and zero otherwise. (This is a definition of $\mathbf{S}$). Suppose $\mathbf{k} \in \mathbf{S}$ implies $-\mathbf{k} \in \mathbf{S}$. We can eliminate immediate reversals by defining an fcrw with $p(\mathbf{k}, s | t) = 1/(m - 1)$ when $k = s \neq -t$ and zero otherwise. The more general case of a walk with restricted reversals has been considered in the literature, see Domb and Fisher [3]. Here the steps in the various directions are allowed to have different probabilities but these probabilities are reduced by an amount δ when a step forms a direct reversal of the previous step. Clearly this is also an fcrw and $\delta = 1/(m - 1)$ gives the previous example with no immediate reversals. We shall return to restricted reversal walks in a subsequent example with δ defined as here.

We now indicate how the results of [2], hereafter referred to by III, apply. Familiarity with III is needed in what follows. Unfortunately recapitulating the concepts and theorems of that paper here would require extensive space. Following Temperly [11], define

$$T_{ij} = \textstyle\sum_{\mathbf{k}} p(\mathbf{k}, j | i)\mathbf{x}^{\mathbf{k}}$$

where $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \cdots x_d^{k_d}$. Also set $C_{ij} = q_i$, the starting probability. It is easily seen that the conditions in Definition III.1 hold with $\alpha = \mathbf{S}$, $\mathbf{r}$ and n_0 arbitrary, and p dependent on an examination of T. (This uses Condition B above.) Since $\sum \Pr(\mathbf{k}; n, j | i)\mathbf{x}^{\mathbf{k}} = T_{ij}^{(n)}$, Condition A implies that Λ in Definition III.2 is d-dimensional. Then Theorem III.1 and Lemmas III.2 and III.4 apply with $a_n(\mathbf{k}) = p_n(\mathbf{k})$. Theorem III.1 is a local limit theorem for matrix recursions modulo a lattice which shows not only that the probability that an n-step walk is at a point tends with n to a multivariate normal distribution (which follows from classical results of Kolmogorov [8] or see Hammersley [7]), but also that when the points lie in a given coset of an appropriate sublattice, the density function is asymptotically a multivariate normal. Much literature, beginning with Kolmogorov [8], has been devoted to the study of conditions under which the local limit theorem is valid. For many natural fcrw's, such as the classical nearest neighbor walk, there is not a local limit theorem. See Renshaw and Henderson [11] for a complete discussion of a certain correlated random walk which is shown to have an asyptotically normal distribution and for references to other correlated random walks and their applications.

Our first result is a generalization to fcrw's of Polya's theorem that return to the origin is persistent for the classical nearest neighbors walk if and only if the dimension is less than or equal to two. It follows easily from Feller's theorem [5, Theorem XIII.3.2], Theorem III.1, Lemmas III.2 and III.4 that:

THEOREM 1. *Return to the origin is persistent if and only if the* fcrw *is drift-free and of dimension* 1 *or* 2.

Asymptotic estimates for the average number of points visited at least r times by an n-step random walk have been obtained by Dvoretzky and Erdos [4]. We show:

THEOREM 2. *Let $S_n^{(r)}$ denote the expected number of points visited at least r times after n steps by an* fcrw. *Then as $n \to \infty$ and r is fixed*

$$S_n^{(r)} \sim \begin{cases} Cn^{1/2} & d = 1 \\ Cn/\log n & d = 2 \\ Cn & d = 3 \end{cases}$$

where C is independent of r for $d = 1, 2$ and a strictly decreasing function of r for $d \geq 3$.

Determination of C is generally difficult, even for $d = 1$ and 2 the value of C will depend on the lattice. See Montroll and Weiss [9] for some values.

Theorem 1 can be generalized somewhat. The requirement that $p(\mathbf{k}, s \mid t)\mathbf{x}^{\mathbf{k}}$ have finite support can be relaxed to the requirement $p(\mathbf{k}, s \mid t)\mathbf{x}^{\mathbf{k}}$ converge in a neighborhood of 1. The results of Foster and Good [6] suggest that, in fact, $\sum |\mathbf{k}|^2 p(\mathbf{k}, s \mid t) < \infty$ suffices, however, our methods appear inadequate to prove this. Barber and Ninham [1, Section 2.3] discuss extensions of Polya's theorem to other random walks. Nash-Williams [10] has given a characterization of a geometrical nature for a random walk to be persistent. The fact that a random walk can be recurrent only if it is drift-free follows from the strong law of large numbers for functions of Markov chains. The proof of Theorem 2 parallels that of Montroll and Weiss's treatment [9] of uncorrelated case, to which we refer the reader for further historical background, and so is omitted. Since an fcrw is really a semi-Markov chain (indeed the special case which can be reduced to a finite Markov chain) it seems likely that both Theorems 1 and 2 can be generalized via more orthodox probabilistic techniques. The specific questions that Theorems 1 and 2 treat seem not to have been answered in such generality in spite of a vast literature concerning Markov and semi-Markov chains. In our proofs Theorem III.1 of [2] is essential.

The determination of asymptotic estimates for the probability that n-step walk is at a point far from the mean seems to have been almost ignored (totally ignored in the case when there is not a local limit theorem). We now discuss this much more thoroughly than in [2].

By (III.4.3) with $\mathbf{r}$ given by $m(\mathbf{r}) = \mathbf{k}/n$,

$$p_n(\mathbf{k}) = \frac{A + O(1)}{n^{d/2}\mathbf{r}^{\mathbf{k}}} \lambda(\mathbf{r})^n, \tag{1.1}$$

where A depends on $\mathbf{r}$, $\mathbf{k} + \Lambda$ and n modulo pq. The explicit dependence can be calculated as discussed in III.

We only illustrate the determination of $\mathbf{r}$ and λ by an example here. Consider random walks with restricted reversals. In this case, $\mathbf{S} \subseteq \mathbf{Z}^d$ has cardinality s and $\mathbf{k} \in S$ implies $-\mathbf{k} \in S$. We have

$$p(\mathbf{k}, \mathbf{k} \,|\, \mathbf{j}) = \begin{cases} \varepsilon \text{ if } \mathbf{j} = -\mathbf{k}, \text{ a reversal,} \\ \varepsilon + \delta \text{ otherwise.} \end{cases}$$

Clearly $s\,\varepsilon + (s - 1)\delta = 1$. Domb and Fisher [3, (16)] obtain

$$\sum p_n(\mathbf{k})\mathbf{x}^{\mathbf{k}} z^n = \frac{1 - \delta f z}{1 - (1 + \delta) f z + \delta^2} \tag{1.2}$$

where

$$f = \frac{1}{s} \textstyle\sum_{\mathbf{k} \in S} \mathbf{k}^{\mathbf{x}}.$$

As noted in Section III.5, λ is given by $1/z$ where z is a zero of the denominator of (1.2). Thus

$$\lambda = \frac{1}{2s} (1 + \delta) \textstyle\sum \mathbf{r}^{\mathbf{k}} + \sqrt{(1 + \delta)^2 (\sum \mathbf{r}^{\mathbf{k}})^2 - 4\delta s^2}. \tag{1.3a}$$

and

$$\frac{k_i}{n} = \frac{\partial \log \lambda}{\partial \log r_i} = \frac{(1 + \delta) \sum k_i\, \mathbf{r}^{\mathbf{k}}}{2\lambda s - (1 + \delta) \sum \mathbf{r}^{\mathbf{k}}}. \tag{1.3b}$$

A one-dimensional walk with unrestricted reversals ($\delta = 0$) and unit steps ($\mathbf{S} = \{1, -1\}$) is one simple case of (1.3). In this case we easily obtain the result obtained by applying Stirling's formula to the exact expression

$$\Pr(X_n = k) = 2^{-n} \binom{n}{n(n - k)/2}.$$

Generally $\mathbf{r}$ and λ can only be obtained numerically in terms of $\mathbf{k}/n$. Since

$$\mathbf{k}/n = 2 \log \lambda / 2 \log \mathbf{r} \tag{1.4}$$

and $\lambda(\mathbf{r})$ is an eigenvalue of $T(\mathbf{r})$, it is fairly straightforward to carry out the numerical calculations. In fact, if tabular results are desired, we can specify $\mathbf{r}$, solve the eigenvalue equation for λ, and then find $\mathbf{k}/n$ from (1.4).

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