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A UNIFIED APPROACH TO A CLASS OF BEST CHOICE PROBLEMS WITH AN UNKNOWN NUMBER OF OPTIONS

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This article tries to unify best choice problems under total ignorance of both the candidates, quality distribution and the distribution of the number of candidates. The result is what we shall call the e^{-1}-*law* because of the multiple role which is played by e^{-1}, and this in a more general context as only in the solution of the classical secretary problem. The unification is possible whenever best choice problems can be redefined as continuous time decision problems on conditionally independent arrivals. We shall also give several examples to illustrate how the approach and its implications compare with other models.

1. Introduction. We start by recalling the classical secretary problem (CSP) or marriage problem, to which we shall frequently refer:

Suppose we want to appoint a secretary, knowing that there are n candidates. We suppose that, if we knew them all in advance, then we would be able to classify them in a unique order in accordance to our own criteria from the best, $\langle 1 \rangle$, down to the worst, $\langle n \rangle$. Only the respective ranks count; the candidates' qualities are neglected. We can interview the candidates one by one, but after each interview we have to decide whether to accept or refuse. Recalls are inadmissible. If each permutation of the arrival order is equally probable, what is the strategy which maximizes the probability of accepting the best candidate?

The solution may be found in Lindley (1961) or, as a special case of a general Markov chain stopping time problem, in Dynkin and Juschkewitsch (1969). The optimal strategy is to pass over a certain number $k^*(n)$ of candidates and then to accept the first candidate which is better than all preceding ones. $k^*(n)n^{-1} \to e^{-1}$ and the corresponding success probability tends to e^{-1} as $n \to \infty$.

There are many interesting modifications of this problem. For a complete review of the published work to date, see Freeman (1983). We confine our attention to the case where the number of candidates N is a random variable, and where the quality distribution is unknown.

Presman and Sonin (1972) seem to be the first to have published results on this problem followed by Gianini and Samuels (1976), Stewart (1981) and others. We shall refer to several of these papers in more detail since their results allow for an interesting comparison to ours. Common to all these papers is that the distribution g of the number N of candidates is supposed to be known, or that at least the class of g is specified.

The essential difference in our approach is that we suppose g to be unknown.

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Maintaining all other CSP conditions, we want to maximize the probability of accepting the best candidate. We consider the following model:

2. The model. Let $F(z)$ be a distribution function on the real time interval $[0, t]$; let $Z_1, Z_2, \cdots$ be independent random variables each having continuous distribution function F and let N be a nonnegative integer-valued random variable independent of all Z_k's. Z_k is thought of as the arrival time of applicant k and N represents the total number of applicants which will decide to show up. Associated with each applicant is a different quality. We suppose that given $N = n$, each arrival order of ranks $\langle 1 \rangle, \langle 2 \rangle, \cdots, \langle n \rangle$ has the probability $1/n!$

Motivation. The essential part of the model assumptions is the independence condition for the Z_k's. Their identical distribution is then imposed by the last assumption which comes from the CSP. However, the no-recall-condition is only meaningful if simultaneous arrivals are prevented (a.s.) and so it is convenient to suppose that F is continuous.

As examples which satisfy our hypotheses, we may think of Poisson processes on $[0, t]$ or nonhomogeneous Poisson arrival processes. However, our hypotheses are more general since we cannot describe each experiment producing i.i.d. random variables as a suitable Poisson process. The question whether the proposed model is an interesting alternative to existing ones is discussed in Section 6.

3. Waiting time policies. We want to maximize the probability of choosing the best candidate, and so it only makes sense to accept a candidate which is better than all preceding ones. Such a candidate is usually called *leading* candidate. For convention, we suppose that the first candidate to arrive is leading by definition.

In order to decide whether a candidate is leading, all previous ones must have been evaluated. In the continuous time case this implies that, if a candidate is accepted at time τ, say, $\tau \in [0, t]$, then the interval $[0, \tau]$ has been continuously observed on all its subintervals where F is strictly increasing. This together with the fact that any accepted candidate must be leading leads us to consider the following class of strategies:

DEFINITION. The (x, r)-strategy on $[0, t]$ is a policy to act as follows:

1. To observe and rank all incoming candidates up to time x without accepting a candidate.

2. To accept the rth leading candidate arriving after time x, i.e. the first to be r ranks superior to the best of those which arrived in $[0, x]$, if it exists, and to refuse all candidates if not.

The time x will be called *waiting time.* For $r = 1$, we use the notation *x-strategy.*

THEOREM. *For any distribution g with $P(N > 0) > 0$, there exists a waiting time z^* maximizing the success probability of the x-strategy. Moreover, for all*

$\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $N \geq m$ implies $z^* \in [e_F^{-1} - \varepsilon, e_F^{-1}]$, where $e_F^{-1} = \inf\{z \mid F(z) = e^{-1}\}$.

PROOF. For the following, it is convenient to introduce a change of time

$$x = F(z), \quad z \in [0, t] \tag{1}$$

such that, in the x time scale time runs from 0 to 1 and such that each $X_k = F(Z_k)$ is uniform on [0, 1].

If $N = 0$, then every strategy leads to a trivial success. If $N = 1$, then the candidate will be accepted if he arrives after time x, thus $x^* = 0$ and the corresponding success probability equals 1. Suppose now $N \geq 2$. Given $N = n$, the x-strategy yields a success if the best candidate, $\langle 1 \rangle$, arrives in $]x, 1]$ before all other candidates arriving in $]x, 1]$ which are better than the best of those which arrived in $[0, x]$. Consider the $k + 1$ best candidates. According to the model assumptions, $\langle k + 1 \rangle$ arrives in $[0, x]$ and the k best ones in $]x, 1]$ with probability $x(1 - x)^k$. Since $\langle 1 \rangle$ arrives before $\langle 2 \rangle, \cdots, \langle k \rangle$ with probability $1/k$, we obtain

$$\begin{aligned} p_n(x) &= P(\text{success of } x\text{-strategy} \mid N = n) \\ &= x \textstyle\sum_{k=1}^{n-1} (1/k)(1 - x)^k + (1 - x)^n/n, \quad \text{for} \quad n = 2, 3, \cdots. \end{aligned} \tag{2}$$

We recognize the Taylor expansion of $-\ln x$ in the sum term. Since

$$p_n(x) - p_{n+1}(x) = (1 - x)^{n+1}/(n(n + 1)) \geq 0, \quad \text{for} \quad x \in [0, 1], \tag{3}$$

we obtain

$$p_n(x) \searrow p(x) = -x \ln x \quad \text{as} \quad n \to \infty. \tag{4}$$

The function $p(x)$ is maximized by $x = e^{-1}$.

Furthermore,

$$dp_n(x)/dx = -1 + \textstyle\sum_{k=1}^{n-1} (1 - x)^k/k \tag{5}$$

and

$$d^2p_n(x)/dx^2 = -(1 - (1 - x)^{n-1})/x \leq 0, \quad \text{for} \quad x \in [0, 1], \tag{6}$$

so $p_n(x)$ has a unique maximum x_n. It is clear from (5) that

$$x_n \nearrow e^{-1}. \tag{7}$$

Now let

$$G_m(x) = \textstyle\sum_{n \geq m} p_n(x) P(N = n). \tag{8}$$

It follows from (7) that, if there exists a value x^* which maximizes $G_m(x)$, then necessarily $x^* \in [x_m, e^{-1}]$. However, the convergence in (4) and (8) is uniform and so $G_m(x)$ is continuous, i.e. x^* exists. Using (1) and the continuity of F completes the proof. □

The theorem allows for two powerful corollaries which we want to call together the *e^{-1}-law*.

COROLLARY 1. *The e_F^{-1}-strategy has success probability $\geq e^{-1}$ whatever the distribution of N might be.*

PROOF. According to (4), we have $p_n(x) \searrow p(x) = -x \ln x$ on [0, 1] and so, by (1) and (8),

$$G_0(e^{-1}) \geq \sum_{n=0}^{\infty} p(e^{-1})P(N = n) = p(e^{-1}) = e^{-1}. \quad \square \tag{9}$$

COROLLARY 2. *The e_F^{-1}-strategy is the only waiting time policy with the property described in Corollary 1.*

PROOF. From (6) and (7), it is evident that it is the only x-strategy with this property. If $r > 1$, then a necessary condition for a possible success is the existence of at least r leading candidates. Thus, for any distribution g with $P(1 \geq N < r) > 1 - e^{-1}$, the success probability of the (x, r)-strategy is smaller than e^{-1} for all x. In particular, if $P(1 \leq N < r) = 1$, then the latter equals 0. $\square$

4. The best choice problem as a two person game. Let us consider the following game. Given the arrival time distribution function F on $[0, t]$, player A wants to maximize the probability of accepting the best candidate and player B plays the role of a sinister adversary by choosing g, the distribution of N, in the most disadvantageous way possible. It is interesting to find a stopping time τ^* and a distribution g^* which solve the minimax problem

$$p(\tau^*, g^*) = \inf_g \sup_\tau p(\tau, g) \tag{10}$$

where $p(\tau, g)$ denotes the success probability corresponding to the choice of τ and g.

The game allows for three different interesting variants:

1. A is obliged to tell B what he will do, but not vice versa.
2. The case opposite to 1.
3. A and B take their decisions secretly.

If A wants to apply a waiting time policy with waiting time τ_W, then, by the theorem and by Corollary 2,

$$p(\tau_W^*, g^*) = \inf_g \sup_{\tau_W} p((\tau_W, r), g) = \begin{cases} e^{-1}, & \text{if } r = 1, \\ 0, & \text{if } r > 1. \end{cases} \tag{11}$$

It is not difficult to check that, in all three cases, A should apply the e_F^{-1}-strategy and that B can do nothing else than reducing A's success probability to e^{-1} by concentrating g on $N = \infty$. It is conjectured that the e_F^{-1}-strategy is even the best of all conceivable strategies for A under the given assumptions, but the proofs would require more machinery. The asymptotic optimality of the e_F^{-1}-strategy is easy to show (see Section 5).

5. Upper bounds. For the problem as we posed it, the study of waiting time policies was quickly confined to the one of x-strategies whereas $(x, r \geq 2)$-strategies are not optimal (Corollary 2). This may change as soon as partial information about g becomes available. To give an example, let us consider the deterministic case $N = 3$. An analysis similar to (2) shows that the (0, 2)-strategy is optimal with success probability 0.5. It coincides with the optimal stopping time strategy for the CSP according to which one has to observe the first candidate and then to choose, if possible, the next better one. Indeed, in our model, this means to choose, if possible, the second leading candidate after time 0.

The preceding example is extreme in the sense that g is completely specified and, moreover, concentrated in one point. In addition, this example represents the case where the difference between the success probabilities of the $k^*(n)$ stopping time strategy (0.5) and the e_F^{-1}-strategy (0.3902) is maximal (see Table 1). After all, knowing $N = 3$, we would of course directly apply the $k^*(3)$ stopping time strategy since we know that it is optimal.

Still, the given example raises the following question: how much information about g is needed such that it can be worth considering $(x, r \geq 2)$-strategies, i.e. such that their success probabilities may exceed e^{-1}?

To get an idea, we define $L(N) = I(1) + I(2) + \cdots + I(N)$ where $I(k) = 1$ if the kth arrival is a leading candidate and 0 otherwise, and $L(0) = 0$.

A necessary condition for a $(x, r \geq 2)$-strategy to succeed with probability $> e^{-1}$ is evidently $P(L(N) \geq r) > e^{-1}$. It is known (see e.g. [2], page 83) that the $I(k)$ are independent random variables with $P(I(k) = 1) = k^{-1}$. This implies $E(L(N) \mid N = n) = 1 + 2^{-1} + \cdots + n^{-1} \sim \ln n$ and $\mathrm{Var}(L(N) \mid N = n) = 2^{-1} - 2^{-2} + \cdots + n^{-1} - n^{-2} \sim \ln n$ (= 0 if $n < 2$). Using Chebychev's inequality with the above estimates leads to a necessary condition in terms of $E(\ln N)$. Thus essential hypotheses about g would be needed, which we forewent for the sake of applicability. This is why we concentrate on x-strategies, and according to the corollaries, on the e_F^{-1}-strategy.

Now, let $\bar{p}_n$ denote the success probability of the $k^*(n)$ stopping time strategy for the CSP. It is known that this strategy is optimal with $\bar{p}_n = e^{-1} + O(1/n)$, where $O(1/n) \geq 0$ for all $n = 1, 2, \cdots$. We may imbed the CSP in our model associating the n candidates with independent arrival times on $[0, t]$ with distribution function F. From (2), (6) and Corollary 1, we obtain

$$(12) \qquad e^{-1} \leq p_n(e^{-1}) \leq p_n(x_n) \leq \bar{p}_n = e^{-1} + O(1/n), \quad n = 1, 2, \cdots,$$

and thus the e_F^{-1}-strategy is asymptotically optimal if N becomes large in

TABLE 1

$N = n$	1	2	3	5	10	15	$\rightarrow$
x_n	0	0	0.2679	0.3489	0.3670	0.3678	e^{-1}
$p_n(x_n)$	1	0.5	0.3987	0.3723	0.3680	0.3678	e^{-1}
$p_n(e_F^{-1})$	0.6321	0.4323	0.3902	0.3718	0.3680	0.3678	e^{-1}
maximal loss	<0.368	<0.068	<0.009	$<10^{-3}$	$<10^{-5}$	$<10^{-8}$	

The first row gives the values x_n maximizing $p_n(x)$. The second and third rows show the success probabilities of the x_n-strategy and the e_F^{-1}-strategy respectively. Their difference in the last row is the maximal loss for $P(N \geq n) = 1$. The number e^{-1} could be made to appear a fourth time, namely as the probability of accepting no candidate at all.

probability. On the other hand, $p_n(x_n)$ converges very quickly to e^{-1}. The rough estimate $p_n(x_n) - e^{-1} < 0.68^n/n$ for $n > 3$ can easily be derived from (2).

The convergence speed of $(p_n(x_n) - p_n(e^{-1})) \to 0$ is truly surprisingly high (see Table 1). This is central in the paper since it shows that e^{-1} is an excellent approximation of x^* if $P(N \leq 2)$ is not large such that a g-dependent maximization problem $(G_0(x)\max!)$ still makes sense in our model.

6. Discussion and examples. The question whether the proposed model is an interesting alternative to existing ones must be differentiated. From the mathematical point of view, it is just an alternative. Whether g or F is easier to estimate depends on the context of the practical problem.

For applications, the answer should be yes. Time intervenes in most real world decision problems, and, if not, it often may be simulated (see Example 4). Secondly, there are many situations where one may have absolutely no information about the distribution of the *number* of candidates, but quite essential indications about *when* they should appear more likely, if there are any. A simple example of such a situation is the following problem:

Suppose we want to hire a secretary within the next t weeks, because then we urgently need one. We ordered a newspaper to advertise until recall in the Saturday editions. Whatever g might be, we can expect any call more likely after the weekends than during other days, and it is a fair guess that peaks will decrease with the number of Saturdays.

The idea is to draw such a likelihood curve over $[0, t]$, norm it to a density f, say, and to use $F(x) = \int_0^x f(\tau)\, d\tau$. Here we can also count on a smoothing effect of the integral such that a bad estimate of f does not necessarily imply a bad estimate of e_F^{-1} (see Example 4).

In the following, we shall give examples to display the unifying character of our approach, under which different models may coincide.

EXAMPLE 1 (CSP). There is no loss of generality to associate candidates with i.i.d. arrival times on $[0, t]$. Let $N(\tau)$ be the number of arrivals up to time τ. Then the limit relation $k^*(n)/n \to e^{-1}$ follows from the law of large numbers since $N(e_F^{-1})/N(t) \to e^{-1}$ a.s. as $N(t) \to \infty$.

EXAMPLE 2. Presman and Sonin (1972) showed that, if N is Poisson distributed with parameter λ, then the optimal stopping time limit relation is $k^*(\lambda)/\lambda \to e^{-1}$. Associate arrival times and note that $N(\tau)$ is Poisson implies that the unordered conditional arrival times are i.i.d. and uniform on $[0, t]$. Thus $e_F^{-1} = e^{-1}t$ for any $\lambda > 0$, and this waiting time policy can be applied without knowing λ. To see the speed of convergence, suppose a young lady wants to find the best husband within the next t years. She assumes the arrival process to be Poisson with $\lambda t > 100$, say, and so she applies the $e^{-1}t$-strategy. However N turns out to be 3 only. Nevertheless, her strategy is excellent if the Poisson assumption is correct. She only looses 0.009 of success probability compared with what she could have attained at best (see Table 1).

EXAMPLE 3. Stewart (1981) (who pointed out the advantages of a continuous time model before) studied the exponential interarrival time process. But such an arrival process is Poisson such that, with respect to our model, this is equivalent to Example 2.

We can generalize the result: consider a nonhomogeneous Poisson process with intensity $\lambda(s)$. The unordered arrival times are i.i.d. random variables with conditional distribution function $F(x) = m(x)/m(t)$, where $m(x) = \int_0^x \lambda(s)\, ds$. Thus we solve $m(x) = e^{-1}m(t)$ and apply the x-strategy.

EXAMPLE 4. There are solutions to specific best choice problems which may look incompatible with the e^{-1}-law. Abdel-Hamid et al. (1982) say "if N is unknown, then the best choice problem has many reasonable solutions". This is correct, but with respect to real world problems, it is more precise to say that it may have different solutions according to different hidden restrictions. For instance, Presman and Sonin proved that, if N is uniform on $\{1, 2, \cdots, n\}$, then the optimal strategy is to pass over the first $k^*(n) \sim [e^{-2}n]$ candidates and then to choose the next leading one. The corresponding success probability tends to $2e^{-2} \approx 0.2707$ as $n \to \infty$. As a secretary problem, this makes sense if the N candidates do not appear simultaneously, since otherwise recalls would be possible. Thus time intervenes and we could make assumptions about the conditional arrival times like in the discussion rather than about g. Indeed assumptions would have to be rather wrong in order to do worse (probabilistically) with the e_F^{-1}-strategy than with the $k^*(n)$-strategy since $-x \ln x \geq 2e^{-2}$ for all $x \in [e^{-2}, 0.6661]$!

One may argue that there are situations where time does not intervene. However it is difficult to think of any practical situation where we would not be able to simulate it. Indeed, suppose that your assistant tells you that n candidates turned up for an interview. But he chose at random a ball from balls numbered $1, 2, \cdots, n$ and limited the number of candidates to the drawn number N. These N candidates are waiting in front of your office and you have to find the best one under the no-recall condition. Here we have indeed the mentioned problem in purified form, since g is known and time is eliminated. However, you can do the trick by saying "send them in alphabetic order", because your phone book on the table will give you at once an idea about the lexicographical family name distribution and your waiting time will be the last family name in the first 37% names. Similarly, if you have to choose the "best" ball in an urn with $N \leq n$ balls, you may add $m \gg n$ blank balls, mix them, and choose the first leading ball after the $[e^{-1}m]$th draw. Of course, if any redefinition or simulation is explicitly forbidden by definition of the problem, then the optimal strategy is the $k^*(n)$ policy mentioned before. This is what was meant by hidden restrictions.

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