The Annals of Probability
1984, Vol. 12, No. 3, 912–917

AN EXAMPLE ON THE CENTRAL LIMIT THEOREM FOR ASSOCIATED SEQUENCES

BY NORBERT HERRNDORF

Mathematisches Institut der Universität Köln

We construct a strictly stationary associated sequence $(X_n)_{n\in\mathbb{N}}$ with $EX_n = 0$, $0 < EX_n^2 < \infty$ such that $K(R) = \text{Cov}(X_1, X_1) + \sum_{j=2}^{R} \text{Cov}(X_1, X_j) \sim \log R$ as $R \to \infty$, but $\sum_{j=1}^{n} X_j/(nK(n))^{1/2}$ does not converge to $\mathscr{N}(0, 1)$ in distribution. This is a counterexample to a conjecture of Newman and Wright (1981).

1. Introduction. In the last years there has been growing interest in sequences of random variables which fulfill a condition of positive dependence called association. A finite collection of random variables $X_1, \cdots, X_m$ is said to be associated if for any two coordinatewise nondecreasing functions f_1, f_2 on R^m such that $f_j(X_1, \cdots, X_m)$ has a finite variance for $j = 1, 2$, $\text{Cov}(f_1(X_1, \cdots, X_m), f_2(X_1, \cdots, X_m)) \geq 0$; an infinite collection is said to be associated if every finite subcollection is associated. This definition was introduced in Esary, Proschan and Walkup (1967). The following central limit theorem for associated sequences is contained in the invariance principle of Newman and Wright (1981).

THEOREM 0. *Suppose $(X_n)_{n\in\mathbb{N}}$ is a strictly stationary associated sequence with $EX_1 = 0$, $0 < EX_1^2 < \infty$, and assume*

$$\sigma^2 = \text{Cov}(X_1, X_1) + 2 \sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) < \infty. \tag{1.1}$$

Then $\sum_{j=1}^{n} X_j/(\sigma n^{1/2})$ converges in distribution to $\mathscr{N}(0, 1)$.

At the end of their paper Newman and Wright state a conjecture, which implies the following central limit theorem.

CONJECTURE. *Assume all the hypotheses of Theorem* 0 *except* (1.1). *Let*

$$K(R) = \text{Cov}(X_1, X_1) + 2 \sum_{j=2}^{R} \text{Cov}(X_1, X_j), \quad R \in \mathbb{N}, \tag{1.2}$$

and suppose

$$K(R) \to \infty, \quad K(R) \quad \textit{is slowly varying as} \quad R \to \infty. \tag{1.3}$$

Then $\sum_{j=1}^{n} X_j/(nK(n))^{1/2}$ converges in distribution to $\mathscr{N}(0, 1)$.

It is the purpose of this paper to construct a counterexample to the above conjecture. We state the main result of our construction in a theorem.

Received April 1983.
AMS 1980 *subject classifications*. Primary 60F05.
Key words and phrases. Central limit theorem, strictly stationary associated sequences.

THEOREM 1. *There exists a strictly stationary associated sequence* $(X_n)_{n\in\mathbb{N}}$ *with* $EX_1 = 0$, $0 < EX_1^2 < \infty$ *such that* $K(R)$ *defined by* (1.2) *satisfies* $K(R) \sim \log R$ *as* $R \to \infty$, *but* $\sum_{j=1}^n X_j/(K(n)n)^{1/2}$ *does not have a nondegenerate limit distribution.*

We note that the general idea of the construction in this paper is similar to the method in [3], but the details are quite different.

2. The construction of the spectral density. A nonnegative integrable function f on $[-\frac{1}{2}, \frac{1}{2}]$ is called a spectral density of the wide sense stationary sequence $(X_n)_{n\in\mathbb{N}}$, if

$$\operatorname{Cov}(X_n, X_{n+k}) = \int_{-1/2}^{1/2} e^{2\pi ikt} f(t)\, dt, \quad n \in \mathbb{N}, \quad k \in \mathbb{N} \cup \{0\}.$$

In Lemma 2.1 a function f is constructed which can be written as a countable sum of spectral densities of certain moving average sequences.

2.1 LEMMA. *There exist functions* f_k: $[-\frac{1}{2}, \frac{1}{2}] \to [0, \infty)$, $k \in \mathbb{N}$, *with the following properties:*

(2.2) *For each* $k \in \mathbb{N}$, f_k *can be written as*

$$f_k(t) = (a_{0,k} + 2\textstyle\sum_{n=1}^{N(k)} a_{n,k}\cos(2\pi nt))^2$$

with

$$N(k) \in \mathbb{N} \cup \{0\}, \quad a_{n,k} \in [0, \infty) \quad \textit{for } n = 0, \cdots, N(k).$$

$$f_k(t) \le 1 \quad \textit{for all} \quad k \in \mathbb{N}, \quad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]. \tag{2.3}$$

For $f = \sum_{k=1}^\infty f_k$ *holds*

$$f(t) \le -\log|t| \quad \textit{for all} \quad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]. \tag{2.4}$$

$$f(t/n)/\log n \to 1 \quad \textit{as} \quad n \to \infty, \quad t \in [-n/2, n/2]. \tag{2.5}$$

PROOF. For $k \in \mathbb{N}$ let $g_k(t) = (k+1)^{-1}(1 - |t|)^{k+1}$, $h_k(t) = g_k(t)^{1/2} = (k+1)^{-1/2}(1 - |t|)^{(k+1)/2}$, $t \in [-\frac{1}{2}, \frac{1}{2}]$. h_k can be developed as a uniformly convergent Fourier series.

$$h_k(t) = a_{0,k} + 2\textstyle\sum_{n=1}^\infty a_{n,k}\cos(2n\pi t)$$

with

$$a_{n,k} = \int_{-1/2}^{1/2} h_k(t)\cos(2\pi nt)\, dt. \tag{1}$$

The uniform convergence of the Fourier series can be proved by an application of Theorem 2.11 of Kufner and Kadlec (1971). Now we choose $N(k) \in \mathbb{N} \cup \{0\}$

such that

(2) $$\sup_t |h_k(t) - (a_{0,k} + 2\sum_{n=1}^{N(k)} a_{n,k}\cos(2\pi nt))| \le \varepsilon_k = \tfrac{1}{6}2^{-k}.$$

Put $f_k(t) = (a_{0,k} + 2\sum_{n=1}^{N(k)} a_{n,k}\cos(2\pi nt))^2$. To prove (2.2), it remains to show $a_{n,k} \ge 0$. Since h_k is bounded, nonincreasing and convex on $[0, \tfrac12]$ and $h_k(t) = h_k(-t)$, $a_{n,k} \ge 0$ follows from (1) by an easy computation, which is carried out in Lemma 2.6. From (2) we obtain for all $t \in [-\tfrac12, \tfrac12]$

(3) $$\begin{aligned} |g_k(t) - f_k(t)| &\le |h_k(t) - f_k(t)^{1/2}|(h_k(t) + f_k(t)^{1/2}) \\ &\le \varepsilon_k(2h_k(t) + \varepsilon_k) \le 3\varepsilon_k. \end{aligned}$$

Now (2.3) can be easily checked. (2.4) and (2.5) follow from (3) and $\sum_{k=1}^{\infty} g_k(t) = -\log|t| - (1 - |t|)$ by routine calculus.

2.6 LEMMA. *Let h: $[0, \tfrac12] \to [0, \infty)$ be bounded, nonincreasing and convex. Then*

$$\int_0^{1/2} h(t)\cos(2\pi nt)\,dt \ge 0 \quad \textit{for all} \quad n \in \mathbb{N} \cup \{0\}.$$

PROOF. The case $n = 0$ is trivial. If $n = 2n'$, $n' \in \mathbb{N}$, then

(1) $$\int_0^{1/2} h(t)\cos(2\pi nt)\,dt = \sum_{m=0}^{n'-1} \int_{m/n}^{(m+1)/n} h(t)\cos(2\pi nt)\,dt.$$

If $n = 2n' + 1$, $n' \in \mathbb{N} \cup \{0\}$, then

(2) $$\begin{aligned} &\int_0^{1/2} h(t)\cos(2\pi nt)\,dt \\ &\quad = \sum_{m=0}^{n'-1} \int_{m/n}^{(m+1)/n} h(t)\cos(2\pi nt)\,dt + \int_{n'/n}^{1/2} h(t)\cos(2\pi nt)\,dt. \end{aligned}$$

Using the convexity of h, we obtain

(3) $$\begin{aligned} &\int_{m/n}^{(m+1)/n} h(t)\cos(2\pi nt)\,dt \\ &\quad = \int_0^{1/4n} \left(h\left(\frac{m}{n} + t\right) - h\left(\frac{m}{n} + \frac{1}{2n} - t\right)\right. \\ &\qquad \left. - h\left(\frac{m}{n} + \frac{1}{2n} + t\right) + h\left(\frac{m+1}{n} - t\right)\right)\cos(2\pi nt)\,dt \ge 0. \end{aligned}$$

For n even the assertion follows from (1) and (3). If n is odd, then we can use the assumption that h is nonincreasing to estimate the last summand on the r.h.s. of (2). We obtain

$$\begin{aligned}
&\int_{n'/n}^{1/2} h(t)\cos(2\pi nt)\,dt \\
&\quad = \int_0^{1/4n} h\left(\frac{n'}{n} + t\right)\cos(2\pi nt)\,dt + \int_{1/4n}^{1/2n} h\left(\frac{n'}{n} + t\right)\cos(2\pi nt)\,dt \\
&\quad \geq h\left(\frac{n'}{n} + \frac{1}{4n}\right)\left(\int_0^{1/4n} \cos(2\pi nt)\,dt + \int_{1/4n}^{1/2n} \cos(2\pi nt)\,dt\right) \\
&\quad = 0.
\end{aligned}$$

In 2.7 we show that a positively correlated stationary sequence with spectral density f satisfies the assumption (1.3).

2.7 LEMMA. *Let f: $[-\frac{1}{2}, \frac{1}{2}] \to [0, \infty]$ be a nonnegative integrable function satisfying* (2.4) *and* (2.5). *Let $(X_n)_{n\in\mathbb{N}}$ be a mean zero, wide sense stationary sequence with spectral density f. Then holds*

(2.8) $$\sigma_n^2 = E\left(\textstyle\sum_{j=1}^n X_j\right)^2 \sim n \log n \quad as \quad n \to \infty.$$

If $\mathrm{Cov}(X_1, X_n) \geq 0$ *for all* $n \in \mathbb{N}$, *then* (2.8) *implies*

(2.9) $$K(R) \sim \log R \quad as \quad R \to \infty,$$

where $K(R)$ is defined in (1.2).

PROOF. We use the formula

$$\sigma_n^2 = \int_{-1/2}^{1/2} \frac{\sin^2(n\pi t)}{\sin^2(\pi t)} f(t)\,dt,$$

and substitute $s = nt$. This yields

$$\frac{\sigma_n^2}{n \log n} = \int_{-n/2}^{n/2} \frac{\sin^2(\pi s)}{n^2 \sin^2(\pi s/n)} \frac{f(s/n)}{\log n}\,ds \quad \text{for} \quad n \geq 2.$$

Using (2.4), (2.5) and applying Lebesgue's theorem of dominated convergence, we obtain

$$\frac{\sigma_n^2}{n \log n} \to \int_{-\infty}^{\infty} \frac{\sin^2(\pi s)}{\pi^2 s^2}\,ds = 1,$$

which is (2.8). To prove (2.9) let $m \in \mathbb{N}$, $m \geq 2$ be arbitrary. We can estimate

$$\begin{aligned}\frac{\sigma_n^2}{n} &\leq K(n) = \mathrm{Cov}(X_1, X_1) + 2 \textstyle\sum_{j=2}^{n} \mathrm{Cov}(X_1, X_j)\\ &\leq \mathrm{Cov}(X_1, X_1) + 2 \frac{m}{m-1} \textstyle\sum_{j=2}^{n} \mathrm{Cov}(X_1, X_j)\left(1 - \frac{j-1}{mn}\right)\\ &\leq \frac{m}{m-1}\left(\mathrm{Cov}(X_1, X_1) + 2 \textstyle\sum_{j=2}^{mn} \mathrm{Cov}(X_1, X_j)\left(1 - \frac{j-1}{mn}\right)\right)\\ &= \frac{m}{m-1}\, \sigma_{mn}^2/(mn).\end{aligned}$$

Using (2.8), we obtain

$$1 \leq \frac{\liminf_{n\in\mathbb{N}} K(n)}{\log n} \leq \frac{\limsup_{n\in\mathbb{N}} K(n)}{\log n} \leq \frac{m}{m-1}.$$

Since $m \in \mathbb{N}$, $m \geq 2$ is arbitrary, (2.9) follows.

3. The example. Let $f_k(t) = (a_{0,k} + 2 \sum_{n=1}^{N(k)} a_{n,k}\cos(2\pi nt))^2$ be the functions which have been constructed in 2.1. For $k \in \mathbb{N}$ let

$$\alpha_k = 2^k(1 + \exp(k^2))(1 + 2N(k)).$$

Let $\xi_{n,k}$, $n \in \mathbb{Z}$, $k \in \mathbb{N}$, be independent random variables with

$$P\{\xi_{n,k} = \pm\alpha_k^{1/2}\} = 1/(2\alpha_k, \quad P\{\xi_{n,k} = 0\} = 1 - 1/\alpha_k.$$

Define

$$X_{n,k} = \textstyle\sum_{j=-N(k)}^{N(k)} a_{|j|,k}\xi_{j+n,k}, \quad k, n \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, $(X_{n,k})_{n\in\mathbb{N}}$ is a strictly stationary sequence with $EX_{n,k} = 0$. According to Doob (1953) page 499, f_k is the spectral density of $(X_{n,k})_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, $(X_{n,k})_{k\in\mathbb{N}}$ is an independent sequence, and

$$\textstyle\sum_{k=1}^{\infty} EX_{n,k}^2 = \sum_{k=1}^{\infty} \int_{-1/2}^{1/2} f_k(t)\, dt = \int_{-1/2}^{1/2} f(t)\, dt < \infty.$$

Hence the random series $\sum_{k=1}^{\infty} X_{n,k}$ is convergent a.s. and with respect to $\| \ \|_2$, and it is possible to define

$$X_n = \textstyle\sum_{k=1}^{\infty} X_{n,k}, \quad n \in \mathbb{N}.$$

Then $(X_n)_{n\in\mathbb{N}}$ is a strictly stationary sequence with $EX_n = 0$, and it is straightforward to check that $f = \sum_{k=1}^{\infty} f_k$ is the spectral density of $(X_n)_{n\in\mathbb{N}}$. We will prove that (X_n) has the following properties.

(3.1) $(X_n)_{n\in\mathbb{N}}$ is associated.

(3.2) $K(R)$ defined in (1.2) satisfies $K(R) \sim \log R$ as $R \to \infty$.

(3.3) There exist positive integers $n(1) < n(2) < \cdots$ such that

$$\sum_{j=1}^{n(k)} X_j/(n(k)\log n(k))^{1/2} \to 0$$

in measure as $k \to \infty$.

To (3.1): Since $a_{n,k} \geq 0$ it follows from P4 of Esary, Proschan and Walkup (1967) that $\{X_{n,k}: n, k \in \mathbb{N}\}$ is associated. By the same argument, for each $m \in \mathbb{N}$ $\{\sum_{k=1}^{m} X_{n,k}: n \in \mathbb{N}\}$ is associated. Since for each $n \in \mathbb{N}$ $(\sum_{k=1}^{m} X_{j,k})_{1\leq j\leq n} \to (X_j)_{1\leq j\leq n}$ in distribution as $m \to \infty$, Theorem 3.3 of Esary et al. (1967) implies that $\{X_n: n \in \mathbb{N}\}$ is associated.

(3.2) follows from Lemma 2.7.

To (3.3): We consider $n(k) = \min\{j \in \mathbb{N}: j \geq \exp(k^2)\}$, $k \in \mathbb{N}$. For $n, k \in \mathbb{N}$ let $\sigma_{n,k}^2 = E(\sum_{j=1}^{n} X_{j,k})^2$. One can estimate

$$\sigma_{n,k}^2 = \int_{-1/2}^{1/2} \frac{\sin^2(n\pi t)}{\sin^2(\pi t)} f_k(t)\, dt \leq \sup_t f_k(t) n \leq n.$$

Now we write $\sum_{j=1}^{n(k)} X_j = V_k + W_k$ with $V_k = \sum_{r=1}^{k} \sum_{j=1}^{n(k)} X_{j,r}$, $W_k = \sum_{r=k+1}^{\infty} \sum_{j=1}^{n(k)} X_{j,r}$, and estimate V_k and W_k separately.

$$(1) \qquad \frac{EV_k^2}{n(k) log\, n(k)} = \frac{\sum_{r=1}^{k} \sigma_{n(k),r}^2}{n(k)\log n(k)} \leq \frac{k}{\log n(k)} \leq k^{-1}.$$

$$(2) \qquad P\{W_k \neq 0\} \leq \sum_{r=k+1}^{\infty} n(k) P\{X_{1,r} \neq 0\}$$

$$\leq \sum_{r=k+1}^{\infty} n(r)(2N(r) + 1)\alpha_r^{-1} \leq \sum_{r=k+1}^{\infty} 2^{-r} \leq 2^{-k}.$$

(1) and (2) imply $\sum_{j=1}^{n(k)} X_j/(n(k)\log n(k))^{1/2} \to 0$ in measure as $k \to \infty$.

REMARK. The above construction proves Theorem 1. It shows that in the situation of Newman and Wright's conjecture the "natural" standardization σ_n^{-1}, which is equivalent to $(nK(n))^{-1/2}$, does not lead to a nondegenerate limit distribution of the partial sums. Possibly there exists a different standardization which yields asymptotic normality.

REFERENCES

[1] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[2] ESARY, J., PROSCHAN, F., and WALKUP, D. (1967). Association of random variables with applications. *Ann. Math. Statist.* **38** 1466–1474.

[3] HERRNDORF, N. (1983). Stationary strongly mixing sequences not satisfying the central limit theorem. *Ann. Probab.* **11**, 809–813.

[4] KUFNER, A. and KADLEC, J. (1971). *Fourier Series.* Iliffe Books, London.

[5] NEWMAN, C. M. and WRIGHT, A. L. (1981). An invariance principle for certain dependent sequences. *Ann. Probab.* **9** 671–675.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT KÖLN
WEYERTHAL 86-90
D-5000 KÖLN 41
FEDERAL REPUBLIC OF GERMANY